Dense and high degree structures in graphs with chromatic number equal to maximum degree

by

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Abstract

It is well-known that for any graph G, $\chi(G) \leq \Delta(G) + 1$; Brooks' Theorem says that all graphs meeting this upper bound must contain either $K_{\Delta(G)+1}$ or be an odd cycle. The Borodin-Kostochka Conjecture, from 1977, posits that all graphs with $\chi(G) = \Delta(G)$ (with $\Delta(G) \geq 9$) must contain a $K_{\Delta(G)}$. This thesis has two main lines of work, both of which are related to the Borodin-Kostochka Conjecture. In the first, we focus on vertex-critical graphs with $\chi(G) = \Delta(G) = 9$ and prove that, given some forbidden subgraph conditions, we can get close (in some sense) to containing a $K_{\Delta(G)}$. In the second main topic of this thesis, we prove that every graph G with $\chi(G) = \Delta(G)$ (and $\Delta(G) \neq 5, 6$) contains either a "high" $K_{\omega(G)}$ or a "high" odd hole, where "high" means that the vertices have average degree at least $\Delta(G) - 1$. This theorem is a weakening of the Borodin-Kostochka Conjecture when $\Delta(G) \geq 9$; when $\Delta(G) \leq 8$ there are examples showing that both the high odd hole and the high $K_{\omega(G)}$ are needed in the statement of the theorem.

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Chapter 1

Introduction

In this dissertation all graphs are simple. This introductory chapter will include a number of definitions which are fundamental to the content of this thesis; for terms not defined here, we follow [23].

A k-*coloring* of a graph G is an assignment of the colors $\{1, 2, \ldots, k\}$ to the vertices of G so that adjacent vertices receive different colors. The *chromatic number* of a graph G, denoted $\chi(G)$, is the minimum k such that G has a k-coloring. Given any graph G, it is easy to show that $\omega \leq \chi(G) \leq \Delta + 1$, where $\Delta = \Delta(G)$ is the maximum degree of G and $\omega = \omega(G)$ is the clique number of G, denoted $\omega = \omega(G)$. A cornerstone of chromatic graph theory is Brooks' Theorem [2], which says that if $\chi = \Delta + 1$ then G contains $K_{\Delta+1}$ or is an odd cycle. In this thesis we focus on graphs with chromatic number one less than this, that is, graphs with $\chi(G) = \Delta(G)$. Looming large in the study of such graphs is the following famous conjecture from 1977.

Conjecture 1.1 (Borodin and Kostochka [1])**.** *Let G be a graph with $\Delta \geq 9$. Then $\chi = \Delta$ implies that G contains K_Δ as a subgraph.*

We will discuss what is currently known towards the Borodin-Kostochka Conjecture in subsection 1.1. For now, let us just say that it is very much open, and that results presented in this thesis can be seen as progress towards it.

In Chapter 3 of this thesis we prove several different results about graphs with $\chi = \Delta \geq 9$ which do not contain certain forbidden substructures; we will describe this in more detail in subsection 1.2. It is worth noting that our work in Chapter 3 will require the technique of *Mozhan Partitions*, which will be introduced in Chapter 2.

Chapter 4 of this thesis is mainly focused on the proof of one theorem, which says that: any graph with $\chi = \Delta \neq 5, 6$ either contains a K_ω with average degree at least $\Delta - 1$ or an *odd hole*, that is a chordless odd cycle of length at least 5, with average degree at least $\Delta - 1$. We will provide more context for this result in subsection 1.3.

1.1 The Borodin Kostochka Conjecture

The first comment to make about Conjecture 1.1 is that the $\Delta \geq 9$ condition is necessary: the graph $C_5(C_3)$ obtained by blowing up C_5 with triangles – that is, replacing each vertex in C_5 with a triangle and replacing each edge with the complete bipartite graph $K_{3,3}$ – has $\chi = \Delta = 8$ but does not contain K_8.

Conjecture 1.1 is known to be true when $\Delta > 10^{14}$ (Reed [20]). It is also known to be true when induced copies of certain small graphs are forbidden, in particular: the claw $K_{1,3}$ (Cranston and Rabern [7]); P_6 (Wu and Wu [24]), and P_5 and C_4 (Gupta and Pradhan [12], with an improvement by Cranston, Lafayette and Rabern [6]). For our work in this thesis, the following approximation results towards the Borodin-Kostochka Conjecture will turn out to be very important; note that these build on prior work by Kostochka [15] and Mozhan [18].

Theorem 1.2 (Cranston and Rabern [9])**.** *Let G be a graph with $\chi(G) = \Delta(G)$.*

(a) If $\Delta(G) \geq 13$ then G contains $K_{\Delta(G)-3}$.

(b) If G contains no $K_{\Delta(G)}$, then it contains a $K_{\Delta(G)-5}$ where every vertex in the clique has degree $\Delta(G)$.

In order to make progress on Conjecture 1.1 many authors assume that Δ is larger than its minimum possible value of nine, for example Theorem 1.2(a) above, also Kostochha [15], Mozhan [18], and Haxell and MacDonald [14]. One important one reason for this will become apparent in Chapter 2. However Catlin [3] and Kostochka [15] both independently proved that it suffices to prove Conjecture 1.1 for $\Delta = 9$ only. When G is a graph with $\chi = \Delta = 9$ then G is known to contain K_5 (Borodin and Kostochka [1]); such a G is also known to contain a

K_4 in which all the vertices of the K_4 have degree Δ (Cranston and Rabern [9]). Cranston and Rabern have shown that it is not necessary to find a full K_9 in such graphs however, and in fact the following conjecture is completely equivalent to Conjecture 1.1. Note that here and in what follows, given graphs G_1, G_2, by $G_1 \vee G_2$ we mean the graph obtained by taking disjoint copies of G_1, G_2 and then adding all possible edges between G_1 and G_2. By E_t we mean the edgeless graph on t vertices.

Conjecture 1.3 (Cranston and Rabern [8])**.** *Any graph with $\chi = \Delta = 9$ contains $K_3 \vee E_6$ as a subgraph.*

We have now seen that in order to prove Conjecture 1.1 we may restrict ourselves to graphs with $\chi = \Delta = 9$ and we need not find a full K_9, but rather $K_3 \vee E_6$ suffices — and there is yet one more significant restriction we can make. It is easy to show, using Brooks' Theorem, that it suffices to prove Conjecture 1.1 (and Conjecture 1.3) for *vertex-critical* graphs, that is, for graphs G where $\chi(G - v) < \chi(G)$ for all $v \in V(G)$. Of course, thinking about Conjecture 1.1, if G is vertex-critical with $\chi = \Delta = 9$ and it contains K_9 as a subgraph, then since $\chi(K_9) = 9$ we get $G = K_9$, which is a contradiction since $\Delta(K_9) = 8$. So another completely equivalent form of Conjecture 1.1 (and Conjecture 1.3) is that there *are no* vertex-critical graphs with $\chi = \Delta = 9$.

1.2 Forbidden Substructures

Here we outline in detail the results we will present in Chapter 3 of this thesis, all of which will all have the following form: *if G is a vertex-critical graph with $\chi = \Delta = 9$ which does not contain some special forbidden substructures, then G contains $K_3 \vee E_6$ (or is close to containing $K_3 \vee E_6$).*

The forbidden substructures we consider all have one of four basic forms – F, R, Q or S, as depicted in Figure 1.1. In each of these figures, a path of length two with endpoints joined by a dotted line indicates that the path is part of an odd hole in the larger graph G. The rest of the structure need not be an induced subgraph in G, but we do specify the degree of certain vertices to be either Δ or $\Delta - 1$ in G. These restrictions are given in Table 1.1; vertices not listed in the table can have any degree in G. As an example, a graph G contains R_2 if it contains R as a subgraph, where: (v_1, v_6, v_2) happens to be part of an odd hole in G; v_1 and v_2 can have any degree in G; v_6 has degree $\Delta - 1$ in G, and; all other vertices in R_2 have degree Δ in G. It may be helpful to keep in mind that even vertices allowed to have any degree in G only have two choices: it is easy to show that in any vertex-critical graph G with $\chi = \Delta$, all vertices must have degrees Δ or $\Delta - 1$.

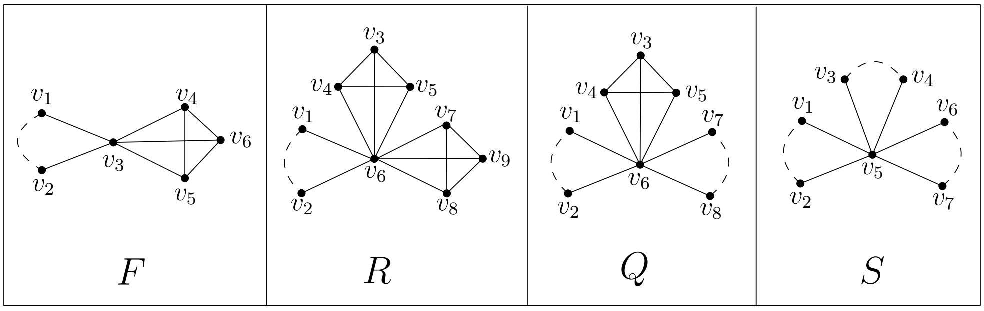

Figure 1.1: The forbidden substructures F, R, Q, and S.

	Δ-Vertices	$(\Delta - 1)$-Vertices
F	$v_1, \ldots, v_6$	
R_1	$v_1, \ldots, v_5, v_7, v_8, v_9$	v_6
R_2	$v_3, v_4, v_5, v_7, v_8, v_9$	v_6
R_3	$v_3, v_4, v_5, v_7, v_8, v_9$	v_1, v_6
R_4	v_3, v_4, v_5	v_1, v_6
R_5	v_3, v_4, v_5	v_6, v_7
R_6	v_3, v_4, v_5	v_6
R_7	v_1, v_2	v_6, v_7
R_8	v_3, v_4	v_1, v_5, v_6
R_9	v_3, v_4	v_5, v_6, v_7
R_{10}	$v_1, \ldots, v_5$	v_6, v_7
Q	$v_1, \ldots, v_5$	v_6
S	v_3, v_4	v_5, v_6

Table 1.1: Vertices in the F, R, Q, S forbidden substructures that are required to have degree Δ or $\Delta - 1$ in G. Vertices not listed can have any degree in G.

By forbidding certain groups of these structures, we can get that our target G contains all of $K_3 \vee E_6$; specially we can prove the following theorem.

Theorem 1.4. *Let G be a vertex-critical graph with $\chi = \Delta = 9$ which does not contain any of $F, R_1, R_4, R_5, R_7, R_8, R_9$. Then G contains $K_3 \vee E_6$ as a subgraph.*

The graphs R_4, R_5, $R_7 - R_9$ all contain S, since each contains three triangles intersecting at a vertex of degree $\Delta - 1$ (with one of the triangles containing another vertex of degree $\Delta - 1$ and another of the triangles containing two vertices of degree Δ). So Theorem 1.4 also yields the following corollary.

Corollary 1.5. *Let G be a vertex-critical graph with $\chi = \Delta = 9$ which does not contain any of F, R_1, S. Then G contains $K_3 \vee E_6$ as a subgraph.*

If we are okay with some edges of $K_3 \vee E_6$ being missing, then we can get get more relaxed forbidden subgraph conditions. In particular, we get the following, where by H^{-k} we mean a graph obtained from H by removing k edges.

Theorem 1.6. *Let G be a vertex-critical graph with $\chi = \Delta = 9$ which does not contain any of F, R_2, R_5. Then G contains $(K_3 \vee E_6)^{-4}$ as a subgraph.*

Theorem 1.7. *Let G be a vertex-critical graph with $\chi = \Delta = 9$ which does not contain any of F, R_1, R_3, R_{10}. Then G contains $(K_3 \vee E_6)^{-6}$ as a subgraph.*

Theorems 1.6 and 1.7 also yield the following corollaries (since both R_2 and R_5 contain R_6, and since R_1, R_3, R_{10} all contain Q).

Corollary 1.8. *Let G be a vertex-critical graph with $\chi = \Delta = 9$ which does not contain any of F, R_6. Then G contains $(K_3 \vee E_6)^{-4}$ as a subgraph.*

Corollary 1.9. *Let G be a vertex-critical graph with $\chi = \Delta = 9$ which does not contain any of F, Q. Then G contains $(K_3 \vee E_6)^{-6}$ as a subgraph.*

Observe that none of the forbidden substructures in our above theorems are necessarily contained within either a K_5, or a K_4 having all vertices of degree Δ – which is important, since we know those structures *must* be present (in any graph with $\chi = \Delta = 9$) by the above-mentioned work of Borodin and Kostochka [1] and Cranston and Rabern [9], respectively. It is not obvious that a graph with $\chi = \Delta = 9$ should be *able* to forbid all of the substructures listed in our results above: but it can. In the final section of Chapter 3, we present an infinite family of graphs with $\chi = \Delta = 9$ which contain none of the substructures listed in Table 1 (or listed in our above results). Of course, our infinite family of graphs are not vertex-critical: as discussed above, finding a vertex-critical graph with $\chi = \Delta = 9$ would amount to disproving Conjecture 1.1. However, our forbidden substructure assumptions do not seem to get in the way of vertex-criticality. In particular, Kostochka and Yancey [16] have established a lower bound on the average degree for vertex-critical graphs; this bound implies that a vertex-critical graph with $\chi = 9$ must have average degree at least 8.75. All the members of our infinite family have average degree higher than this, even while they forbid all our listed substructures.

1.3 High cliques and high odd holes

We say a subgraph H of a graph G is *high* if the vertices of H have average degree at least $\Delta(G) - 1$ in G, where $\Delta(G)$ denotes the maximum degree of G. In particular, this means that if $K_{\Delta(G)}$ is a subgraph of G, then it is a high subgraph. On the other hand, a copy of $K_{\omega(G)}$ in G, with $\omega(G)$ denoting the clique number of G, is not necessarily high when $\omega(G) < \Delta(G)$. In Chapter 4 of this thesis, we prove the following theorem.

Theorem 1.10. *Let G be a graph with $\chi(G) = \Delta(G)$ and $\Delta(G) \neq 5, 6$. Then G contains either a high $K_{\omega(G)}$ or a high odd hole.*

In fact we conjecture that Theorem 1.10 holds for $\Delta(G) = 5$ and $\Delta(G) = 6$ as well; we will see later why our current methods don't extend to those cases.

When $\Delta(G) \geq 9$ our Theorem 1.10 is a (significant) weakening of the Borodin-Kostochka Conjecture. When $3 \leq \Delta(G) \leq 8$ there are examples showing that G need not contain a $K_{\Delta(G)}$ when $\chi(G) = \Delta(G)$ (although it trivially does for $\Delta(G) \leq 2$). The afore-mentioned $C_5(K_3)$ is the only known example for $\Delta(G) = 8$, but there are more for smaller values of $\Delta(G)$: Catlin [3] defined seven examples for $\Delta(G) = 7$ along with an infinite family for $\Delta(G) = 6$, and it is not hard to construct examples for $3 \leq \Delta(G) \leq 5$. The graph $C_5(K_3)$ and the examples by

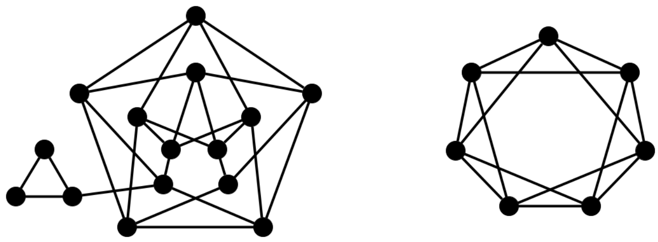

Figure 1.2: Two graphs with $\chi = \Delta = 4$, one with a high odd hole but no high K_ω (left), and one with a high odd hole and no high K_ω (right).

Catlin contain both a high $K_{\omega(G)}$ and a high odd hole. However, the high odd hole is needed in Theorem 1.10: the graph G on the left of Figure 1.2 was obtained from a construction of Chvátal [5] by adding a pendant K_3 and has $\chi(G) = \Delta(G) = 4$ but does not contain a high $K_3 = K_{\omega(G)}$ (it does contain a high odd hole). The high $K_{\omega(G)}$ is also needed in Theorem 1.10: plentiful examples can be made by taking a $K_{\omega(G)}$ and attaching one or more pendant structures, and the graph on the right in Figure 1.2 is another type of example by Xie [25].

