Some Conditions for Hamiltonicity in Tough Graphs

by

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Abstract

This dissertation has two focuses, both concerning Hamiltonicity conditions in tough graphs. First, for any integer $t \geq 4$, we present a t-closure lemma that generalizes a closure lemma of Bondy and Chvátal from 1976. In 1995, Hoàng generalized Chvátal's degree sequence condition for Hamiltonicity in 1-tough graphs and posed two t-tough analogues for any positive integer $t \geq 1$. Hoàng confirmed his conjectures respectively for $t \leq 3$ and $t = 1$. We apply our t-closure lemma to confirm the two conjectures for all $t \geq 4$.

Second, we present that all 11-tough $(2P_2 \cup P_1)$-free graphs of order at least three are Hamiltonian. This research is inspired by Chvátal's conjecture that there exists a constant t_0 such that all t_0-tough graphs of order at least three are Hamiltonian. This conjecture is still open, but work has been done to find such a t_0 for certain graph classes. With our result, the conjecture is confirmed for all R-free graphs where R is any five-vertex linear forest excepting P_5.

Artificial Intelligence (AI) Use Disclosure Statement

In the preparation of this dissertation, no Artificial Intelligence (AI) tools were used.

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Soli Deo Gloria

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Chapter 1

Introduction

1.1 Definitions and Notation

All graphs considered in this dissertation are simple and finite. Let $G = (V(G), E(G))$ be a graph. We use $V(G)$ and $E(G)$ to denote the vertex set and edge set of G, respectively. Let $v \in V(G)$ be a vertex. The set of neighbors of v in the graph G is denoted by $N_G(v)$. The *degree of* v in G is the size of $N_G(v)$ and is written as $\deg_G(v)$. The *minimum degree* of G is defined to be $\delta(G) = \min\{\deg(v) : v \in V(G)\}$. We let $\Delta(G) = \max\{\deg(v) : v \in V(G)\}$ be the *maximum degree* of G. For a subset $W \subseteq V(G)$, we let $N_G(W) = \bigcup_{w \in W} N_G(w)$. For a pair of vertices $u, v \in V(G)$, we write $u \sim_G v$ to mean that u and v are adjacent in G. When it is clear from the context that G is the graph in question, we omit the subscript G from the foregoing notation.

Let $H \subseteq G$ be a subgraph of G. We define $\deg(v, H) = |N_G(v) \cap V(H)|$. Let u and v be distinct, non-adjacent vertices of G. We let $G + uv = (V(G), E(G) \cup \{uv\})$. For $S \subseteq V(G)$, let $G[S]$ and $G - S$ denote the subgraph of G induced on S and $V(G) \setminus S$, respectively. For $v \in V(G)$, we write $G - v$ to mean $G - \{v\}$. Given $F \subseteq E(G)$, we define $G - F = (V(G), E(G) \setminus F)$. Let $V_1, V_2 \subseteq V(G)$ be disjoint. Then $E_G(V_1, V_2)$ denotes the set of edges of G with once endvertex in V_1 and one one endvertex in V_2. For two integers p and q, we let $[p, q] = \{i \in \mathbb{Z} : p \leq i \leq q\}$.

Let P be a path between two vertices u and v. We say P is a (u, v)-path. If $w, x \in V(P)$, we write wPx or xPw to mean the subpath of P with endpoints w and x. If yQz is a path distinct from wPx and xy is an edge, we define $wPxyQz$ to be the concatenation of wPx and yQz through the edge xy. Let C be a cycle and fix an orientation for C. Unless otherwise noted, all cycles in this dissertation will be oriented such that clockwise is the positive direction. If $a, b \in V(C)$, we let $a\overrightarrow{C}b$ denote the (a, b)-path of C which begins at a and proceeds to b along the positive orientation of C. We use $a\overleftarrow{C}b$ to denote the (a, b)-path of C which begins at a and proceeds to b along the negative orientation of C. The paths $a\overrightarrow{C}b$ and $a\overleftarrow{C}b$ are depicted in Figure 1.1.

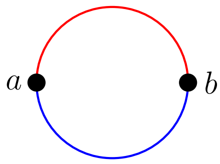


Figure 1.1: The cycle C oriented clockwise, with the top half in red corresponding to the path $a\overrightarrow{C}b$ and the bottom half in blue corresponding to the path $a\overleftarrow{C}b$.

Let $c(G)$ denote the number of components of graph G. Let $t > 0$ be a real number. The *toughness* of a graph G is denoted by $\tau(G)$ and is defined as

$$\tau(G) = \min\left\{\frac{|S|}{c(G-S)} : S \subseteq V(G),\ c(G-S) \geq 2\right\},$$

if G is not complete. If G is complete, we define $\tau(G) = \infty$. We say that G is *t-tough* if $\tau(G) \geq t$.

1.2 Background

In this subsection, we provide some background on Hamilton cycles and Chvátal's toughness conjecture.

1.2.1 Hamilton cycles

Let G be a graph. A cycle C of G is called *Hamiltonian* or simply *Hamilton* if $V(C) = V(G)$. A graph containing a Hamiltonian cycle is also called Hamiltonian. A path P of G which contains every vertex of G is a *Hamiltonian* or *Hamilton path*. We say G is *Hamiltonian-connected* if there exists a Hamilton path between every pair of vertices of G.

The problem of deciding whether a graph has a Hamilton cycle is NP-complete [18]. Thus, much work is focused on finding sufficient conditions for the existence of Hamilton cycles.

Two well known early results were given by Dirac [10] and Ore [21]. Observe that Ore's theorem implies Dirac's.

Theorem 1.2.1 ([10, Theorem 3]). *Let G be a graph on $n \geq 3$ vertices. If $\delta(G) \geq \frac{n}{2}$, then G has a Hamilton cycle.*

Theorem 1.2.2 ([21, Theorem 2]). *Let G be a graph on $n \geq 3$ vertices. If for all pairs of distinct, non-adjacent vertices $u, v \in V(G)$ it holds that $\deg(u) + \deg(v) \geq n$, then G has a Hamilton cycle.*

In 1962, Pósa [23] proved a theorem which generalizes Theorems 1.2.1 and 1.2.2 using degree sequences. Chvátal [7] generalized Pósa's Theorem in 1972. Given a graph G on n vertices, the sequence of the degrees of its vertices $d_1, d_2, \ldots, d_n$ written in a non-decreasing order is called the *degree sequence* of G.

Theorem 1.2.3 ([23]). *Let G be a graph on $n \geq 3$ vertices with degree sequence $d_1, d_2, \ldots, d_n$. If for all $i < \frac{n}{2}$, $i < d_i$, then G is Hamiltonian.*

Theorem 1.2.4 ([7, Theorem 1]). *Let G be a graph on $n \geq 3$ vertices with degree sequence $d_1, d_2, \ldots, d_n$. If for all $i < \frac{n}{2}$, $d_i \leq i$ implies $d_{n-i} \geq n - i$, then G is Hamiltonian.*

1.2.2 Closures and closure lemmas

Given an n-vertex graph G, Bondy and Chvátal [5] defined the *closure* G to be the graph G^* found by iteratively adding edges between non-adjacent vertices x and y such that $\deg(x) + \deg(y) \geq n$ until no such x and y remain. It is easily shown that G^* is well-defined, regardless of the order in which vertices are joined. Thus, from the following theorem, we may conclude that G is Hamiltonian if and only if its closure G^* is Hamiltonian.