Recall now the statement of Theorem 1.2, presented in subsection 1.1. The (b) part of this theorem immediately implies the truth of Theorem 1.10 when $\omega(G) \leq \Delta(G) - 5$ (in a very strong way). Of course, such a (very) high $K_{\Delta(G)-5}$ may not have many vertices in common with any $K_{\omega(G)}$'s. In any case, we will use both parts (a) and (b) of Theorem 1.2 in our proof of Theorem 1.10; they will allow us to assume that $\omega(G) \geq \Delta - 4$ and that $\omega(G) \geq \Delta - 3$ when $\Delta(G) \geq 13$.

We can prove that Theorem 1.10 holds for $\Delta(G) \leq 4$ using an elementary, self-contained argument. But our proof that it holds for $\Delta(G) \geq 7$ requires both Theorem 1.2 and another result concerning odd holes. Here and in what follows, a vertex v in a graph G is *critical* if $\chi(G - v) < \chi(G)$.

Theorem 1.11. *Let G be a graph with $\chi(G) = \Delta(G) \geq 7$, and suppose that G does not contain $K_{\Delta(G)}$. Then the following are both true.*

(a) (Xie [26]) *Every critical vertex in G is contained in some odd hole.*

(b) (Chen, Lan, Lin, and Zhou [4]) *G contains an odd hole.*

Theorem 1.11 (b) says that any counterexample to the Borodin-Kostochka Conjecture contains an odd hole – but of course it gives us no indication of the vertex degrees in the odd hole. In fact, we won't find occasion to use Theorem 1.11(b) in our proof, but Theorem 1.11(a) will be very important for us. It is worth noting that the required bound here of $\Delta(G) \geq 7$ is the main reason why we need to exclude $\Delta(G) \in \{5, 6\}$ from our Theorem 1.10. There is some hope to overcome this limitation in the future however: although the number 7 is present in both Theorem 1.11(a) and (b), it is not certain to be required, at least not for Theorem 1.11(a) [25]. The graph on the right in Figure 1.2 (due to Xie [25]) does show that neither part of Theorem 1.11 holds for $\Delta(G) = 4$.

Before proceeding to the main content of this thesis let us share one small result which is similar in nature to Theorem 1.10 but which we have not otherwise seen in the literature. Namely, we prove the following strengthening of a classic theorem on chromatic number (due independently to Gallai [11], Hasse [13], Roy [21] and Vitaver[22]).

Theorem 1.12. *If G is a graph with $\chi(G) = t$ for some positive integer t, then G contains a path of length t whose vertices all have degree at least $t - 1$.*

Proof. Consider the set of all vertices with degree at least $t - 1$ in G, and let H be the graph they induce. If H does not contain a copy of P_t, then $\chi(H) \leq t - 1$ by the Gallai–Hasse–Roy–Vitaver Theorem; let φ be a $(t-1)$-coloring of H. But then since every vertex outside of H has degree at most $t - 2$, we can greedily extend φ to a $(t - 1)$-coloring of G, contradiction. □

It is worth noting that Theorem 1.12 finds structures where all vertices have large degree, while our Theorem 1.10 finds a structure with high average degree. Theorem 1.2(b) finds that any graph G with $\chi(G) = \Delta(G)$ contains a $K_{\Delta(G)-5}$ where every vertex has degree $\Delta(G)$, indeed every vertex is very high. Cranston and Rabern state in [8] that they suspect Theorem 1.2(b) could be improved to a $K_{\Delta(G)-4}$. This has been confirmed for several cases: when $\Delta(G) \equiv 1 (mod 3)$ [8], when $\Delta(G) = 6$ [19], and when $\Delta(G) = 7$ [17]. While Theorem 1.2(b) does find a clique where every vertex is very high, the size of said clique may be well below the clique number.

In contrast, when our Theorem 1.10 produces a clique with high average degree, we know the clique is as large as possible. Our proof methods for Theorem 1.10 limit us to only finding structures with high average degree, but we have not yet encountered an example which shows that our structures do not have every vertex high. We have however found many examples with $\chi(G) = \Delta(G)$ where every $K_{\omega(G)}$ and odd hole subgraph contains one or more vertices of degree $\Delta(G) - 1$, the simplest of which is any clique with an isolate appended. See also the graph O_5 in [8] and the aforementioned graphs defined by Catlin [3] with $\Delta = 6, 7$. Therefore Theorem 1.10 cannot be strengthened to find a subgraph where every vertex has degree $\Delta(G)$, but we do conjecture that the following may be true: *G a graph with $\chi(G) = \Delta(G)$. G contains either a $K_\omega(G)$ where every vertex is high or an odd hole where every vertex is high.*

Chapter 2

Mozhan Partitions

We now introduce *Mozhan Partitions,* a technique which will be the main tool used to prove our results in Chapter 3. This method was first developed by Mozhan [18], and we follow the notation used by Kostochka, Rabern, and Stiebitz in [17], with a few minor simplifications.

Let G be a vertex-critical graph with $\chi = 1 + t_1 + \cdots + t_p \geq 4$, for some positive integers $t_1, \ldots, t_p, p$. Then a $(t_1, \ldots, t_p)$*-partition* of G is a sequence $(v, X_1, \ldots, X_p)$ where $v \in V(G)$ is called the *special vertex*, $X_1, \ldots, X_p$ is a partition of $V(G) \setminus \{v\}$, and $\chi(G[X_i]) = t_i$ for $i = 1, \ldots, p$. Such a partition is easily obtained from any χ-coloring of G where one color class has size one (always possible since G is vertex-critical), by simply taking v to be this special color class of size one, and then grouping t_i of the color classes together to form X_i, for all i. This way of making a $(t_1, \ldots, t_p)$-partition of G means that the only edges in $G[X_i]$ are between the t_i color classes of X_i. A $(t_1, \ldots, t_p)$-partition of G is said to be *optimal* if $|E[X_1]| + \cdots + |E[X_p]|$ is minimum over all $(t_1, \ldots, t_p)$-partitions of G, where $E[X] = E(G[X])$.

Given any vertex-critical graph G with $\chi \geq 1$, an old result of Dirac [10] tells us that G is $(\chi - 1)$-edge-connected, and hence that the minimum degree in G is at least $\chi - 1$. We call a vertex in G *high* if its degree is $\geq \chi$ in G, and *low* if its degree is $< \chi$. Note that in the case $\chi = \Delta$ we are interested in, our high vertices will all have degree Δ and our low vertices will all have degree $\Delta - 1$. An optimal $(t_1, \ldots, t_p)$-partition of G is said to be *proper* if the special vertex v is low.

The following result is the cornerstone of the Mozhan partition method and is fundamental to our work; a short proof, which uses Brooks' Theorem, is written in [17]. Here and in what follows, for a vertex $a \in V(G)$ and a vertex set $X \subseteq V(G)$, we let $N_X(a) = N(a) \cap X$ and we let $d_X(a) = |N_X(a)|$. If $(v, X_1, \ldots, X_p)$ is a $(t_1, \ldots, t_p)$-partition of G, then by X_i^* we mean the component of $G[\{v\} \cup X_i]$ containing v, for any $1 \leq i \leq p$.

Theorem 2.1 (Mozhan [18]). *Let G be a vertex-critical graph and let $(v, X_1, \ldots, X_p)$ be an optimal $(t_1, \ldots, t_p)$-partition of G. Then the following statements hold:*

1. $\chi(X_i^*) = t_i + 1$ *and* $d_{X_i}(v) \geq t_i$ *for all* i.

2. *If v is a low vertex of G, then* $d_{X_i}(v) = t_i$ *for all* i.

3. *If $d_{X_i}(v) = t_i$ for some i, then either $X_i^* = K_{t_i+1}$ or $t_i = 2$ and X_i^* is an odd cycle.*

In terms of how to picture proper $(t_1, \ldots, t_p)$-partitions, see Figure 2.1 for an example of a proper $(t_1, \ldots, t_p)$-partition where all $t_i = \chi(X_i) \geq 3$ except for $t_3 = \chi(X_3) = 2$; note that each of the components X_i^* are cliques, except for X_3^*, which is an odd cycle.

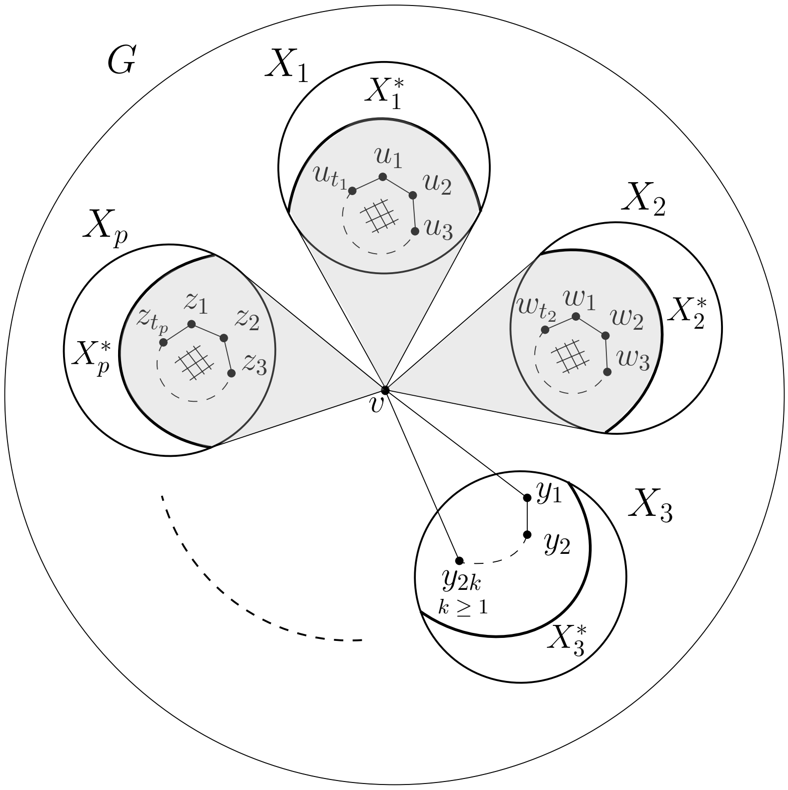

Figure 2.1: General Mozhan partition

If a vertex-critical graph G has $\chi \geq 10$, then we can always get a (t_1, t_2, t_3)-partition of G where $t_i \geq 3$ for all i (via our above discussion) and hence Theorem 2.1(3) always gives a clique instead of odd cycle. This is why it is convenient to assume $\chi \geq 10$ when using the method of Mozhan partitions, but we will not have that luxury in this thesis.

In addition to Theorem 2.1, we will also use two other theorems from [17] concerning these partitions.

Theorem 2.2 (Kostochka, Rabern, and Stiebitz [17]). *Let G be a vertex-critical graph and let $(v, X_1, \ldots, X_p)$ be a proper $(t_1, \ldots, t_p)$-partition of G. If $y \in V(X_i^*) \setminus v$ is a low vertex of G for some i, and y has a neighbor in $X_j^* \setminus \{v\}$ for some $j \neq i$, then $N_{X_j}(v) = N_{X_j}(y)$.*

Let $P = (v, X_1, \ldots, X_p)$ be an optimal $(t_1, \ldots, t_p)$-partition of G and say vertex $y \in X_j$ for some j. Let $Y_i = X_i$ if $i \neq j$ and let $Y_j = (X_j \cup \{v\})/\{y\}$; we say that $P' = (y, Y_1, \ldots, Y_p)$ is obtained by *swapping* v with y in P. We can see that P' is a $(t_1, \ldots, t_p)$-partition of G if and only if $\chi(G[Y_j]) = t_j$. The following theorem (in fact a combination of two theorems from [17]) provides conditions for when swapping maintains the property of being optimal, or even being proper.

Theorem 2.3 (Kostochka, Rabern, and Stiebitz [17]). *Let G be a vertex-critical graph, let $P = (v, X_1, \ldots, X_p)$ be a $(t_1, \ldots, t_p)$-partition of G, and let $y \in V(X_j^*) \setminus v$.*

1. *If P is optimal and $d_{X_j}(v) = t_j$, then swapping v for y in P yields an optimal $(t_1, \ldots, t_p)$-partition of G.*

2. *If P is proper and y is low, then swapping v for y in P yields a proper $(t_1, \ldots, t_p)$-partition of G.*

Chapter 3

Chromatic number and maximum degree both equal to nine

Sections 3,1, 3.2, and 3.3 present the proofs of Theorems 1.4, 1.6, and 1.7, respectively. Section 3.4 provides a description of an infinite family of graphs as promised in the introduction.

3.1 Containing $K_3 \vee E_6$

Theorem 1.4. *Let G be a vertex-critical graph with $\chi = \Delta = 9$ which does not contain any of $F, R_1, R_4, R_5, R_7, R_8, R_9$. Then G contains $K_3 \vee E_6$ as a subgraph.*

Proof. Since G is vertex-critical with $\chi = 9$, we may choose an optimal $(2, 3, 3)$-partition of G, say $P = (v, X, Y, Z)$.

Claim 1. *We may assume that the special vertex v of P is low, and hence that P is proper.*

Proof of Claim. By Theorem 2.1(1), v has at least two edges into X and at least three edges into both Y and Z. Suppose, for a contradiction, that v is high – then it has degree 9, so it has one additional edge to some part X, Y, or Z. Since x does not meet the hypothesis for Theorem 2.1(3) for some part, the corresponding graph may not be a clique or an odd cycle. However, Theorem 2.1(3) does apply for the other two parts and hence the corresponding graphs there are cliques or odd cycles. But then G contains the basic structure of F with $v_3 = v$ a high vertex. Since G does not contain F, at least one of v_1, v_2, v_4, v_5, v_6 must be a low vertex. But then by Theorem 2.3(1), we can swap v with this low neighbor, obtaining an optimal (2,3,3)-partition where the special vertex is low. □

Consider now Algorithm 1. This algorithm performs a sequence of vertex swaps to gradually change our proper (2,3,3)-partition $P = (v, X, Y, Z)$, while maintaining that we indeed have a proper (2,3,3)-partition. Once a pair of vertices are swapped in the algorithm, we say that both of them have *moved*, a designation they will carry for the remainder of the algorithm. Each time a vertex moves, we create a new partition. We begin the algorithm with our original P and denote the partition when a certain vertex v_i is special by $P_i = (v_i, X_{v_i}, Y_{v_i}, Z_{v_i})$. Furthermore, we use $X^*_{v_i}, Y^*_{v_i}, Z^*_{v_i}$ to denote the components of $G[X_{v_i} \cup \{v_i\}], G[Y_{v_i} \cup \{v_i\}], G[Z_{v_i} \cup \{v_i\}]$ containing v_i, respectively. Note that in our original partition $P = (v, X, Y, Z)$ (which we label as $P = (v_1, X_{v_1}, Y_{v_1}, Z_{v_1})$), since $v = v_1$ is low, Theorem 2.1(2) and Theorem 2.1(3) implies that $X^*_{v_1}$ is an odd cycle and that $Y^*_{v_1}, Z^*_{v_1}$ are cliques (each of size four). The algorithm only ever swaps a pair of low vertices so by Theorem 2.3(2), $(v_i, X_{v_i}, Y_{v_i}, Z_{v_i})$ will be a proper (2,3,3)-partition throughout the algorithm, and hence by Theorem 2.1(3) we know $X^*_{v_i}$ will be an odd cycle and $Y^*_{v_i}, Z^*_{v_i}$ will be cliques (each of size four) throughout the algorithm. Given any P_i, we may refer to X_{v_i} as the *X-set* or the *cycle part*, while we may refer to Y_{v_i}, Z_{v_i} as the the Y-set or Z-set, respectively, or as the *clique parts*.

Algorithm 1 begins by swapping the special vertex $v = v_1$ with any low neighbor of v_1. After that, the algorithm alternates between a swap involving the Y-set, and a swap involving either the X-set or the Z-set (with a preference for the X-set). Note that the condition for swapping with an X-set or Z-set is stricter than the condition for swapping with the Y-set. See Figure 3.1 for an example of of a sequence of three swaps in Algorithm 1. Depicted in Figure 3.1 are the following swaps: iteration i we imagine that v_i has just been swapped out of the X-set in the previous step, so we swap v_i with some $w_i \in Y$; in iteration $i + 1$ we swap the new v_{i+1} with $w_{i+1} \in Z$ (meaning v_{i+1} must have a low neighbor in the X-set that has already moved), and; in iteration $i + 2$ we swap the new v_{i+2} with $w_{i+2} \in Y$.

Algorithm 1 makes sense as written because we have forbidden R_1, R_5 and R_7. First, we need to ensure that in iteration 1, the vertex v_1 *has* a low neighbor, and forbidding R_1 accomplishes this. Secondly, in iteration $i(2)$, we need to ensure that the condition on the low vertices in the X-set or Z-set is not met vacuously, that is, we need to ensure that if one of these conditions is met then there is at least one low neighbor in the X-set or Z-set, respectively, to swap with. So if the special vertex v_i has just swapped out of the Y-set, we want it to have at least one low neighbor in both the X-set and the Z-set. In this scenario, v_i has a low neighbor in the Y-set (the vertex it just swapped with), and forbidding R_5 and R_7 forces v_i (v_6) to have a low neighbor in both the X-set and the Z-set.