Theorem 1.2.5 (Closure Lemma, [5]). *Let G be a graph on n vertices and let x and y be distinct, non-adjacent vertices such that $\deg(x) + \deg(y) \geq n$. Then G is Hamiltonian if and only if $G + xy$ is Hamiltonian.*

This theorem is useful because it means that when looking for Hamilton cycles, we may search in the closure of the graph. If we find such a cycle, we know one must have existed in the original graph. The closure of G is often easier to work with (provided that whatever structure being considered is preserved under closure) because whenever two vertices u and v have $\deg(u) + \deg(v) \geq n$, we have $u \sim v$.

Hoàng and Robin [15] modified the definition of a closure in the context of tough graphs. For a real number $t \geq 0$, the t-*closure* of the n-vertex graph G is the graph G^{*t} found by iteratively adding edges between non-adjacent vertices x and y such that $\deg(x) + \deg(y) \geq n - t$ until no such x and y remain. As with a closure, the t-closure of a graph is well-defined. Hoàng and Robin proved a toughness closure lemma similar to Bondy and Chvátal's.

Theorem 1.2.6 ([15, Lemma 1.7]). *Let $t \geq 2$ and G be a $\frac{3t-1}{2}$-tough graph. Then G is Hamiltonian if and only if its t-closure G^{*t} is Hamiltonian.*

Theorem 1.2.6 is useful when the toughness t' of G is greater than the value of t used to construct G^{*t}, and was used by Hoàng and Robin to confirm Conjecture 1.2.10 when $t = 4$. However, we proved a more natural extension of Theorem 1.2.5, which first appeared in [28]. We provide the proof here in Chapter 2.

Theorem 1.2.7 (Toughness Closure Lemma, [28, Theorem 5]). *Let $t \geq 4$ be a rational number, G be a t-tough graph on $n \geq 3$ vertices, and let $x, y \in V(G)$ be distinct and non-adjacent with $\deg(x) + \deg(y) \geq n - t$. Then G is Hamiltonian if and only if $G + xy$ is Hamiltonian.*

Note that Theorem 1.2.7 gives that the Toughness Closure Lemma holds for $t \geq 4$. It is an open problem whether the statement holds for $1 < t < 4$. However, it is certainly false for $t = 1$, as shown by the following theorem, which first appeared in [27].

Theorem 1.2.8 ([27, Theorem 7]). *For any integer $n \geq 7$, there exists a graph G on n vertices with the following properties:*

1. *There exist distinct, non-adjacent vertices $x, y \in V(G)$ such that $\deg(x) + \deg(y) = n - 1$,*

2. *$G + xy$ is Hamiltonian but G is not Hamiltonian, and*

3. *$\tau(G) = 1$.*

Furthermore, the bound $n \geq 7$ in Theorem 1.2.8 is best possible, as first proved in [27].

Theorem 1.2.9 ([27, Theorem 8]). *For any integer $n \in [3, 6]$, if G is a 1-tough, n-vertex graph with distinct, non-adjacent $x, y \in V(G)$ for which $\deg(x) + \deg(y) \geq n - 1$, then G is Hamiltonian if and only if $G + xy$ is Hamiltonian.*

We conclude Chapter 2 by proving Theorems 1.2.8 and 1.2.9.

1.2.3 Degree sequences in tough graphs

In 1995, Hoàng [14] conjectured that if a graph is t-tough, then the condition on the degree sequence given in Theorem 1.2.4 can be loosened while still guaranteeing Hamiltonicity. We define the following useful predicate, then state Hoàng's first conjecture. Let $P(t)$ be defined for a non-negative integer t and a degree sequence $d_1, d_2, \ldots, d_n$:

$$P(t): \text{ For all } i, \text{ if } i < \frac{n}{2}, d_i \leq i \text{ implies } d_{n-i+t} \geq n - i.$$

Conjecture 1.2.10 ([14, Conjecture 1]). *Let t be a positive integer and G be a t-tough graph on $n \geq 3$ vertices with degree sequence $d_1, d_2, \ldots, d_n$. If G satisfies $P(t)$, then G is Hamiltonian.*

In the same paper, Hoàng [14, Theorem 3] confirmed his conjecture for $t \leq 3$. Recently, Hoàng and Robin [15, Theorem 1.6] confirmed the conjecture when $t = 4$ using Theorem 1.2.6. Combining the two results gives the following statement.