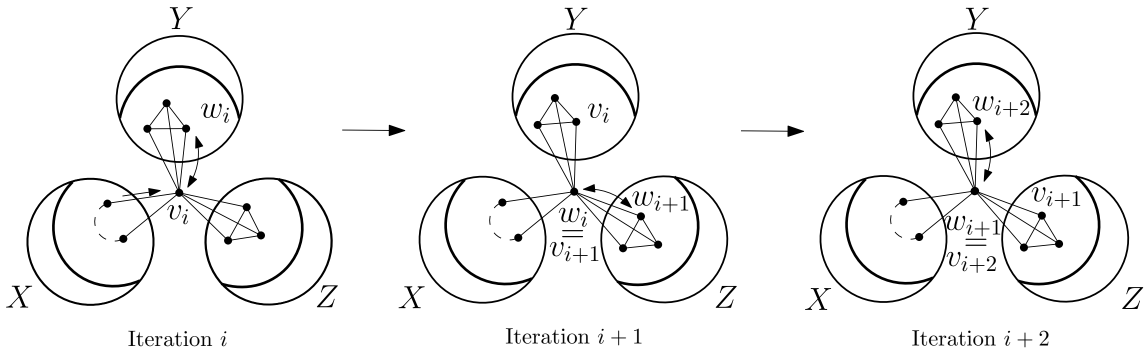

Figure 3.1: Example of three successive iterations of Algorithm 1

Algorithm 1.

• *Initialize:* $P_1 = P$, $v_1 = v$, and $i = 1$.

• *Iteration 1:* In P_1, swap v_1 with a low neighbor w_1 in one of $X_{v_1}, Y_{v_1}, Z_{v_1}$, setting $j_1 = 1, 2,$ or 3, respectively. Set $v_2 = w_1$ and let P_2 be the resulting proper (2,3,3)-partition $(v_2, X_{v_2}, Y_{v_2}, Z_{v_2})$.

• *Iteration i, $i \geq 2$:*

(1) *If $j_{i-1} \in \{1, 3\}$:*

In P_i, if there exists a low vertex $w_i \in Y^*_{v_i} \setminus \{v_i\}$ that has never moved, then swap v_i with w_i, set $j_i = 2$, set $v_{i+1} = w_i$, let P_{i+1} be the resulting proper (2,3,3)-partition, and iterate. Otherwise, terminate the algorithm.

(2) *If $j_{i-1} = 2$:*

In P_i, if every low neighbor of v_i in $X^*_{v_i}$ has never moved, then swap v_i with some such low neighbor w_i, set $j_i = 1$, set $v_{i+1} = w_i$, let P_{i+1} be the resulting proper (2,3,3)-partition, and iterate. Otherwise, if every low vertex in $Z^*_{v_i} \setminus \{v_i\}$ has never moved, then swap v_i with some such low vertex w_i, set $j_i = 3$, set $v_{i+1} = w_i$, let P_{i+1} be the resulting proper (2,3,3)-partition, and iterate. Otherwise, terminate the algorithm.

In Algorithm 1, each time we swap a pair of vertices, we are swapping the special vertex with a vertex that has never moved before. Since the graph is finite, this means that the algorithm must terminate eventually. We divide the remainder of our proof into two cases, depending on whether our algorithm terminated due to the stopping condition in (1) or in (2) above. In both cases we will show the existence of the desired subgraph $K_3 \vee E_6$.

Case 1: *At termination,* $j_{i-1} = 2$.

In this case, at termination, we know that v_i was swapped out of the Y-set in the previous step (since $j_{i-1} = 2$). Moreover, we know that the special vertex v_i has a low neighbor $a \in X_{v_i}$ and a low neighbor $b \in Z_{v_i}$, both of which have already moved. Suppose that a moved before b. (The argument of this case is almost exactly the same if b moved before a, so we omit those details to avoid repetition.) Since a moved before b, we know that b cannot be the initial special vertex v_1. So given the algorithm's alternating pattern, and the fact that $b \in Z_{v_i}$, we know that b must have swapped out of the Y-set in a previous step. Let b' be the vertex that was special when this swap occurred; note that after this swap b' remains in the Y-set until the algorithm terminates.

Consider the point in the algorithm when a was special. Since a moves before both v_i and b, it must be that $v_i, b \in Y_a$. Furthermore, since $a \sim v_i \sim b$, we have $v_i, b \in Y_a^*$. In addition, a must have one more neighbor in the clique Y_a^*, call it m. So $V(Y_a^*) = \{a, v_i, b, m\}$.

Claim 2. *The vertex m does not move after a moves.*

Proof of Claim. Assume for contradiction that m did move at some point in the algorithm after a moves. First, we assume that m moves after a but before b. Then consider the point in the algorithm when m is special (so immediately after it swapped out of the Y-set). Since $m \sim a$ and a moves into the X-set, $a \in X_m^*$. Since a is a neighbor of m that has already moved, m must move into the Z-set. But then when b becomes the special vertex later in the algorithm, we would have $a \in X_b^*$ and $m \in Z_b^*$ (since $b \sim a, m$), and both a, m have moved. So the algorithm should have terminated when b was the special vertex, contradiction. We may now assume that m moves after both a and b. But then when m is special, we would already have $a \in X_m^*$ and $b \in Z_m^*$, and the algorithm should have terminated at this point, contradiction. $\square$

Claim 3. $v_i \sim b'$.

Proof of Claim. Consider the point in the algorithm when b was special. We know that v_i has not moved out of the Y-set yet (since it is special at termination). So since $b \sim v_i$, we get $v_i \in Y_b^*$. In addition, b' has just swapped into the Y-set (because b' is the vertex that swaps with b to make b special). So $b' \in Y_b^*$ as well, meaning that $v_i \sim b'$. $\square$

Claim 4. $m, b' \in Y_{v_i}^*$.

Proof of Claim. Consider the final partition P_i at termination. Since $m \sim v_i$ and m does not move after a (by Claim 2), we get that $m \in Y_{v_i}^*$. On the other hand, since $v_i \sim b'$ (by Claim 3) and since b' remains in the Y-set until termination, we also get that $b' \in Y_{v_i}^*$. $\square$

Let us now assemble everything we know about the partition P_i with special vertex v_i. We know by assumption that $a \in X_{v_i}^*$, $b \in Z_{v_i}^*$, and Claim 4 tells us that $m, b' \in Y_{v_i}^*$. Let us label now the other vertices adjacent to v_i in $X_{v_i}^*, Y_{v_i}^*$ and $Z_{v_i}^*$: let u_1 be this other vertex in the X-set, let u_2 be this other vertex in the Y-set, and let u_3, u_4 be these other two vertices in the Z-set. See the leftmost picture in Figure 3.2, where edges we currently know exist are indicated with thin edges, and edges we are about to find (in Claim 5) are indicated with thick edges. The thin edges in the picture come from $X_{v_i}^*, Y_{v_i}^*, Z_{v_i}^*$ as just described, plus the triangle (a, b, m) which is a part of the clique Y_a^*, and the edge $b \sim b'$. Our final goal is the $K_3 \vee E_6$ pictured on the right of Figure 3.2. In particular, we have already established that $\{v_i, m, b\}$ induces a K_3, and we have already established that v_i is adjacent to all of the vertices in our proposed E_6 (namely $a, b', u_1, u_2, u_3, u_4$). We already know that $m \sim a, b', u_2$ and that $b \sim a, u_3, u_4$. So it remains only for us to show that $m \sim u_1, u_3, u_4$ and that $b \sim u_1, u_2$. That is, Claim 5 will complete this case (of $j_{i-1} = 2$).

Claim 5. $m \sim u_1, u_3, u_4$ *and* $b \sim u_1, u_2$.

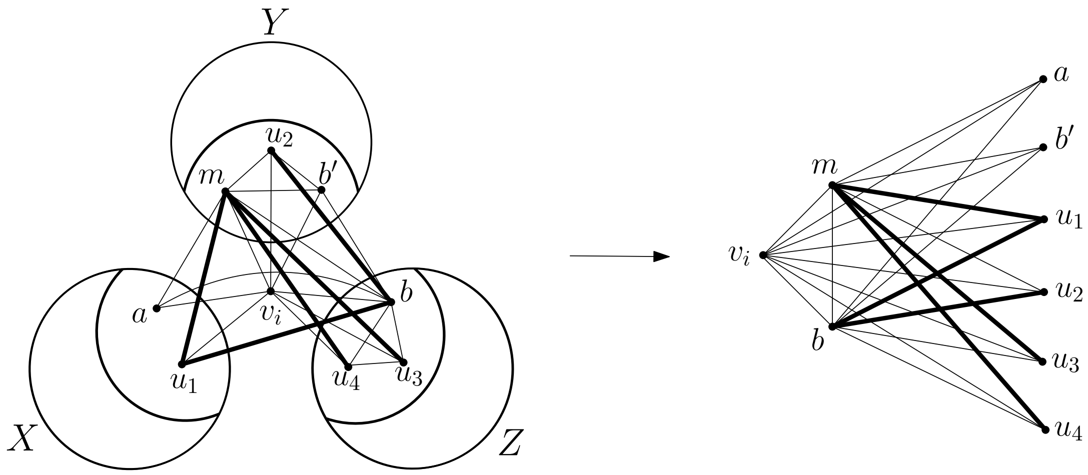

Figure 3.2: On the left we see the partition P_i at the termination of Algorithm 3.1; the graph to the right illustrates the final result: a $K_3 \vee E_6$.

Proof of Claim. In order to show that $b \sim u_2$, we apply Theorem 2.2: since b is low and $b \in Z^*_{v_i} \setminus \{v_i\}$, and since $b \sim m$ with $m \in Y^*_{v_i} \setminus \{v_i\}$, we have that $N_{Y_{v_i}}(v_i) = N_{Y_{v_i}}(b)$, and thus $b \sim u_2$ since $v_i \sim u_2$ (see Figure 3.2).

In order to show $m \sim u_3, u_4$, we again apply Theorem 2.2: m is low and $m \in Y^*_{v_i} \setminus \{v_i\}$, and since $m \sim b$ (by the last paragraph) and $b \in Z^*_{v_i} \setminus \{v_i\}$, we have that $N_{Z_{v_i}}(v_i) = N_{Z_{v_i}}(m)$, and thus $m \sim u_3, u_4$ since $v_i \sim u_3, u_4$ (see Figure 3.2).

It remains now to show that $u_1 \sim m, b$, for which we again apply Theorem 2.2: m, b are both low with $m \in Y^*_{v_i} \setminus \{v_i\}$ and $b \in Z^*_{v_i} \setminus \{v_i\}$, and since $m, b \sim a$ with $a \in X^*_{v_i} \setminus \{v_i\}$, we have that $N_{X_{v_i}}(v_i) = N_{X_{v_i}}(m) = N_{X_{v_i}}(b)$, and thus $m, b \sim u_1$ since $v_i \sim u_1$ (see Figure 3.2). □

Case 2: *At termination,* $j_{i-1} \in \{1, 3\}$.

In this case, at termination, we know that v_i was either swapped out of the X-set or the Z-set in the previous step (since $j_{i-1} \in \{1, 3\}$). Moreover, every low neighbor of v_i in Y_{v_i} has already moved. Since v_i either swapped out of the X-set or the Z-set, it certainly has a low neighbor in whichever set it just swapped out of (the vertex that it swapped with to become special). Since G does not contain R_4, R_5, we cannot have that all the neighbors of v_i in the Y-set are high. Since G does not contain R_8, R_9, we cannot have that exactly two of the neighbors of v_i in the Y-set are high. Thus, v_i has two low neighbors in Y_{v_i}, call them a', b'.

By the assumption on this case, a', b' have already moved; let a, b be the vertices that a', b' swapped with, respectively, when they swapped into the Y-set. Note that $a, b \neq v_i$, since by the assumption of this case, v_i just swapped out of the X-set or the Z-set, not the Y-set. Without loss, say that a' swapped into the Y-set before b'. In fact, assume that a' was the first of any of the low vertices in $Y_{v_i}^*$ to swap into the Y-set, followed by b' second. In particular, this means that a' was already in the Y-set when b' moved in.

Claim 6. *The vertices $\{a, a', b, b'\}$ induce a clique with the possible exception of the edge ab'. (In fact we will find later that ab' exists.)*

Proof of Claim. Edges aa' and bb' exist by definition of a, b. As $a', b' \in Y_{v_i}^*$, which is a clique, they are adjacent. Consider the point at which b' was special: at this point we must have had a', b in the Y-set, and since both are adjacent to b', we get that $a', b \in Y_{b'}^*$ which is a clique, so $a' \sim b$. Similarly, we consider the point where a' was special: at this point we must have had that a, b in the Y-set, and since both are adjacent to a', we get that $a, b \in Y_{a'}^*$ which is a clique, so $a \sim b$. □

We know that a, b were swapped out of the Y-set by a', b', respectively. The following claim gives more information about what happened immediately after each one of those swaps.

Claim 7. *After a swaps out of the Y-set, it must swap into the X-set. After b swaps out of the Y-set, it must swap into the Z-set.*

Proof of Claim. When a is special, it has a low neighbor a' in the Y-set. Since G does not contain R_7, it must be that a has some low neighbor in the X-set at this point. Since a just swapped out of the Y-set (it swapped with a'), the algorithm will look to move a into the X-set if possible. If it is not possible, it is because some low neighbor of a in the X-set has already moved; assume for a contradiction that this is the case. Since we guaranteed that this condition cannot be met vacuously, we may assume that when a was special, it had some low neighbor c in X_a^* which had already moved, and so a swapped into the Z-set.

Consider the point in the algorithm when c was special. This must be before a, b swap out of the Y-set. At this point, $a \in Y_c^*$, and as a and b are adjacent, $b \in Y_c^*$, so we have $b \sim c$.

Consider now the (later) point in the algorithm when b was special. This was after a, c had both moved. At this point, $c \in X_b^*$ and $a \in Z_b^*$. Then since a, c are low neighbors of b that have already moved, the algorithm should have terminated here by the stopping condition in (2). This is a contradiction since we assumed the algorithm terminated by the stopping condition in (1). So the first sentence of our claim statement is true.

Consider now the point in the algorithm when b was special. We know that b just swapped out of the Y-set (swapped with b'), so the algorithm will first look to swap b into the X-set if possible. But this is not possible here, because we know that a is in the X-set at this point (by the first sentence of our claim), and $a \sim b$ by the previous claim, and a has already moved. So, the algorithm will next look to swap b into the Z-set if possible. But we know it must be possible, since the algorithm doesn't terminate at this point (it terminates when $v_i \neq b$ is the special vertex). So we also get that the second sentence of our claim statement is true. □

Consider the point in the algorithm when b' is special, just before it swaps b out of the Y-set. We know that a' must have swapped into the Y-set at some prior step (swapping a out). Since $b' \sim a', b$ by Claim 6, we get that $a', b \in Y_{b'}^*$. Then $Y_{b'}^*$ is a K_4 made of b', a', b, and some other vertex m. We now analyze this vertex m.

Claim 8. *The vertex m never moves, and $m \sim a$.*

Proof of Claim. Assume to the contrary that m did move at some step before termination. First we assume m has already moved when b' is special. Then since $m \in Y_{b'}$, it must remain in the Y-set until the algorithm terminates. At termination, $a', b' \in Y_{v_i}^*$ and since $a', b' \sim m$, we have $m \in Y_{v_i}^*$. But since m swapped into the Y-set before b', we contradict our assumption that b' was the second low neighbor of v_i in Y_{v_i} to swap into the Y-set, after a'.

We next assume for contradiction that m moved after b' was special. We consider the partition when a' was special; we know this is before b' was special because we have assumed that a leaves the Y-set before b. At this step, none of the vertices a, b, m have moved yet, so they are all in the Y-set. Then $a, b, m \in Y_{a'}^*$, so in particular $m \sim a, b$. Since we are assuming that m moves after b', it must move after a and b, so we consider the later point in the algorithm when m is special. By Claim 7, a would be in the X-set and b in the Z-set by this point. Since m just moved out of the Y-set, the algorithm will look to swap m into either the X-set or Z-set. But since m has a low neighbor in both of these sets that has already moved (a, b), the algorithm would terminate by the stopping condition in (2), which is a contradiction since we assumed the algorithm terminated by the stopping condition in (1).

Since we have established that m never moves, consider the point at which a' is special. Then $a \in Y^*_{a'}$, and since $m \sim a'$ and $m \in Y_{a'}$, we get that $m \in Y^*_{a'}$, and hence $m \sim a$. $\square$

We now return our focus to the final partition at termination. Here v_i is special with $a', b' \in Y^*_{v_i}$ and $a, b,$ in the X-set and Z-set, respectively, by Claim 7. Since m never moves, it remains in the Y-set, and since $m \sim a', b'$, we have $m \in Y^*_{v_i}$. In order to achieve our final structure, we propose making one additional swap. We swap v_i with a', making a' the special vertex. As a' is low, this new partition is still proper by Theorem 2.3(2). Call this new partition $(a', X_{a'}, Y_{a'}, Z_{a'})$.

Let us now assemble everything we know about this new partition with special vertex a'. We know that $v_i, m, b' \in Y^*_{a'}$, and we know that $a \in X^*_{a'}$ since $a \sim a'$ by definition and $b \in Z^*_{a'}$ since $a' \sim b$ by Claim 6. Let us label now the other vertices adjacent to a' in $X^*_{v_i}$ and $Z^*_{v_i}$: let u_1 be this other vertex in the X-set and let u_2, u_3 be these other two vertices in the Z-set. See the leftmost picture in Figure 3.3, where edges we currently know exist are indicated with thin edges, and edges we are about to find (in Claim 9) are indicated with thick edges. The thin edges in the picture come from $X^*_{v_i}, Y^*_{v_i}, Z^*_{v_i}$ as just described, plus $m \sim a$ (by Claim 8), $a \sim b$ (by Claim 6), and $b \sim b'$. Our final goal is the $K_3 \vee E_6$ pictured on the right of Figure 3.3. In particular, we have already established that $\{a', b', b\}$ induces a K_3 by Claim 6, and we have already established that a' is adjacent to all of the vertices in our proposed E_6 (namely v_i, m, a, u_1, u_2, u_3). We already know that $b' \sim v_i, m, b$ and that $b \sim a, u_2, u_3, b'$. So it remains only for us to show that $b' \sim a, u_1, u_2, u_3$ and that $b \sim v_i, m, u_1$. That is, Claim 9 will complete this case (of $j_{i-1} \in \{1, 3\}$).

Claim 9. $b' \sim a, u_1, u_2, u_3$ *and* $b \sim v_i, m, u_1$.

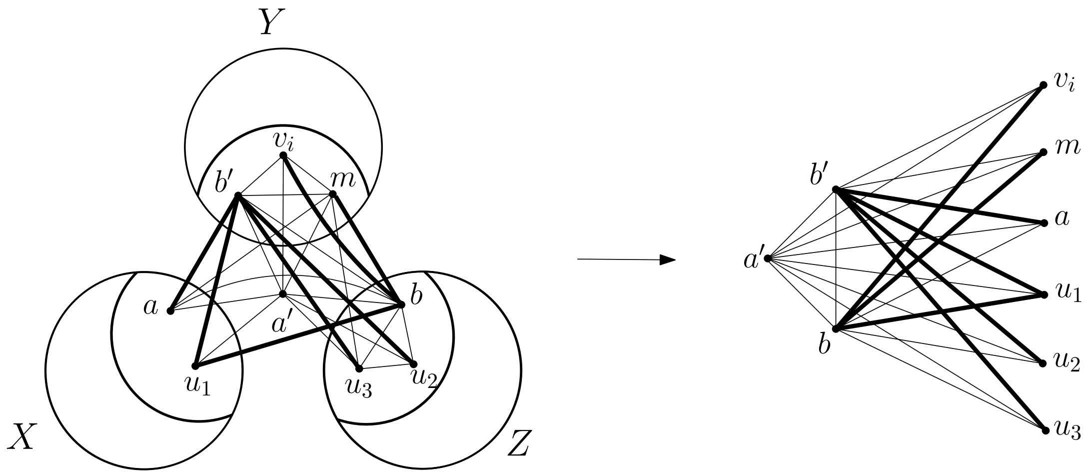

Figure 3.3: On the left we see the partition $(a', X_{a'}, Y_{a'}, Z_{a'})$; the thick edges are those found in Claim 9; the graph to the right illustrates the final result: a $K_3 \vee E_6$.

Proof of Claim. In order to show that $b \sim v_i, m$, we apply Theorem 2.2: since b is low and $b \sim b'$ with $b' \in Y^*_{a'} \setminus \{a'\}$, we have that $N_{Y_{a'}}(a') = N_{Y_{a'}}(b)$, and thus $b \sim v_i, m$ because $a' \sim v_i, m$. (See Figure 3.3).

In order to show that $b' \sim u_2, u_3$, we apply Theorem 2.2 again: since b' is low and $b' \sim b$ with $b \in Z^*_{a'} \setminus \{a'\}$, we have that $N_{Z_{a'}}(a') = N_{Z_{a'}}(b')$, and thus $b' \sim u_2, u_3$ because $a' \sim u_2, u_3$. (See Figure 3.3).

To show that $b' \sim a$, we again apply Theorem 2.2: since a is low and $a \sim m$ with $m \in Y^*_{a'} \setminus \{a'\}$, we have that $N_{Y_{a'}}(a') = N_{Y_{a'}}(a)$, and thus $a \sim b'$ since $a' \sim b'$. (See Figure 3.3).

It remains to show that $b, b' \sim u_1$, for which we again apply Theorem 2.2: since b, b' are both low and $b, b' \sim a$ (by our last line) with $a \in X^*_{a'} \setminus \{a'\}$, we have that $N_{X_{a'}}(a') = N_{X_{a'}}(b') = N_{X_{a'}}(b)$, and thus $b, b' \sim u_1$ since $a' \sim u_1$. (See Figure 3.3). □

□

3.2 Containing $(K_3 \vee E_6)^{-4}$

This section contains the proof of Theorem 1.6, which is restated below.

Theorem 1.6. *Let G be a vertex-critical graph with $\chi = \Delta = 9$ which does not contain any of F, R_2, R_5. Then G contains $(K_3 \vee E_6)^{-4}$ as a subgraph.*

Proof. Since G is vertex-critical with $\chi = 9$, we may choose an optimal (2,3,3)-partition of G, say $P = (v, X, Y, Z)$. We use the same argument as in Claim 1 of Theorem 1.4 to assume that the special vertex v is low and hence that P is a proper $(2, 3, 3)$-partition of G.

Consider now Algorithm 2, which we will discuss using the same notation and terminology as Algorithm 1. In fact, the two algorithms are very similar, with two key differences. The first difference is that switches into the X-set (i.e. the cycle part) are not allowed in Algorithm 2. In particular, this affects iteration 1, where we look to swap into either the Y-set or the Z-set, and also iteration i(2), where we look only to swap into the Z-set. The second key difference between Algorithm 1 and Algorithm 2 is that in iteration i(1), we raise the bar for the swapping condition into the Y-set – we now need *every* low vertex in $Y_{v_i^*} \setminus v_i$ to have never moved, rather than just having one that has never moved. Note that this means we have symmetric swapping conditions in $i(1)$ and $i(2)$, which is different from Algorithm 1.

Algorithm 2 makes sense as written because we have forbidden R_2 and R_5. First, we need to ensure that in iteration 1, the vertex v_1 *has* a low neighbor in one of the clique parts, and forbidding R_2 accomplishes this. Secondly, in iteration i, we need to ensure that the condition on the low vertices in the Y-set or the Z-set is not met vacuously, that is, we need to ensure that if the condition in (1) or (2) is met, then there is at least one low vertex in the Y-set or the Z-set, respectively, to swap with. So if the special vertex v_i has just swapped out of the Y-set, we want it to have at least one low neighbor in the Z-set. In this scenario, v_i has a low neighbor in the Y-set (the vertex it just swapped with), and forbidding R_5 forces v_i (v_6) to have a low neighbor in the Z-set. Similarly, a special vertex v_i which has just swapped out of the Z-set will be forced to have a low neighbor in the Y-set.

In Algorithm 2, just as in Algorithm 1, we get termination because swaps must always involve a vertex that has never moved before. In the proof of Theorem 1.4, we needed two different cases to find our target structure, depending on whether Algorithm 1 stopped due to condition (1) or condition (2). However now in Algorithm 2, these two conditions are symmetric, and indeed the sets Y, Z can be relabelled without loss of generality if we wish. Hence, we may assume without loss that Algorithm 2 terminates by the stopping condition in (1), that is, when $j_{i-1} = 2$.

Algorithm 2.

• *Initialize:* $P_1 = P$, $v_1 = v$, and $i = 1$.

• *Iteration 1:* In P_1, swap v_1 with a low neighbor w_1 in Y_{v_1} or Z_{v_1}, setting $j_1 = 1$ or 2, respectively. Set $v_2 = w_1$ and let P_2 be the resulting proper $(2,3,3)$-partition $(v_2, X_{v_2}, Y_{v_2}, Z_{v_2})$.

• *Iteration* i*,* $i \geq 2$:

(1) *If* $j_{i-1} = 2$*:*
In P_i, if every low vertex in $Y^*_{v_i} \setminus v_i$ has never moved, then swap v_i with some such low vertex w_i, set $j_i = 1$, set $v_{i+1} = w_i$, let P_{i+1} be the resulting proper partition $(2,3,3)$-partition, and iterate. Otherwise, terminate the algorithm.

(2) *If* $j_{i-1} = 1$*:*
In P_i, if every low vertex in $Z^*_{v_i} \setminus v_i$ has never moved, then swap v_i with some such low vertex w_i, set $j_i = 2$, set $v_{i+1} = w_i$, let P_{i+1} be the resulting proper partition $(2,3,3)$-partition, and iterate. Otherwise, terminate the algorithm.

Consider the special vertex v_i at termination. Since $j_{i-1} = 2$, v_i has just swapped out of the Z-set and has a low neighbor $a' \in Y^*_{v_i}$ which has already moved. Let one of the other vertices in $Y^*_{v_i} \setminus \{a'\}$ be called m. Then, in particular, $m \sim a'$. When a' moved into the Y-set in some prior step, it must have swapped a vertex out, call it a. Then because of our algorithm's alternating pattern, we know a must have swapped into the Z-set, where it remains at termination.

Claim 10. *The vertex* m *never moves, and* $a \sim m$*.*

Proof of Claim. Assume to the contrary that m moves at some step in the algorithm. We know that m is in the Y-set at termination, so it must swap into the Y-set either before or after a' does.

First assume for a contradiction that m swapped into the Y-set before a' did. We consider the step in the algorithm when a' was special, just before it swapped into the Y-set. Since $m \sim a'$, we have that $m \in Y^*_{a'}$, but then the algorithm should have terminated at this step by the stopping condition in (1) since m has already moved. Since $a' \neq v_i$, this is a contradiction.

Now assume for a contradiction that m swapped into the Y-set after a' did. We consider the step in the algorithm when m was special, just before it swapped into the Y-set. We find the same contradiction as in the first case: $m \sim a'$ implies that $a' \in Y^*_m$. Then since a' has already moved, the algorithm should have terminated at this point, but $m \neq v_i$, contradiction. Thus m never moves.

Since m never moves, it resides in the Y-set for every step of the algorithm. Thus, it is in the Y-set with a before a moves. In particular, since $a \sim a' \sim m$, we get $a, m \in Y^*_{a'}$, so $a \sim m$. □

We consider again the final partition at termination. Here v_i is special, $m, a' \in Y^*_{v_i}$, and a is in the Z-set. In order to achieve our final structure, we perform one additional swap. We swap a' with v_i so that a' is special and v_i is in the Y-set, and a, m remain where they are. Since a' is low, this new partition is still proper by Theorem 2.3(2). Call this new partition $(a', X_{a'}, Y_{a'}, Z_{a'})$.

We now assemble everything we know about this new partition with special vertex a'. We know that $v_i, m \in Y^*_{a'}$ and $a \in Z^*_{a'}$. Let us label now the other vertices adjacent to a' in $X^*_{a'}, Y^*_{a'}$ and $Z^*_{a'}$: let u_1, u_2 be these other vertices in the X-set, let u_3 be this other vertex in the Y-set, and let u_4, u_5 be these other two vertices in the Z-set. As $Y^*_{a'}$ and $Z^*_{a'}$ are cliques, we have that $\{m, v_i, u_3\}$ and $\{a, u_4, u_5\}$ induce cliques, and we also know that $a \sim m$ by Claim 10. Our final goal is a $(K_3 \vee E_6)^{-4}$. In particular, we will show that $\{a', a, v_i\}$ can play the role of K_3 for this structure, and that $\{m, u_1, \ldots, u_5\}$ can play the role of E_6, with the four potentially missing edges being those between $\{u_1, u_2\}$ and $\{v_i, a\}$. We have already established that a'

is adjacent to all of the vertices in our proposed E_6 (namely $m, u_1, \ldots, u_5$). We also already know that $v_i \sim m, u_3$ and that $a \sim m, u_4, u_5$. So it remains only for us to show that $a \sim v_i, u_3$ and $v_i \sim u_4, u_5$. That is, Claim 11 will complete the proof of Theorem 1.6.

Claim 11. $a \sim v_i, u_3$ *and* $v_i \sim u_4, u_5$.

Proof of Claim. In order to show that $a \sim v_i, u_3$, we apply Theorem 2.2: since a is low and $a \sim m$ with $m \in Y^*_{a'} \setminus \{a'\}$, we have that $N_{Y_{a'}}(a') = N_{Y_{a'}}(a)$, and thus $a \sim v_i, u_3$ since $a' \sim v_i, u_3$. To show that $v_i \sim u_4, u_5$, we apply Theorem 2.2 a second time: since v_i is low and $v_i \sim a$ (by our previous line) with $a \in Z^*_{a'} \setminus \{a'\}$, we have that $N_{Z_{a'}}(a') = N_{Z_{a'}}(v_i)$, and so $v_i \sim u_4, u_5$ because $a' \sim u_4, u_5$. □

□

3.3 Containing $(K_3 \vee E_6)^{-6}$

We now prove Theorem 1.7, which is restated below.

Theorem 1.7. *Let G be a vertex-critical graph with $\chi = \Delta = 9$ which does not contain any of F, R_1, R_3, R_{10}. Then G contains $(K_3 \vee E_6)^{-6}$ as a subgraph.*

Proof. Since G is vertex-critical with $\chi = 9$, we may choose an optimal (2,3,3)-partition of G, say $P = (v, X, Y, Z)$. We use the same argument as in Claim 1 of Theorem 1.4 to assume that the special vertex v is low and hence that P is a proper $(2, 3, 3)$-partition of G.

Consider now Algorithm 3, which we will discuss using the same notation and terminology as Algorithms 1 and 2. Algorithm 1 allows the special vertex to swap with any of the three parts (X-set, Y-set, Z-set) in the first iteration, and then alternates between swaps with the Y-set and swaps with the X-set or Z-set. Algorithm 2 maintains the same pattern, except it allows no swaps at all with the X-set. Algorithm 3 again allows swaps with any of the three parts, but it strictly prioritizes swaps with the clique sets (Y-set and Z-set), preferring to alternate swaps between these two sets only, and if forced to swap with the X-set, looking to terminate the algorithm immediately unless an "ideal next swap" presents itself. Let us now say more about how this all works.

Algorithm 3.

• *Initialize:* $P_1 = P$, $v_1 = v$, and $i = 1$.

• *Iteration 1:* In P_1, swap v_1 with a low neighbor w_1 in one of $X_{v_1}, Y_{v_1}, Z_{v_1}$, prioritizing a w_1 in a clique part. Moreover, if w_1 is from a clique part, prioritizing w_1 which has a low neighbor in the *other* clique part. Set $j_1 = 1, 2,$ or 3, respectively, set $v_2 = w_1$ and let P_2 be the resulting proper (2,3,3)-partition $(v_2, X_{v_2}, Y_{v_2}, Z_{v_2})$.

• *Iteration* i*,* $i \geq 2$:

(1) *If* $j_{i-1} = 2$ $[j_{i-1} = 3]$:

(a) In P_i, if there is a low vertex in $Z^*_{v_i} \setminus v_i$ $[Y^*_{v_i} \setminus v_i]$:

If every low vertex in $Z^*_{v_i} \setminus v_i$ $[Y^*_{v_i} \setminus v_i]$ has never moved, then swap v_i with such a low w_i, prioritizing a w_i which has a low neighbor in $Y_{v_i}[Z_{v_i}]$. Set $j_i = 3(2)$, set $v_{i+1} = w_i$, let P_{i+1} be the resulting proper (2,3,3)-partition, and iterate. Otherwise, terminate the algorithm.