Theorem 1.2.11. *Let $t \leq 4$ be a positive integer and G be a t-tough graph on $n \geq 3$ vertices with degree sequence $d_1, d_2, \ldots, d_n$. If G satisfies $P(t)$, then G is Hamiltonian.*

In his 1995 paper, Hoàng [14, Conjecture 4] conjectured a further generalization of Conjecture 1.2.10 and confirmed a weaker version of the statement for $t = 1$ by allowing G to have one pair of exceptional indices $(i, (n - t + i))$ for some $i < \frac{n}{2}$. (Note that in his paper, Hoàng wrote $d_j + d_{n-j+1} \geq n$ in Conjecture 4, but we believe this to be a typo due to the statement of Conjecture 5 of the same paper.)

Conjecture 1.2.12 ([14, Conjecture 4]). *Let $t \geq 1$ be a positive integer and G be a t-tough graph on $n \geq 3$ vertices with degree sequence $d_1, d_2, \ldots, d_n$. Suppose for each $i \in [1, \lfloor \frac{n-1}{2} \rfloor]$, if $d_i \leq i$ and $d_{n-i+t} < n - i$ implies $d_j + d_{n-j+t} \geq n$ for all $j \in [i+1, \lfloor \frac{n-1}{2} \rfloor]$, then G is Hamiltonian.*

Theorem 1.2.13 ([14, Theorem 6]). *Let $t = 1$ and G be a t-tough graph on $n \geq 3$ vertices with degree sequence $d_1, d_2, \ldots, d_n$. Suppose for each $i \in [1, \lfloor \frac{n-1}{2} \rfloor]$, if $d_i \leq i$ and $d_{n-i+t} < n - i$ implies $d_j + d_{n-j+t} \geq n$ for all $j \in [i+1, \lceil \frac{n}{2} \rceil]$, then G is Hamiltonian.*

In Chapter 3, we confirm Conjectures 1.2.10 and 1.2.12 for $t \geq 4$. The proofs originally appeared in [28] and [27], respectively.

Theorem 1.2.14 ([28, Theorem 3]). *Let $t \geq 4$ be a positive integer and G be a t-tough graph on $n \geq 3$ vertices with degree sequence $d_1, d_2, \ldots, d_n$. If G satisfies $P(t)$, then G is Hamiltonian.*

Theorem 1.2.15 ([27, Theorem 4]). *Let $t \geq 4$ be a positive integer and G be a t-tough graph on $n \geq 3$ vertices with degree sequence $d_1, d_2, \ldots, d_n$. Suppose for each $i \in [1, \lfloor \frac{n-1}{2} \rfloor]$, if $d_i \leq i$ and $d_{n-i+t} < n - i$ implies $d_j + d_{n-j+t} \geq n$ for all $j \in [i+1, \lfloor \frac{n-1}{2} \rfloor]$, then G is Hamiltonian.*

In the case of Conjecture 1.2.10, combining Theorem 1.2.14 with Theorem 1.2.11 fully confirms Conjecture 1.2.10.

Theorem 1.2.16. *Let $t \geq 1$ be a positive integer and G be a t-tough graph on $n \geq 3$ vertices with degree sequence $d_1, d_2, \ldots, d_n$. If G satisfies $P(t)$, then G is Hamiltonian.*

1.2.4 Chvátal's Toughness Conjecture

Toughness was defined by Chvátal in [8], in which he observed that every Hamiltonian graph is 1-tough. He conjectured that the converse was true for some sufficiently large real toughness.