(b) In P_i, if there is no low vertex in $Z^*_{v_i} \setminus v_i$ $[Y^*_{v_i} \setminus v_i]$:

If every low neighbor of v_i in $X^*_{v_i}$ has never moved, then swap v_i with such a low w_i, set $j_i = 1$, $v_{i+1} = w_i$, let P_{i+1} be the resulting proper (2,3,3)-partition, and iterate. Otherwise, terminate the algorithm.

(2) *If* $j_{i-1} = 1$*:*

In P_i, if every low vertex in $(Y^*_{v_i} \cup Z^*_{v_i}) \setminus v_i$ has never moved, then swap v_i with such a low w_i, prioritizing a w_i which has a low neighbor in the opposite clique part. Set $j_i = 2$ or 3 if w_i is in Y_{v_i} or Z_{v_i}, respectively. Set $v_{i+1} = w_i$, let P_{i+1} be the resulting proper (2,3,3)-partition, and iterate. Otherwise, terminate the algorithm.

In iteration 1 of Algorithm 3, v_1 swaps into the X-set only if it has no low neighbors in either the Y-set or Z-set (where Algorithm 1 expressed no preference between the three sets).

Moreover, even if Algorithm 3 is able to swap into the Y-set or Z-set, it prioritizes specially picking a vertex to swap with that will allow the next iteration to go smoothly as well: namely, if possible it chooses a vertex in the Y-set or Z-set that has a low neighbor in the *other* clique part. Note that this means in iteration 2, the special vertex will be able to swap with this neighbor in the other clique part, beginning this desired alternating pattern of swaps with the Y-set and Z-set.

Suppose we are at a point in Algorithm 3 where we have just been forced to swap into the X-set (i.e. $j_{i-1} = 1$), then we set the bar very high for continuing the algorithm: we only swap (as opposed to terminate) if every low vertex in *all* of $(Y^*_{v_i} \cup Z^*_{v_i}) \setminus v_i$ has never moved. If this bar is met, then just as in iteration 1, we choose (if possible) a vertex in the Y-set or Z-set that has a low neighbor in the *other* clique part.

In any iteration $i \geq 2$ where $j_{i-1} \neq 1$ (i.e. we have not just swapped with the X-set), we have the same condition for swapping into a set. In particular, whether a vertex is going to swap into the X-set, Y-set, or Z-set, it must satisfy the following condition: no low neighbor of v_i in $X^*_{v_i}$, $Y^*_{v_i}$, or $Z^*_{v_i}$, respectively can have already moved. This is actually the condition for swaps that Algorithm 2 has; in Algorithm 1 there was a lower bar to meet for swapping into the Y-set. Also, similarly to iteration 1, if v_i is going to swap into the Y-set or Z-set, then Algorithm 3 prioritizes specially picking a vertex to swap with that has a low neighbor in the *other* clique part.

Algorithm 3 makes sense as written because we have forbidden R_1, R_3, R_{10}. First, we need to ensure that in iteration 1, the vertex v_1 *has* a low neighbor, and forbidding R_1 accomplishes this. Secondly, in iteration $i(1)(b)$, we need to ensure that the condition on the low vertices in the X-set is not met vacuously, i.e. that there is some low vertex in the X-set to swap with. The situation here is that v_i has just swapped out of either the Y-set or the Z-set (so it has one low neighbor there) but has no low neighbors in the other clique part. So forbidding R_{10} forces v_i (v_6) to have a low neighbor in the X-set. Finally, for iteration $i(2)$, we'll ensure that a special vertex v_i which has just swapped out of the X-set has at least one low neighbor in the Y-set or the Z-set. In this scenario, v_i has a low neighbor in the X-set (the vertex it just swapped with), and forbidding R_3 forces v_i (v_6) to have a low neighbor in either the Y-set or the Z-set.

In Algorithm 3, as in Algorithms 1 and 2, we get termination because swaps must always involve a vertex that has never moved before. We divide the remainder of our proof into two cases. If the algorithm terminates due to the stopping condition in $(1b)$, we will find a contradiction. If it terminates due to the stopping condition in either $(1a)$ or (2), we show that G contains $(K_3 \vee E_6)^{-6}$ as desired.

Case 1: *Algorithm 3 terminates due to the stopping condition in (1b).*

In this case, at termination, we know that v_i was either just swapped out of the Y-set or the Z-set (since $j_{i-1} \in \{2, 3\}$), and that v_i has a low neighbor $a \in X_{v_i}$ that has already moved. Given the pattern of Algorithm 3, we know that a either swapped out of one of the clique parts in a previous step, or that a was the initial special vertex v_1. We will consider both possibilities, and in fact show that neither is possible.

We first assume, for a contradiction, that a was in one of the clique parts before swapping into the X-set. We may assume, without loss of generality, that a was in the Y-set previously. We claim that this means v_i was not in the Y-set previously. If it was, then a, v_i were both in the Y-set before a moved. Let a' be the vertex that swapped into the Y-set and swapped a out. Consider the point in the algorithm when a' was special. At this point, $a \in Y^*_{a'}$ and $a \sim v_i$, so $v_i \in Y^*_{a'}$, and thus $v_i \sim a'$. Then after a' swapped into the Y-set (swapped with a), we know that a' remained in the Y-set until termination. Now we look to a later iteration. We consider the vertex that swapped into the Y-set and swapped v_i out, call this vertex v_i'. At the step when v_i' was special, $v_i \in Y^*_{v_i'}$, and since $v_i \sim a'$, we also have $a' \in Y^*_{v_i'}$, so $v_i' \sim a'$. But since a' is a low vertex that has already moved, the algorithm should have terminated at this point by the stopping condition in either $(1a)$ or (2), contradicting our assumption that it stopped by the condition in $(1b)$. So we establish our claim, and hence we know that v_i was in the Z-set prior to becoming the special vertex (not the Y-set). But then consider the step in the algorithm when a was special, just after a swapped out of the Y-set. At this point, $v_i \in Z^*_a$ since $v_i \sim a$ and a

moves before v_i. Since a has a low neighbor (v_i) in the Z-set, the algorithm should have either swapped a with a low neighbor in the Z-set or terminated, as described in $(1a)$. Since we know that a actually swapped into the X-set, we have a contradiction.

We now assume, for a contradiction, that a was the initial special vertex v_1. Since $j_{i-1} \in \{2, 3\}$, we know that v_i was in one of the clique parts before becoming special; in particular, v_i was in one of the clique parts in the initial partition $P_1 = (a = v_1, X_{v_1}, Y_{v_1}, Z_{v_1})$. So then when a was special, it had a low neighbor v_i in one of the clique parts. Thus, iteration 1 dictates that a should have swapped into either the Y-set or the Z-set. But we know that $a \in X^*_{v_i}$ at termination, so a swapped into the X-set, contradiction.

Case 2: *Algorithm 3* terminates due to the stopping condition in $(1a)$ or (2).

In this case, at termination, v_i has just swapped out of either the X-set, Y-set, or the Z-set (since $j_{i-1} \in \{1, 2, 3\}$). Regardless, by either of these stopping conditions, v_i has a low neighbor a' in either the Y-set or the Z-set that has already moved. Without loss of generality suppose that $a' \in Y^*_{v_i}$. Then since the algorithm stops with v_i looking into the Y-set at a', we know that v_i cannot have swapped out of the Y-set, so $j_{i-1} \in \{1, 3\}$ and v_i either swapped out of the X-set or the Z-set. Let one of the other neighbors of v_i in $Y^*_{v_i} \setminus \{a'\}$ be called m. Let the vertex that a' swapped with when it moved into the Y-set be called a. Note that a moves before v_i in the course of the algorithm.

Claim 12. *The vertices $\{m, a', v_i\}$ induce a clique. Also, $a \sim m$, $a \sim a'$, and m never moves.*

Proof of Claim. We know that these three vertices induce a clique because $m, a', v_i \in Y^*_{v_i}$. Also $a' \sim a$ by definition. Applying the same argument as in Claim 10 of the proof of Theorem 1.6 shows that $a \sim m$ and that m never moves. □

Since a' swapped into the Y-set and swapped a out, we know that a either swapped into the X-set or the Z-set and remained there until termination. The argument is easier in the latter case, where we in fact find a slightly stronger subgraph.

Claim 13. *If a is in the Z-set at termination, then G contains $(K_3 \vee E_6)^{-4}$.*

Proof of Claim. We consider the partition $(v_i, X_{v_i}, Y_{v_i}, Z_{v_i})$ at termination. Here we know $a', m \in Y^*_{v_i}$ (by definition) and $a \in Z_{v_i}$ by assumption. To achieve our final structure, we propose making one additional swap: we swap v_i with a', making a' special and swapping v_i into the Y-set. Note that after this swap, m and a are still in the Y-set and Z-set respectively. Since a' is low, this new partition is proper by Theorem 2.3 (2). Call the new partition $(a', X_{a'}, Y_{a'}, Z_{a'})$.

We now assemble everything that we know about the partition $(a', X_{a'}, Y_{a'}, Z_{a'})$. By Claim 12, $m, v_i \in Y^*_{a'}$ and $a \in Z^*_{a'}$. We now label the other vertices adjacent to a' in $X^*_{a'}, Y^*_{a'}$, and $Z^*_{a'}$: let u_1, u_2 be these other vertices in the X-set, let u_3 be this other vertex in the Y-set, and let u_4, u_5 be these other vertices in the Z-set. As $Y^*_{a'}$ and $Z^*_{a'}$ are cliques, we have that $\{v_i, m, u_3\}$ and $\{a, u_4, u_5\}$ induce cliques. We also have $a \sim m$ by Claim 12.

We will show that $\{a, a', v_i\}$ induce a K_3 and that $\{m, u_1, \ldots u_5\}$ can play the role of E_6, with the four potentially missing edges being those between $\{v_i, a\}$ and $\{u_1, u_2\}$. We have already established that a' is adjacent to all of the vertices in our proposed E_6 (namely $m, u_1, \ldots u_5$). We already know that $v_i \sim m, u_3$ and $a \sim u_4, u_5, m$. It remains for us to show that $a \sim v_i, u_3$ and $v_i \sim u_4, u_5$.

In order to show that $a \sim v_i, u_3$, we apply Theorem 2.2: since a is low and $a \sim m$ with $m \in Y^*_{a'} \setminus \{a'\}$, we have that $N_{Y_{a'}}(a') = N_{Y_{a'}}(a)$, and thus $a \sim v_i, u_3$ since $a' \sim v_i, u_3$. To show that $v_i \sim u_4, u_5$, we apply Theorem 2.2 again: since v_i is low and $v_i \sim a$ (by our previous line) with $a \in Z^*_{a'} \setminus \{a'\}$, we have that $N_{Z_{a'}}(a') = N_{Z_{a'}}(v_i)$, and so $v_i \sim u_4, u_5$ because $a' \sim u_4, u_5$. □

We may now assume that a is in the X-set at termination. We will show that G contains $(K_3 \vee E_6)^{-6}$ via a series of claims.

Claim 14. $a \sim v_i$.

Proof of Claim. In $(v_i, X_{v_i}, Y_{v_i}, Z_{v_i})$, we propose swapping v_i with a' so that a' is special and v_i is in the Y-set. This swap creates a new proper (2,3,3)-partition because a' is low (Theorem 2.3(2)). Note that after this swap, we still have a in the X-set and m in the Y-set. Call this new partition $(a', X_{a'}, Y_{a'}, Z_{a'})$. Since $a' \sim a, m, v_i$ by Claim 12, we have $a \in X^*_{a'}$ and $m, v_i \in Y^*_{a'}$. Applying Theorem 2.2, since a is low and $a \sim m$ (by Claim 12) with $m \in Y^*_{a'}$, we have that $N_{Y_{a'}}(a') = N_{Y_{a'}}(a)$, and so $a \sim v_i$ because $a' \sim v_i$. □

Claim 15. *Before v_i moved, it was in the X-set.*

Proof of Claim. We already know that $j_{i-1} \in \{1, 3\}$; assume for contradiction that v_i was in the Z-set before moving. We consider the point in the algorithm when a was special, just after it swapped out of the Y-set. At this point, $v_i \in Z^*_a$ (since $a \sim v_i$ by our prior claim and a moves before v_i). Thus, the partition (a, X_a, Y_a, Z_a) meets the condition in 1(a): $j_{i-1} = 2$ and there is a low vertex $v_i \in Z^*_a \setminus \{a\}$. So this iteration of the algorithm should have either ended with termination or with a swapping into the Z-set, contradiction. □

Now consider again the step where a was the special vertex. We know that v_i is in the X-set before it moves (Claim 15) and that a moves before v_i, so v_i is in X_a, and since $a \sim v_i$ by Claim 14, we get that v_i is a neighbor of a in X^*_a. Let u_1 be the other neighbor of a in the X-set.

Claim 16. *When a swapped into the X-set, it swapped with v_i.*

Proof of Claim. Assume, for a contradiction, that when a moved into the X-set it swapped with u_1. See Figure 3.4 for a depiction of both before and after this swap. Consider the X-set when u_1 is special (the right picture in Figure 3.4); recall that $X^*_{u_1}$ is an odd hole. Let p be the other neighbor of u_1 in X (besides a), and let P be the induced path between a and p in the X-set. Note that P contains v_i.

As the algorithm progresses, P will remain unchanged in the X-set until some vertex s becomes special and wants to switch with one of the vertices in P. But since X^*_s must be an induced cycle, this means that the two neighbors of s in the X-set must be a, p. At this point however, the algorithm would have stopped according to 1(b), since a has already moved. Hence P remains unchanged in the X-set at termination. This is a contradiction, since v_i is a member of P, and we know that v_i is the special vertex at termination. □

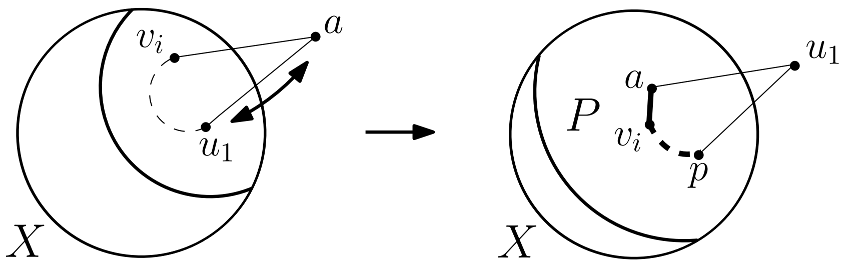

Figure 3.4: Two points in the algorithm as described in Claim 16.

Claim 17. $v_i \sim u_1$.

Proof of Claim. After a swaps with v_i (Claim 16), v_i becomes special, and the resulting partition is actually the partition at termination: $P_i = (v_i, X_{v_i}, Y_{v_i}, Z_{v_i})$. Here we know that a, u_1 are in the X-set and $a' \in Y^*_{v_i}$ (by definition of u_1, a'). We know that $a \sim u_1$; to show that v_i is adjacent to u_1, we suggest a sequence of swaps between low vertices, each of which creates a new proper (2,3,3)-partition by Theorem 2.3 (2).

First we swap v_i into the Y-set and swap a' out. Then since $a' \sim a$ by definition, we can swap a' into the X-set and swap a out. Call this new partition $(a, \tilde{X}_a, \tilde{Y}_a, \tilde{Z}_a)$; note we are using the tilde notation here so as not to confuse this new partition (which is created after termination) with the partition (a, X_a, Y_a, Z_a) (the partition when a was special during Algorithm 3).

Then $a', u_1 \in \tilde{X}^*_a$ (since $a \sim a', u_1$ by definition) and $v_i \in Y^*_a$ by Claim 14. Since v_i is low and $v_i \sim a' \in X^*_a \setminus \{a\}$ by definition, we apply Theorem 2.2 to find that $N_{X_a}(a) = N_{X_a}(v_i)$, and thus $v_i \sim u_1$ since $a \sim u_1$. □

We now assemble everything we know about the partition (a, X_a, Y_a, Z_a) when a was special during Algorithm 3 (just after a' swapped a out of the Y-set). At this point, we know that two neighbors of a in X^*_a are v_i, u_1 (see discussion above Claim 16). We also know that $a' \in Y^*_a$ since a' has just swapped into the Y-set and swapped a out. Moreover, we know that $m \in Y^*_a$ since $a \sim m$ and m never moves (Claim 12). We now label the other vertices adjacent to a in Y^*_a and Z^*_a: let u_2 be this other vertex in the Y-set and let u_3, u_4, u_5 be these other vertices in the Z-set.

As Y_a^* is a clique, we have that $\{a', m, u_2\}$ induce a clique. We also have that $a' \sim v_i$ by definition and $v_i \sim m, u_1$ by Claims 12 and 17. Our final goal is a $(K_3 \vee E_6)^{-6}$. In particular, we have already established that $\{a, a', v_i\}$ induce a K_3, and we will show that $\{m, u_1, \ldots u_5\}$ can play the role of E_6, with the six potentially missing edges being those between $\{v_i, a'\}$ and $\{u_3, u_4, u_5\}$. We have already established that a is adjacent to all of the vertices in our proposed E_6 (namely $m, u_1, \ldots u_5$). We already know that $v_i \sim m, u_1$ and $a' \sim m, u_2$. So it remains only for us to show that $v_i \sim u_2$ and $a' \sim u_1$.

In order to show that $v_i \sim u_2$, we apply Theorem 2.2: since v_i is low and $v_i \sim a'$ with $a' \in Y_a^* \setminus \{a\}$, we have that $N_{Y_a}(a) = N_{Y_a}(v_i)$, and thus $v_i \sim u_2$ since $a \sim u_2$. To show that $a' \sim u_1$, we apply Theorem 2.2 once more: since a' is low and $a' \sim v_i$ with $v_i \in X_a^* \setminus \{a\}$, we have that $N_{X_a}(a) = N_{X_a}(a')$, and so $a' \sim u_1$ because $a \sim u_1$. □

3.4 An Infinite family

In this section we present an infinite family of graphs that is as we promised in the introduction: each member of the family will have $\chi = \Delta = 9$ with average degree above 8.75, but will contain none of the substructures listed in Table 1.

Our infinite family of graphs will be called $\mathcal{H}$. Each member of $\mathcal{H}$ will be made using a basic building block H^*, which is the graph depicted in Figure 3.5. In these figures, a solid jagged line between two sets of vertices indicates a complete bipartite graph between those sets; a dashed jagged line indicates that the two sets are joined by a complete bipartite graph less one perfect matching. Note that the labelled vertices b, h_1, a_1, a_h and the bolded segment they lie on in Figure 3.5 will be used much later.

To see how we can use copies of H^* to make members of our family $\mathcal{H}$, consider the four pairs of K_4's occurring along the outside boundary of H^* in Figure 3.5; we will consider each such pair of K_4's as "free for connection". Take any such free pair of K_4's (say with vertex sets named N_1, N_2) and delete the 4 edges joining them. Then take another copy of H^* and similarly delete the edges from one of its free K_4 pairs (say with vertex sets named N_1', N_2'). Then we connect these two H^* building blocks by joining N_1, N_1' as a pair (with four matching

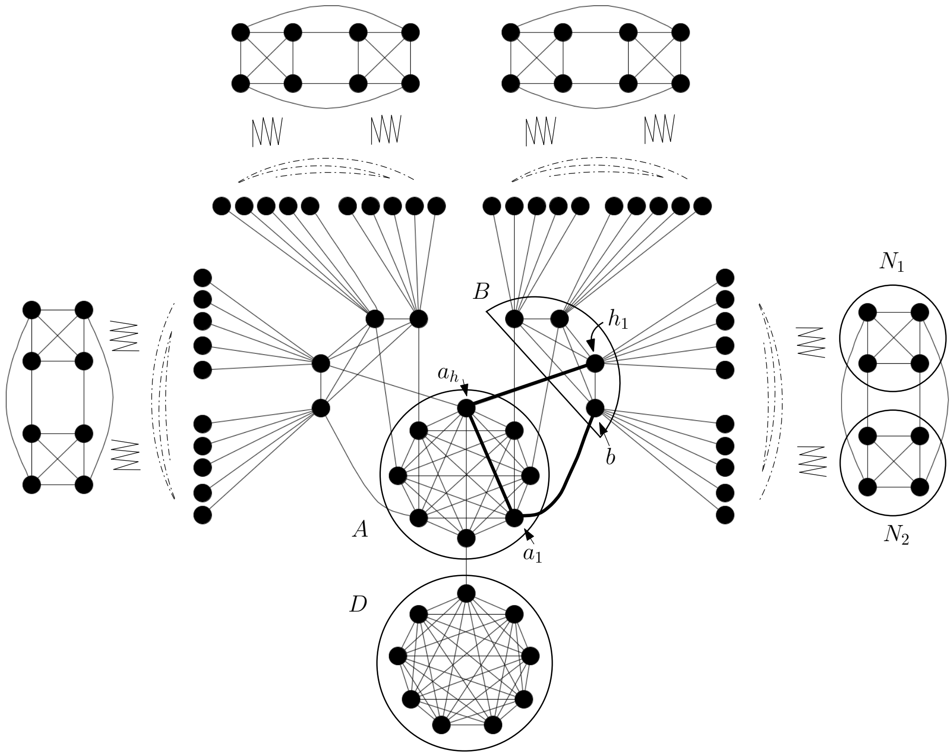

Figure 3.5: An H^* building block.

edges, as used to connect N_1 to N_2), and joining N_2, N_2' as a pair (with four matching edges, as used to connect N_1' to N_2'). See Figure 3.6; the two blocks are connected by edges between N_1 and N_1' and between N_2 and N_2'. The open circles on either side represent the rest of the H^* blocks which are not pictured in detail.

Note that after this procedure, we no longer consider the vertex sets N_1, N_1', N_2, N_2' as corresponding free K_4 pairs, but the resulting graph has exactly 6 pairs of K_4's on the boundary of its structure, and we consider those pairs free for connections. Our family $\mathcal{H}$ consists of H^* plus any number of copies of H^* that have been iteratively joined via the above procedure.

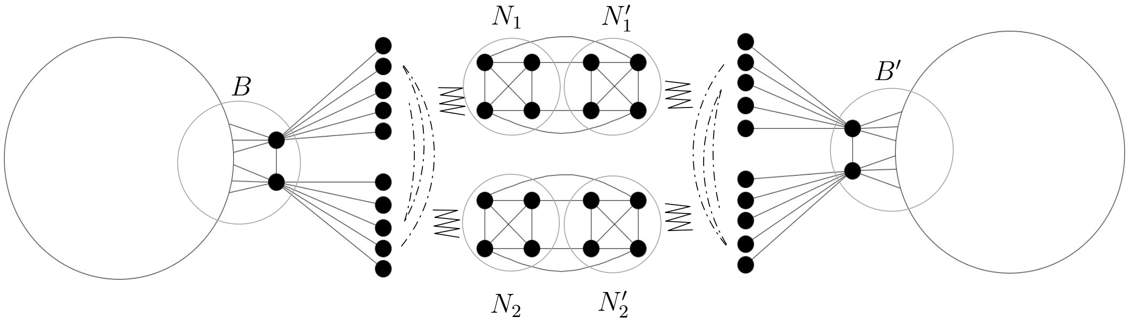

Figure 3.6: A connection point between two H^* building blocks.

Consider any $H \in \mathcal{H}$. Note that $\chi(H) = 9$ since it contains a K_9 (on the vertex set D of H^*, indicated in Figure 3.5). All the vertices in H^* have either degree 9 or 8, with the only low vertices being: the vertices of A (with the exception of its topmost vertex in Figure 3.5), and; the vertices of D (with the exception of its topmost vertex in Figure 3.5). Hence the average degree of H^* is just above 8.845 (H^* contains 15 low vertices and 82 high). The degrees of all vertices remain the same after our combination procedure above, so this average degree is the same for H. (If one desired an average degree that was even higher, it is possible to use a second type of building block in order to make this happen, but we have not included those details here for the sake of brevity.)

Note that H^* is certainly *not* vertex-critical, since one of the vertices in $D \subseteq H^*$ is a cut-vertex, and deleting this vertex from H^* creates two components, both of which have chromatic number 8 (note both contain K_8). On the other hand, if we perform this same deletion in any $H \in \mathcal{H} \setminus H^*$, then at least one of the two components created does contain a K_9 and hence the resulting graph still has chromatic number 9.

It remains to show that if $H \in \mathcal{H}$, then H contains none of our forbidden substructures. It is worth noting how close this is to failing: if we instead had two edges coming out of the K_9 in H^*, then the existence of the two high vertices in D would mean we could find any of R_7, R_8, R_9 or S within D.

Lemma 3.4. *If $H \in \mathcal{H}$, then H does not contain any of $R_1, \ldots, R_{10}, Q$, or S.*

Proof. The structures $R_1, \ldots, R_{10}, Q, S$ all contain an odd hole where one of the vertices in the cycle is low, and its two neighbors on the cycle are high, so it suffices to show that H contains no such structure.

The only low vertices in H come from the cliques A and D (in some copy of H^*). In D, the low vertices each have exactly one high neighbor (the top vertex of D in Figure 3.5), so it is not possible for any of these to be part of an odd hole where their *two* neighbors on the cycle are high.

Let a_1 be a low vertex from A. Note that a_1 has exactly two high neighbors: the high vertex in A, which we'll call a_h, and; a neighbor outside of A, which we'll call b. Note that we may assume that $b \in B$ (see Figure 3.5), since if $b \in D$, then a_1, b are joined along a cut edge, which is certainly not part of any cycle in H.

Assume, for a contradiction, that there is an odd hole C in H containing a_h, a_1, b consecutively in C. Note that since A is a cut-set in the graph H, the other vertex of C adjacent to a_h (i.e besides a_1) must also be in B, say h_1. If not, then C would have to include another vertex from A (a clique) at some point, and this would contradict C being an induced cycle. This means, without loss of generality, that our four consecutive vertices on C (h_1, a_h, a_1, b) must be as labelled in Figure 3.5. But then since h_1, b are both members of B, they are adjacent, and hence we get a copy of C_4 induced by these four vertices. But since C is supposed to be an odd hole, this is a contradiction. □

It remains now to show that no $H \in \mathcal{H}$ contains the structure F (a K_4 and an odd hole meeting at a vertex, with all vertices high). There are a number of $K_4's$ in H^* (and in any given $H \in \mathcal{H}$) but the all-high requirement rules out K_4's from either A or D (see Figure 3.5), and the attached odd hole rules out a K_4 from B (see Figure 3.5). Therefore, if $H \in \mathcal{H}$ contains F, then the K_4-part of this F comes from one of the K_4 pairs that was originally "free for connection" in some copy of H^*. If the K_4 in question was never *used* for a connection, then it does not have an attached odd hole (see Figure 3.5). But this is also true if the K_4 *was* used for a connection: see Figure 3.5, where we can see that N_1 has no odd hole attached to it.

Chapter 4

High cliques and high odd holes in graphs with chromatic number equal to maximum degree

This chapter now proceeds as follows: in Section 4.1 we prove that Theorem 1.10 holds for $\Delta(G) \leq 4$, and in Section 4.2 we prove that it holds for $\Delta(G) \geq 7$.

4.1 Small Maximum Degree

We say a vertex v in a graph G is *high* if $d_G(v) \geq \Delta(G) - 1$. The following lemma has two very basic facts about high vertices and high cliques.

Lemma 4.1. *Let G be a graph with $\chi(G) = \Delta(G)$.*

(a) If G contains $K_{\Delta(G)}$, then this is a high $K_{\omega(G)}$ in G.

(b) If $v \in V(G)$ is critical, then v is high.

Proof. (a) Every vertex in the $K_{\Delta(G)}$ has degree at least $\Delta - 1$. We know that $\omega(G) \leq \chi(G) = \Delta(G)$, and the existence of the clique tells us that $\omega(G) = \Delta(G)$.

(b) If $v \in V(G)$, then $\chi(G - v) = \Delta(G) - 1$ since v is critical. But then v must have degree at least $\Delta(G) - 1$, otherwise the coloring can be extended to G. □

Let us now prove another lemma about minimum counterexamples to Theorem 1.10.

Lemma 4.2. *Let $\Delta \geq 4$. Let G be a graph with the fewest vertices subject to $\chi(G) = \Delta(G) = \Delta$, G contains no high $K_{\omega(G)}$, and G contains no high odd hole.*

(i) If $v \in V(G)$ and $\omega(G - v) = \omega(G)$, then v is critical.

(ii) If A, B are two different copies of $K_{\omega(G)}$ in G, then $|V(A) \setminus V(B)| < \Delta(G) - \omega(G)$.

(iii) If $\omega(G) = \Delta(G) - 1$ then G contains no odd holes.

Proof. (i) Let $v \in V(G)$ and let $G_0 = G - v$. Suppose, for a contradiction, that $\omega(G_0) = \omega(G) = \omega$ but $\chi(G_0) = \chi(G)$. Note that $\Delta(G_0) \in \{\Delta, \Delta - 1\}$.

Suppose first that $\Delta(G_0) = \Delta$. By minimality, G_0 contains either a high K_ω or a high odd hole. The former case is not possible, since G contains no high K_ω. So G_0 must contain a high odd hole that is not an odd hole in G – but again this is impossible, since the only edges we removed were incident to v.

We may now assume that $\Delta(G_0) = \Delta - 1$. Then $\Delta(G_0) = \Delta - 1 = \chi(G_0) - 1 \geq 3$, so Brooks' Theorem says that G_0 contains a $K_{\Delta(G_0)+1} = K_\Delta$. But then this is also a high K_ω by Lemma 4.1(a).

(ii) Let A, B be two different copies of $K_{\omega(G)}$ with $|V(A) \setminus V(B)| = s$. The s vertices in $A \setminus B$ are critical and high by part (i) of this Lemma and by Lemma 4.1(b). Any $v \in V(A) \cap V(B)$ then v has $\omega(G) - 1$ neighbors in B and an additional s neighbors in $A \setminus B$. Since A cannot be a high $K_{\omega(G)}$, we must have $\omega(G) - 1 + s < \Delta(G) - 1$, i.e., $s < \Delta(G) - \omega(G)$.

(iii) Since $\omega(G) = \Delta(G) - 1$, by part (ii) of this Lemma we know that G contains a unique copy K of $K_{\omega(G)}$. We also know that every vertex outside of K is critical and high (by part (i) of this Lemma and by Lemma 4.1(b)). Suppose G also contains an odd hole H, with $|V(H) \cap V(K)| = t$. We cannot have $t = 0$, because then H would be a high odd hole. Since H is a hole, we cannot have $t \geq 3$. So $t \in 1, 2$. Each of these t vertices has $\omega(G) - 1 = \Delta - 2$ neighbors in K, and at least one additional neighbor on H. So H must be high, contradiction. □

We are now ready to prove our result for small maximum degrees.

Theorem 4.3. *Let G be a graph with $\chi(G) = \Delta(G) \leq 4$. Then G contains either a high $K_{\omega(G)}$ or a high odd hole.*

Proof. Let $\Delta = \Delta(G)$, $\omega = \omega(G)$, $\chi = \chi(G)$. If $\Delta = 0$ then $\chi = 1$ and G is a collection of isolates, each of which is a high K_ω. We cannot have $\Delta = 1$, since then G contains a K_2, so $\chi \geq 2$. If $\Delta = 2$ then G contains a $K_2 = K_\Delta$, so it has a high K_ω by Lemma 4.1(a).

Suppose now that $\Delta = 3$. We may assume that G has no K_Δ (by Lemma 4.1(a)), but G does contain a K_2, so we get $\omega(G) = 2$. Since G contains a vertex of degree 3, it contains a K_2 where one vertex has degree 3 and the other has degree at least 1, so the average degree of this K_2 is at least 2, meaning it is a high K_ω.

We may now assume that $\Delta = 4$. Let G be a counterexample to the result with the fewest vertices. If K is a copy of K_ω in G then by Lemma 4.2(i), every vertex $v \in V(G) \setminus V(K)$ is critical. We will apply this repeatedly below.

Note that $\omega \geq 2$, since $\Delta \geq 4$. We may assume that $\omega(G) \leq 3$ since if G contains a $K_4 = K_\Delta$ it is a high K_ω by Lemma 4.1(a).

Suppose first that $\omega = 2$, and consider a vertex v of degree 4. If any of v's neighbors have degree at least two in G, then we get a high K_2 (one vertex of degree 4 and one of degree at least two). So we may assume that every neighbor of v is a leaf, and hence not critical (by Lemma 4.1(b)). But then by choosing K as one of the edges incident to v, we should have that every other neighbor of v is critical, contradiction.

We may now assume that $\omega = 3$. By Lemma 4.2(ii) we know that there is a unique copy K of K_3 in G. We may assume that some vertex $v \in V(K)$ has no neighbor outside of K, since otherwise each vertex in K has degree at least 3, and it is a high K_ω. So v has degree two, and it is not a critical vertex (by Lemma 4.1(b)). Let $G' := G - v$. Then $\chi(G') = \chi = 4$. Since K is the only triangle in G, G' is triangle-free. G' is also odd hole-free, since any such odd hole would also be present in G, and Lemma 4.2(iii) says G has no odd holes in this case. Since $\chi(G') = 4$, G' is not bipartite, and contains a shortest odd cycle C. But a shortest odd cycle is necessarily an odd hole or a triangle, contradiction. □

4.2 Large Maximum Degree

In this section we prove Theorem 1.10 for graphs with $\Delta \geq 7$. We start by proving the following simple coloring lemmas, each of which will be used multiple times in our main result.