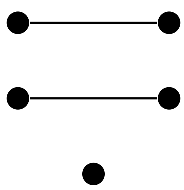
Conjecture 1.2.17 (Chvátal's Toughness Conjecture, [8, Conjecture 2.3]). *There exists a constant t_0 such that every t_0-tough graph on at least three vertices is Hamiltonian.*

In [8], Chvátal showed that if such a t_0 exists, t_0 must be greater than $\frac{3}{2}$. Bauer, Broersma, and Veldman [4] constructed an infinite class of counter-examples for all $t < \frac{9}{4}$, so t_0 must be greater than $\frac{9}{4}$ if it exists. In 2006, Bauer, Broersma, and Schmeichel [2] compiled a survey of graph classes for which the conjecture holds, including planar [31], split [19], co-comparability [9], and chordal graphs [17].

More recently, work has been done studying the conjecture in graphs with certain forbidden linear forests. We define this more rigorously, then list some of these results in Table 1.1. Let R and S be graphs. We say the graph G is R-free if G contains no induced subgraphs isomorphic to R. If G contains no induced subgraph isomorphic to a disjoint union of R and S, we say G is $(R \cup S)$-free. For a positive integer k, we define kR to be a disjoint union of k copies of R. For an integer n, we write K_n to denote the complete graph on n vertices and P_n to denote the path on n vertices. As an example, see Figure 1.2, which depicts a copy of $2P_2 \cup P_1$.

Graph class	Required toughness	Authors and reference
$2K_2$-free	2	Ota & Sanka [22] (See also [6, 24])
R-free, R a 4-vertex linear forest	1	Li, Broersma, & Zhang [20]
$(P_2 \cup kP_1)$-free, for all integers $k \geq 4$	k	Shi & Shan [30] (See also [22, 32])
$(P_4 \cup P_1)$-free	23	Shan [26]
$(P_2 \cup P_3)$-free	15	Shan [25]
$(P_3 \cup 2P_1)$-free	7	Gao & Shan [11]
$(P_2 \cup 3P_1)$-free	3	Hatfield & Grimm [13]

Table 1.1: A list of some results confirming Chvátal's Conjecture for certain forbidden linear forests.



$2P_2 \cup P_1$

Figure 1.2: A copy of $2P_2 \cup P_1$, the substructure forbidden in Theorem 1.2.18.

We now state our result, which first appeared in [29].

Theorem 1.2.18 ([29, Theorem 2]). *Every* 11*-tough* $(2P_2 \cup P_1)$*-free graph on at least three vertices is Hamiltonian.*

If it can be shown that Chvátal's conjecture holds for P_5-free graphs, then Theorem 1.2.18 combined with the final four entries in Table 1.1 would confirm Chvátal's conjecture for R-free graphs where R is a five-vertex linear forest. However, Chvátal's conjecture remains open for P_5-free graphs.

We prove Theorem 1.2.18 in Chapter 4.

Chapter 2

A Toughness Closure Lemma

2.1 Introduction

In this chapter, we prove our Toughness Closure Lemma. As a reminder, we restate the Toughness Closure Lemma here.

Theorem 1.2.7 (Toughness Closure Lemma, [28, Theorem 5])**.** *Let $t \geq 4$ be a rational number, G be a t-tough graph on $n \geq 3$ vertices, and let $x, y \in V(G)$ be distinct and non-adjacent with $\deg(x) + \deg(y) \geq n - t$. Then G is Hamiltonian if and only if $G + xy$ is Hamiltonian.*

To prove Theorem 1.2.7, we first need the following structure theorem, first proved in [28]. To state the theorem, we define some notation. Let C be a cycle and $u \in V(C)$ a vertex. The vertex u^+ is the neighbor of u on C when traveling from u in the positive direction. The vertex u^- is defined likewise when traveling in the negative direction from u. Let $S \subseteq V(C)$. We define $S^+ = \{u^+ : u \in S\}$.

Theorem 2.1.1 ([28])**.** *Let $t \geq 4$ be a rational number and let G be a t-tough graph on $n \geq 3$ vertices. Suppose that G is not Hamiltonian, but there exists $z \in V(G)$ such that $G - z$ has a Hamilton cycle C. Then for any distinct $x, y \in (N(z))^+$, we have that* $\deg(x) + \deg(y) < n - t$.