Lemma 4.4. *Let G be a graph with $\chi(G) = \Delta(G)$ and let u be a critical vertex in G with $d_G(u) = \Delta(G) - 1$. Then there does not exist a path P in G from u to some vertex v where $d_G(v) \leq \Delta(G) - 2$ and $d_G(w) = \Delta(G) - 1$ for all $w \in V(P) \setminus \{u, v\}$.*

Proof. Suppose for a contradiction that such a path exists, say $P = (u_1, u_2, \dots, u_p)$ where $u_1 = u$ and $u_p = v$, so $p \geq 2$. Let φ_1 be a $(\Delta(G) - 1)$-coloring of $G - u$; apply φ_1 to G, leaving u_1 uncolored. Since $d_G(u_1) = \Delta(G) - 1$, we may assume that each neighbor of u in G has different color under φ_1 (otherwise φ_1 can be extended to G). In particular, this means we can modify φ_1 to φ_2 by assigning $\varphi(u_2)$ to u_1, leaving u_2 uncolored. Then again we may assume that each of the $\Delta(G) - 1$ neighbors of u_2 in G has a different color, so we can assign $\varphi(u_3)$ to u_2 to get φ_3, then $\varphi(u_4)$ to u_3 to get φ_4, and so on, shifting the colors along P until we arrive at φ_p which leaves only u_p uncolored. Since $d_G(u_p) \leq \Delta - 2$, φ_p can be extended to G, so we get a $(\Delta - 1)$-coloring of G. □

Lemma 4.5. *Let G be a graph with $\chi(G) = \Delta(G)$, let v be a critical vertex in G with $d_G(v) = \Delta(G)$, and let φ be a $(\Delta(G) - 1)$-coloring of $G - v$. Then:*

(a) We may assume the the neighbors of v in G are $v_0, v_1, \dots, v_{\Delta(G)-1}$ with $\varphi(v_0) = \varphi(v_1) = 1$, and $\varphi(v_i) = i$, $d_G(v_i) \geq \Delta - 1$ for all $2 \leq i \leq \Delta(G) - 1$.

(b) If v_i, v_j are not adjacent for some $2 \leq i < j$, then there exists an odd hole H_{ij} in G which includes the vertices v_i, v_j, v and which is (i, j)-alternating (with the exception of the one uncolored vertex v). Moreover, we also get this conclusion when $i = 1$ provided $d(v_0) \leq \Delta(G) - 2$.

Proof. (a) Each of the $\Delta(G) - 1$ colors of φ must appear on the neighbors of v, otherwise we can color v and extend φ to all of G. So we may assume the the neighbors of v in G are $v_0, v_1, \dots, v_{\Delta(G)-1}$ with $\varphi(v_0) = \varphi(v_1) = 1$, and $\varphi(v_i) = i$ for all $2 \leq i \leq \Delta(G) - 1$.

Suppose that $d_G(v_i) \leq \Delta - 2$ for some $2 \leq i \leq \Delta(G) - 1$. Then modify φ to φ' by removing the color i from v_i and giving it to v. Since $d_G(v_i) \leq \Delta(G) - 2$, we can extend φ' to a $(\Delta(G) - 1)$-coloring to G, contradiction.

(b) Let S be the maximal (i, j)-alternating subgraph containing v_j. Suppose first that S contains v_i. Then choose a shortest (i, j)-alternating path from v_i to v_j, which necessarily has odd length. As v_i, v_j are not adjacent, this path, when combined with the edges v_iv, vv_j, must induce an odd hole – v has no other neighbors colored i or j, and since the path is shortest it prevents any other chords.

We may now assume that S does not contain v_i. If $i \geq 2$, then: swap the colors i and j on all of S, then assign j to v to get complete a $(\Delta(G) - 1)$-coloring of G, contradiction. So it must be the case that $i = 1$, and that S contains v_0 but not v_1. In this case, swap the colors on all of S. Now v_1, v_j are both color 1, and v_0 is the unique neighbor of v colored j. But then if $d(v_0) \leq \Delta(G) - 2$ this contradicts part (a).

□

It is worth noting that in order to prove this section's main result, which appears below, we'll need Lemmas 4.1, 4.2, 4.4, 4.5(a)(b), Theorems 1.2(a)(b), 1.11(a), and we'll also need an instance of Mantel's Theorem.

Theorem 4.6. *Let G be a graph with $\chi(G) = \Delta(G) \geq 7$. Then G contains either a high $K_{\omega(G)}$ or a high odd hole.*

Proof. Fix $\Delta \geq 7$. Let G be a graph with the fewest vertices subject to $\chi(G) = \Delta(G) = \Delta$, G contains no high $K_{\omega(G)}$, and G contains no high odd hole. Let $\omega = \omega(G)$ and $\chi = \chi(G)$.

Claim 18. *If K is a copy of K_ω in G, then every vertex $v \in V(G) \setminus V(K)$ is critical, high, and contained in some odd hole. Moreover, any odd hole in G must contain at least one vertex from K (and no more than 2).*

Proof of Claim. The critical and high parts of the statement follow from Lemma 4.2(i) and Lemma 4.1(b), respectively. If such a critical vertex is not in an odd hole, then G contains

$K_\chi = K_\Delta$ by Theorem 1.11(a), which is a high K_ω by Lemma 4.1(a). If H is an odd hole in G, and $V(K) \cap V(H) = \emptyset$, then H is a high odd hole. On the other hand, $|V(K) \cap V(H)| \leq 2$ since H is a hole. $\square$

In general $\omega \leq \Delta + 1$, but we know that G does not contain a K_Δ by Lemma 4.1(a). If $\omega = \Delta - 1$ then G has no odd holes by Lemma 4.2(iii). But then Claim 18 says that $G = K$, which is impossible when $\omega = \Delta - 1$. So $\omega \leq \Delta - 2$. By Theorem 1.2(b) we may assume that $\omega \geq \Delta - 4$, since we know that G does not contain a K_Δ by Lemma 4.1(a). So in fact ω is equal to $\Delta - 2, \Delta - 3$, or $\Delta - 4$. Let ℓ be the number of different copies of K_ω in G. We split the remainder of our proof into two cases according to whether $\ell \geq 2$ or $\ell = 1$.

Case 1: $\ell \geq 2$

Let A, B be two different copies of K_ω in G.

Consider some vertex $a \in V(A) \setminus V(B)$. Then a is critical, high, and contained in some odd hole H by Claim 18. If H contains no vertices from $A \cap B$, then H is high since every vertex outside some copy of K_ω is high by Claim 18. We also know that H contains at most two vertices from A by Claim 18, and hence at most one vertex from $A \cap B$. Thus H contains exactly one vertex x from $A \cap B$.

Since H is not high and every vertex outside of $A \cap B$ is high, $d_G(x) \leq \Delta - 2$. On the other hand, x is adjacent to all other vertices in A, plus any additional vertices in B, so

$$\Delta - 2 \geq d_G(x) \geq \omega - 1 + |V(B) \setminus V(A)| \geq \omega \geq \Delta - 4.$$

As a is high, it has degree Δ or $\Delta - 1$. If $d_G(a) = \Delta - 1$, then since a is critical and $d_G(x) \leq \Delta - 2$, the path $P = (a, x)$ contradicts Lemma 4.4. So $d_G(a) = \Delta$.

We now apply Lemma 4.5(a) to the critical vertex a. We get that the neighbors of a in G are x and some vertices $v_1, \ldots, v_{\Delta-1}$, and we get that there exits a $(\Delta - 1)$-coloring of $G - a$, φ, with $\varphi(x) = \varphi(v_1) = 1$ and $\varphi(v_i) = i$ for all $2 \leq i \leq \Delta - 1$. Moreover, we get that $d_G(v_i) \geq \Delta - 1$ for all $2 \leq i \leq \Delta - 1$.

Suppose that $\omega \geq 4$. Then the clique A contains at least two other vertices besides a, x, which must be from among $v_2, \ldots, v_{\Delta-1}$, since $\varphi(x) = \varphi(v_1)$ means that v_1 is not in A. Since $d_G(x) \geq \Delta - 4, d_G(a) = \Delta$ it is not possible that all the vertices in $A \setminus a, x$ have degree Δ, since then A would be high. So we may assume that $v_2 \in V(A)$ with $d(v_2) = \Delta - 1$. We know that v_2 is the only vertex adjacent to a having color 2. Obtain φ' from φ by removing the color 2 from v_2 and giving it to a. The existence of φ' shows that v_2 is a critical vertex of G. But then the existence of the path $P = (v_2, x)$ contradicts Lemma 4.4.

We may now assume that $\omega \leq 3$. In fact, since $\omega \geq \Delta - 4$ and $\Delta \geq 7$, we must have $\omega = 3$ and $\Delta = 7$. We know that $x \in V(A)$ and $v_1 \notin V(A)$ (since $\varphi(x) = \varphi(v_1)$), but since a has only one other neighbor in A and $\Delta = 7$, we may assume that $v_2, v_3 \notin V(A)$. Then $d_G(v_2) = d_G(v_3) = \Delta - 1$ since every vertex outside A is high, by Claim 18. If v_2, v_3 are adjacent then a, v_2, v_3 forms a high $K_3 = K_\omega$, contradiction. So v_2, v_3 are not adjacent, and by Lemma 4.5(b) there exists an odd hole H' which includes the vertices v_2, v_3, a. Since H' is not high, it cannot be composed of exclusively high vertices, and therefore must contain at least one vertex of $A \cap B$ (by Claim 18). But then since any such vertex must be adjacent to a, we get that a has three neighbors in H', meaning H' cannot be a hole. This contradiction completes our proof of case 1.

Case 2: $\ell = 1$

Let K be the unique K_ω in G.

Claim 19. *Every odd hole in G contains exactly two vertices in K.*

Proof of Claim. Suppose, for a contradiction, that there exists and odd hole H in G with exactly one vertex $x \in V(K) \cap V(H)$. Then $d_G(x) \leq \Delta - 2$ since H is not high, and every vertex outside of K is high by Claim 18. On the other hand, x has $\omega - 1$ neighbors in K and an additional two neighbors in H. Since $\omega \geq \Delta - 4$ this means that x has degree at least $\Delta - 3$. Both of the neighbors v, v' of x on H cannot have degree Δ, as this would mean H_0 is high. So we can assume without loss that $d_G(v) = \Delta - 1$. We know v is critical (by Claim 18), but then the existence of the path $P = (v, x)$ contradicts Lemma 4.4. □

We will use the following claim to set up our main argument; we will also need to apply the various pieces both within the proof of the claim and at later points in our argument.

Claim 20. *Let H be an odd hole in G containing exactly two vertices $x, y \in V(K)$, with $d_G(x) \leq d_G(y)$. Let x', y' be the neighbors of x, y in $V(H) \setminus V(K)$, respectively. Then:*

(a) We get that $d_G(x) \leq \Delta - 2$ and $d_G(x') = \Delta$. When $\omega \in \{\Delta - 2, \Delta - 3\}$ we get $d_G(y') = \Delta$; when $\omega = \Delta - 4$ we get the same conclusion provided $d_G(y) \leq \Delta - 2$.

(b) We may assume that the neighbors of x' in G are $x, v_1, \ldots, v_{\Delta-1}$ and that there exists φ a $(\Delta - 1)$-coloring of $G - x'$ with $\varphi(x) = \varphi(v_1) = 1$ and $\varphi(v_i) = i$ for all $2 \leq i \leq \Delta - 1$.

(c) If c is a color in φ that does not appear on any vertices of K, then v_c is adjacent to every other vertex in $v_1, \ldots, v_{\Delta-1}$. Moreover, v_1 is also adjacent to every vertex in $v_2, \ldots, v_{\Delta-1}$.

Suppose that v_i, v_j are non-adjacent for some $1 \leq i < j \leq \Delta - 1$. Then:

(d) There exists an odd hole H_{ij} of G containing v_i, v_j, x', containing vertices $w_i, w_j \in V(K)$ with $\varphi(w_i) = i$, $\varphi(w_j) = j$, and which is (i, j)-alternating (other than the one uncolored vertex x').

(e) The vertices v_i, v_j are not in K (so in particular $v_i \neq w_i$ and $v_j \neq w_j$ in part (d)).

(f) There is no $(\Delta - 1)$-coloring φ' of $G - x'$ where $\varphi'(x) = \varphi'(v_i)$ or $\varphi'(x) = \varphi'(v_j)$.

Proof of Claim. (a) Every vertex in $V(H) \setminus \{x, y\}$ is high (and critical) by Claim 18. So since H is not high and $d_G(x) \leq d_G(y)$ we know that $d_G(x) \leq \Delta - 2$. But then $d_G(x') = \Delta$, as otherwise $d_G(x') = \Delta - 1$ and the existence of the path $P = (x', x)$ contradicts Lemma 4.4. If $d_G(y) \leq \Delta - 2$ then we can apply the same argument to get that $d_G(y') = \Delta$; we will show this is the case when $\omega \in \{\Delta - 2, \Delta - 3\}$.

Note that $d_G(x) \geq \omega$, since it is adjacent to all the vertices in K and also to x'. When $\omega = \Delta - 2$ this immediately gives our desired result for y, since otherwise H is high. So we may assume that $\omega = \Delta - 3$ and $d_G(x) = \Delta - 3$. Suppose for a contradiction that $d_G(y) \geq \Delta - 1$. Since H is not high we must in fact have $d_G(y) = \Delta - 1$ since $d_G(x) = \Delta - 3$ and $d_G(x') = \Delta$. But now since y' is critical we can apply Lemma 4.4 to $P = (y', y, x)$ to get that G is $(\Delta - 1)$-colorable, contradiction.

(b) Since x' is a critical vertex (by Claim 18), $d_G(x') = \Delta$ (by part (a)), and $d_G(x) \leq \Delta - 2$ (by part (a)), this follows immediately by Lemma 4.5(a).

We proceed to prove the remaining parts of the claim in the order (d), (c), (f), (e).

(d) Since $d_G(v_0) \leq \Delta - 2$ by (a) we can apply Lemma 4.5(b) to get an odd hole H_{ij} that contains v_i, v_j, x', and which is (i, j)-alternating (other than the one uncolored vertex x'). We know that H_{ij} must intersect K at two vertices (by Claim 19), one with color i and the other with color j.

(c) Let c be a color in φ that does not appear on K, and note that $c \neq 1$ since $x \in V(K)$. Suppose, for a contradiction, that v_c, v_j are distinct non-adjacent vertices for some $j \geq 2$. Part (d) gives the odd hole H_{cj} (or H_{jc}). But the existence of w_c is a contradiction since color c is missing from K.

Suppose now that v_1, v_j are non-adjacent for some $j \geq 2$. We apply part (d) to get H_{1j}. Then $w_1 = x$ since there can only be one vertex of color 1 in K. But then v_1, v_j, x are three different neighbors of x', all of which are in in H_{1j}, which contradicts the fact that H_{1j} is a hole.

(f) Suppose, on the contrary, that such a φ' exists, say with $\varphi'(x) = \varphi'(v_i) = i$. Since G is not $(\Delta - 1)$-colorable, we may assume that $\varphi'(v_t) = t$ for all other t. Since v_i is not adjacent to v_j, part (d) gives us the odd hole H_{ij}. Since there can only be one vertex in K with color i we get that $w_i = x$. But then w_i, v_i, v_j are three different neighbors of x', all of which are in H_{ij}, which contradicts the fact that H_{ij} is a hole.

(e) We obviously cannot have two non-adjacent vertices in K, but suppose for a contradiction that $v_i \in V(K)$. Part (d) gives us the odd hole H_{ij}. Since $v_i \in V(K)$ and there can only be one vertex of color i in K, it must be that $w_i = v_i$. We know that v_i has degree at least ω

because it is adjacent to x' and its neighbors in K. By part (c), v_i is also adjacent to v_1 (which is not in K because x is and $\varphi(x) = 1$), and has an additional neighbor v_c for each color c which does not appear on the vertices of K. There are $\Delta - 1 - \omega$ such colors. Thus, $d_G(v_i) = \Delta$.

Note that H_{ij} contains $v_i, w_j \in V(K)$, and every other vertex in H_{ij} must be outside K and must be high (by Claim 18). Since $d_G(v_i) = d_G(x') = \Delta$, it must be that $d_G(w_j) \leq \Delta - 4$ (otherwise H_{ij} is high). We know that $w_j \neq v_j$ since w_j is adjacent to $w_i = v_i$ and v_i, v_j are non-adjacent. In particular, w_j has a neighbor $w'_j \notin \{v_i, x'\}$ in $H_{ij} \setminus K$. So $d_G(w_j) \geq \omega \geq \Delta - 4$, and hence $d_G(w_j) = \Delta - 4$. But then applying (a) to H_{ij} gives that $d_G(w'_j) = \Delta$. Then H_{ij} is high since $d_G(w_j) = \Delta - 4$ and x', v_i, w'_j each have degree Δ, contradiction. □

We are now ready to start our main argument. Since $G \neq K$ (as $\omega \leq \Delta - 2$), we know by Claim 18 that G contains some odd hole H. By Claim 19 H must contain exactly two vertices x, y in K, say with $d_G(x) \leq d_G(y)$, and let x', y' be the neighbors of x, y in $V(H) \setminus V(K)$, respectively. Label the neighbors of x' as in Claim 20(b) and let φ be the coloring guaranteed there.

Suppose first that $\omega = \Delta - 2$. Then as x, y each have a neighbor outside of K, they have degree at least $\omega = \Delta - 2$. Claim 20(a) gives $d_G(x'), d_G(y') = \Delta$, and every other vertex of H is high by Claim 18, so H is a high odd hole, contradiction. So we may assume that $\omega \in \{\Delta - 4, \Delta - 3\}$. The rest of our argument is split into these two cases.

Case I: $\omega = \Delta - 3$

Vertices x, y each have $\Delta - 4$ neighbors in K, and are also adjacent to x', y', respectively. Thus $d_G(y) \geq d_G(x) \geq \Delta - 3$. Since H is not high and $d_G(x') = d_G(y') = \Delta$ (by Claim 20(a)), we know that $d_G(x) = \Delta - 3$ and $d_G(y) \in \{\Delta - 3, \Delta - 2\}$.

If the vertices $v_2, \ldots v_{\Delta-1}$ induce a clique, then we have a copy of $K_{\Delta-2}$ in G, contradicting $\omega = \Delta - 3$. Thus, we may assume without loss that v_2, v_3 are not adjacent in G. By Claim 20(d) we get the odd hole H_{23}. As w_2, w_3 are in K with at least one neighbor each outside K (in H_{23}), we know that $d_G(w_2), d_G(w_3) \geq \Delta - 3$. We also know that the two vertices on H_{23} adjacent to w_2, w_3 both have degree Δ by applying Claim 20(a) to H_{23}. By Claim 20(e),

$w_2 \neq v_2$ and $w_3 \neq v_3$, so x' is a third vertex on H_{23} of degree Δ. Since H_{23} is not high this means that $d_G(w_2) = d_G(w_3) = \Delta - 3$.

Let T be the maximal $(1, 2)$-alternating subgraph in φ containing v_1. We claim that $v_2 \in V(T)$. If not, then first swap the colors 1 and 2 on all of T. If the color of x is no longer 1 (i.e. if x is in T) then we can now assign 1 to x' to get complete a ($\Delta - 1$)-coloring of G, contradiction. So it must be the x still has color 1 after our switch on T. But now we can pass color 1 from x to x' and then extend the ($\Delta - 1$)-coloring to x (since $d_G(x) = \Delta - 3$). So we indeed get that $v_2 \in V(T)$.

We now claim that $x \in V(T)$. If not, then again first swap the colors 1 and 2 on all of T. Since $x \notin V(T)$, the color of x is still 1, but also color 1 is on v_2 (since $v_2 \in V(T)$), contradicting Claim 20(f). So we indeed get that $x \in V(T)$.

We claim finally that x has a neighbor in T outside of K; suppose not. We know that x is adjacent to $w_2 \in K \cap T$, so this must be x's only neighbor in T. But then w_2 must have at least two neighbors outsides of K: one in H_{23}, which has color 3 since $w_2 \neq v_2$ (by Claim 20(e)), and one in T of color 1. Therefore $d_G(w_2) \geq \omega - 1 + 2 = \Delta - 2$, which contradicts $d_G(w_2) = \Delta - 3$. So we indeed get that x has a neighbor in T outside of K.

The vertex x is adjacent to every other vertex in K, plus the one additional neighbor found in the last paragraph, plus the vertex x' which is not in T since it is uncolored. So $d_G(x) \geq \omega - 1 + 2 = \Delta - 2$. Since we already established that $d_G(x) = \Delta - 3$, this is a contradiction which completes Case I.

Case II: $\omega = \Delta - 4$

Theorem 1.2(a) gives that $\omega \geq \Delta - 3$ when $\Delta \geq 13$, so we may assume that $7 \leq \Delta \leq 12$.

As K has $\Delta - 4$ vertices, there are exactly three colors from φ which do not appear on $V(K)$. Without loss, suppose these three colors are $\Delta - 1, \Delta - 2, \Delta - 3$ (note color 1 appears on K since $x \in V(K)$).

By Claim 20(c), the vertices $v_1, v_{\Delta-1}, v_{\Delta-2}, v_{\Delta-3}$ induce a copy of K_4. In fact Claim 20(c) says that we can extend this clique to a K_5 by also including one other vertex, say $v_{\Delta-4}$ without loss of generality. Since x' is adjacent to all the vertices of this K_5 by definition, in fact we can include x' to get a K_6. So $\omega \geq 6$. But in fact, since K is the unique K_ω in G and $x' \notin V(K)$, we must have have that $\omega \geq 7$. Since $\omega = \Delta - 4$ this means that $\Delta \in \{11, 12\}$ (and $\omega \in \{7, 8\}$). We will be able to get quick contradictions in both cases once we establish the following claim. For here and in what follows, we let $U = \{v_2, \ldots, v_{\Delta-4}\}$, noting that the colors on U are all used on K (in addition to color 1). By $d_U(v)$ we mean the number of neighbors a vertex v has in $U \setminus \{v\}$.

Claim 21. *If $v_i \in U$ then $d_U(v_i) \geq \Delta - 9$. Moreover, for any distinct $v_i, v_j \in U$ with $d_U(v_i) = d_U(v_j) = \Delta - 9$, the vertices v_i, v_j must be adjacent.*

Proof of Claim. Suppose some $v_i \in U$ has $d_U(v_i) \leq \Delta - 10$. As $|U| = \Delta - 5$, this means there are at least four vertices in U to which v_i is not adjacent. Without loss, say these four vertices are v_2, v_3, v_4, v_5. We apply Claim 20(d) to get four different odd holes H_{ci}, with $c \in \{2, 3, 4, 5\}$.

Consider the vertex w_i, which must be common to all four holes since it is the only vertex in K with color i. The vertex w_i therefore has $\Delta - 5$ neighbors in K, as well as four more neighbors outside of K, one from each H_{ci}. Thus $d_G(w_i) \geq \Delta - 1$.

Consider the odd hole H_{2i}, which includes the five distinct vertices v_2, v_i, x', w_2, w_i (by Claim 20(e)). Both w_2, w_i have a neighbor in $H_{2i} \setminus K$, say w_2', w_i', respectively, which makes $d_G(w_2) \geq \Delta - 4$ (we already have a better bound for w_i). By applying Claim 20(a) to H_{2i}, we get that $d_G(w_2') = \Delta$.

We know that every vertex in H_{2i} other than w_2, w_i is high (by Claim 18), and in fact we have seen that w_i is high as well. We also know that w_2', x' are distinct vertices in H_{2i} which both have degree Δ. So since H_{2i} is not high, we must have $d_G(w_2) = \Delta - 4$ and every other vertex on H_{2i} other than w_2', x' has degree exactly $\Delta - 1$, including w_i', w_i. But then the existence of the path $P = (w_i', w_i, w_2)$ contradicts Lemma 4.4.

We have now established the first sentence of the claim. To establish the second, suppose for a contradiction that there exist two non-adjacent vertices $v_i, v_j \in U$ with $d_U(v_i) = d_U(v_j) = \Delta - 9$. Since $|U| = \Delta - 5$, we know that each of v_i, v_j has two non-neighbors in U, in addition to each other, say v_a, v_b and v_c, v_d, respectively. We apply Claim 20(d) to v_i and each of v_a, v_b to get the two odd holes H_{ia}, H_{ib} (it could be that $i > a, b$ but we are not concerned about this here); we get the holes H_{jc}, H_{jd} analogously. We also apply Claim 20(d) to v_i, v_j to get the odd hole H_{ij},

Note that H_{ia}, H_{ib} both include the vertex $w_i \in V(K)$, H_{jc}, H_{jd} both include the vertex $w_j \in V(K)$, and H_{ij} includes both w_i, w_j. Given the different colors on vertices in each of these holes, this means that both w_i, w_j have at least three neighbors outside of K. Since w_i, w_j also have $\Delta - 5$ neighbors in K, $d_G(w_i), d_G(w_j) \geq \Delta - 2$.

By Claim 18, every vertex in H_{ij} besides w_i, w_j is high. By applying Claim 20(a) to H_{ij} we get that the neighbor of w_i in H_{ij} also has degree Δ. Claim 20(e) gives that $w_i \neq v_i$, so x' is not this neighbor, and hence is a second vertex in H_{ij} of degree Δ. But then since $d_G(w_i), d_G(w_j) \geq \Delta - 2$ this means that H_{ij} is high, contradiction. □

Suppose now that $\Delta = 11$. Consider $v_2 \in U$. Claim 21 gives that $d_U(v_2) \geq 2$; suppose without loss that v_3 is a neighbor of v_2. We previously had that $v_1, v_{\Delta-1}, v_{\Delta-2}, v_{\Delta-3}, x'$ induces a K_5 in G. By Claim 20(c) these vertices are all adjacent to v_2, v_3, so in fact we may add these to the K_5 to get a $K_7 = K_\omega$ in G. This contradicts K as the unique K_ω in G since $x' \notin V(K)$.

We may now assume that $\Delta = 12$. We will show that U contains a K_3. If we succeed at this, then we will get copy of $K_8 = K_\omega$ in G which contains x', as in the previous paragraph, leading to our final desired contradiction.

By Claim 21, $d_U(v_i) \geq 3$ for all v_i in U. Consider the graph G_U induced on U, which has 7 vertices. If there are three vertices in U which meet this lower bound with equality, then these three vertices form a K_3, by Claim 21. So G_U has at most two vertices of degree 3, and the rest have degree at least four, hence $|E(G_U)| \geq \frac{1}{2}(5(4) + 2(3)) = 13$. Mantel's theorem says that any 7-vertex graph with more than $4(3) = 12$ edges contains a K_3, so we have just enough edges to guarantee our desired clique. □

Chapter 5

Conclusion

In this dissertation we proved several results related to the Borodin Kostochka Conjecture. Chapter 3 proved that vertex-critical graphs with $\chi = \Delta = 9$ which don't contain certain forbidden substructures either contain $K_3 \vee E_6$, or are close to containing it. Chapter 4 proved that all graphs with $\chi = \Delta \neq 5, 6$ contain either a high K_ω or a high odd hole.

The work we accomplished in Chapter 3 allows much possibility for extension. Each theorem (Theorems 1.4, 1.6, and 1.7) relies on the development of an algorithm which moves the vertices between the parts of a Mozhan partition in a certain pattern. The subgraphs we find are a direct result of the output of these algorithms. Furthermore, most of the structures which we forbid in our results are forbidden to allow the algorithms to run without encountering errors. Hence, a more effectively designed algorithm, or likely an improvement to one of these existing algorithms, could allow us to discover more adjacencies between vertices, resulting in a stronger subgraph containment, and perhaps less restrictive forbidden subgraph conditions.

Concerning the result in Chapter 4, our future work is clear. We conjecture that Theorem 1.10 does in fact hold for graphs with $\Delta = 5$ and $\Delta = 6$, yet our current methods rely on Theorem 1.11(a) which applies for graphs with $\Delta \geq 7$. However, it is not clear that Theorem 1.11(a) does not hold for graphs with $\Delta < 7$. If it could be shown to be true for graphs with these small Δ, then we suspect our Theorem 1.10 could also be extended to $\Delta = 5, 6$ using the arguments we have already developed. On the other hand, there may be a method to prove our remaining cases without relying on Theorem 1.11(a). We also conjecture that Theorem 1.10 could be strengthened to the following: *Graphs with $\chi = \Delta$ either contain a K_ω where every vertex is high or an odd hole where every vertex is high.*

References

[1] O. V. Borodin and A. V. Kostochka, *On an upper bound of a graph's chromatic number, depending on the graph's degree and density*, J. Combinatorial Theory Ser. B **23** (1977), no. 2-3, 247–250.

[2] R. L. Brooks, *On colouring the nodes of a network*, Proc. Cambridge Philos. Soc. **37** (1941), 194–197.

[3] Paul Allen Catlin, *Embedding subgraphs and coloring graphs under extremal degree conditions*, ProQuest LLC, Ann Arbor, MI, 1976, Thesis (Ph.D.)–The Ohio State University.

[4] Rong Chen, Kaiyang Lan, Xinheng Lin, and Yidong Zhou, *Borodin–kostochka conjecture holds for odd-hole-free graphs*, Graphs and Combinatorics **40** (2024), 1–5.

[5] Vasek Chvátal, *The smallest triangle-free 4-chromatic 4-regular graph*, Journal of Combinatorial Theory, Series A **9** (1970), 93–94.

[6] Daniel W. Cranston, Hudson Lafayette, and Landon Rabern, *Coloring* (P_5, gem)*-free graphs with* $\Delta - 1$ *colors*, J. Graph Theory **101** (2022), no. 4, 633–642.

[7] Daniel W. Cranston and Landon Rabern, *Coloring claw-free graphs with* $\Delta - 1$ *colors*, SIAM J. Discrete Math. **27** (2013), no. 1, 534–549.

[8] ———, *Coloring a graph with* $\Delta - 1$ *colors: conjectures equivalent to the Borodin-Kostochka conjecture that appear weaker*, European J. Combin. **44** (2015), 23–42.

[9] ———, *Graphs with* $\chi = \Delta$ *have big cliques*, SIAM J. Discrete Math. **29** (2018), no. 4, 1792–1814.

[10] G. A. Dirac, *The structure of* k*-chromatic graphs*, Fund. Math. **40** (1953), 42–55.

[11] Tibor Gallai, *On directed paths and circuits*, Theory of graphs **38** (1968), 2054.

[12] Uttam K. Gupta and D. Pradhan, *Borodin-Kostochka's conjecture on (P_5, C_4)-free graphs*, J. Appl. Math. Comput. **65** (2021), no. 1-2, 877–884.

[13] Maria Hasse, *Zur algebraischen begründung der graphentheorie. i*, Mathematische Nachrichten **28** (1965), no. 5-6, 275–290.

[14] Penny Haxell and Colter MacDonald, *Large cliques in graphs with high chromatic number*, Discrete Math. **346** (2023), no. 11, Paper No. 113181, 10.

[15] A. V. Kostochka, *Degree, density and chromatic number of graphs*, Metody Diskret. Analiz. (1980), no. 35, 45–70, 104–105.

[16] Alexandr Kostochka and Matthew Yancey, *Ore's conjecture on color-critical graphs is almost true*, Journal of Combinatorial Theory, Series B **109** (2014), 73–101.

[17] Alexandr V. Kostochka, Landon Rabern, and Michael Stiebitz, *Graphs with chromatic number close to maximum degree*, Discrete Math. **312** (2012), no. 6, 1273–1281.

[18] N. N. Mozhan, *The chromatic number of graphs with a density not exceeding two-thirds of the maximal degree*, Metody Diskret. Analiz. (1983), no. 39, 52–65.

[19] Landon Rabern, *Δ-critical graphs with small high vertex cliques*, J. Combin. Theory Ser. B **102** (2012), no. 1, 126–130.

[20] Bruce Reed, *A strengthening of Brooks' theorem*, J. Combin. Theory Ser. B **76** (1999), no. 2, 136–149.

[21] Bernard Roy, *Nombre chromatique et plus longs chemins d'un graphe*, Revue française d'informatique et de recherche opérationnelle **1** (1967), no. 5, 129–132.

[22] LM Vitaver, *Determination of minimal coloring of vertices of a graph by means of boolean powers of the incidence matrix*, Doklady Akademii Nauk, vol. 147, Russian Academy of Sciences, 1962, pp. 758–759.

[23] Douglas B. West, *Introduction to graph theory*, Prentice Hall, Inc., Upper Saddle River, NJ, 1996.

[24] Di Wu and Rong Wu, *Borodin-kostochka conjecture for a family of* p_6*-free graphs*, 2023.

[25] Xinyue Xie, personal communication.

[26] ______, *On the relation among the maximum degree, the chromatic number and the clique number*, University of Waterloo (2024), 1–103.