We first prove Theorem 2.1.1, then Theorem 1.2.7.

2.2 Proof of Theorem 2.1.1

Proof of Theorem 2.1.1. We suppose to the contrary that there exist distinct $x, y \in (N(z))^+$ such that $\deg(x) + \deg(y) \geq n - t$. Because G is not Hamiltonian, G is not a complete graph.

Because G is t-tough, we have $\deg(z) \geq 2t$. Observe that all neighbors of z are on C. Because $V(C) = V(G) \setminus \{z\}$, the path $P := x\overrightarrow{C}y^- z x^- \overleftarrow{C} y$ is a Hamilton path of G. The path P is depicted in Figure 2.1 below.

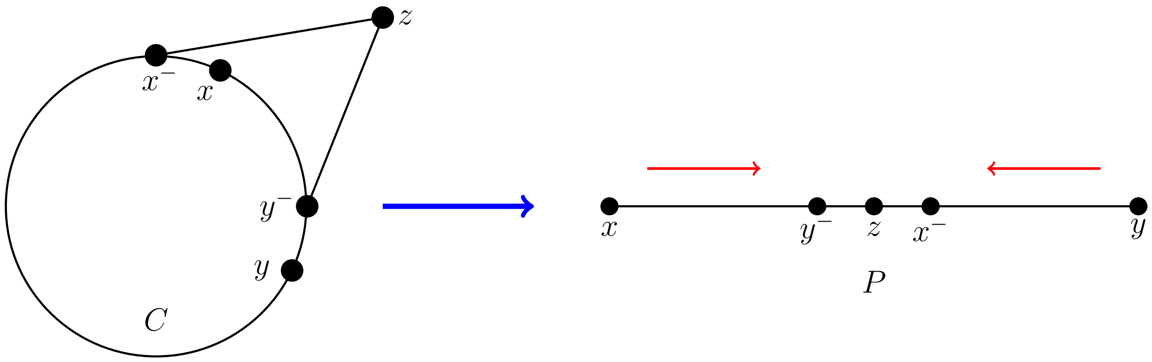


Figure 2.1: A Hamilton (x, y)-path P of G constructed from C, where the first red arrow indicates that the direction from x to y^- on P consists of the path from x to y^- in the clockwise direction of C, and the second red arrow indicates that the direction from x^- to y on P is opposite to the clockwise direction of C.

We fix the forward direction of P to be from x to y. For any $v \in V(P)$, we let v^+ denote the neighbor of v on P contained in $V(vPy)$, and let v^- denote the neighbor of v on P contained in $V(xPv)$. For $S \subseteq V(P)$, we let $S^+ := \{v^+ : v \in S\}$ and $S^- := \{v^- : v \in S\}$. However, for the two special vertices x and y, the superscripts on x^- and y^- are defined with respect to the clockwise direction of C. Thus, we have $z^+ = y^-$ and $z^- = x^-$.

Our goal is to find a cutset S of G which contradicts the toughness of G. To do so, we define some subsets of $V(G)$. The sets we need are as follows:

$$\begin{aligned}
X &:= \{v \in V(P) : v \in N(x)\}, \\
Y &:= \{v^+ \in V(P) : v \in N(y)\}, \\
Z &:= \{v^+ \in V(xPy^-) : v \in N(z)\} \cup \{v^- \in V(x^-Py) : v \in N(z)\} \cup \{x, y\}, \\
R &:= V(G) \setminus (X \cup Y \cup Z).
\end{aligned}$$

We consider some properties of these sets.

Claim 2.2.1. *We have $X \cap Y = \emptyset$.*

Proof of Claim 2.2.1. If $v \in X \cap Y$, then $xvPyv^-Px$ is a Hamilton cycle, a contradiction. $\square$

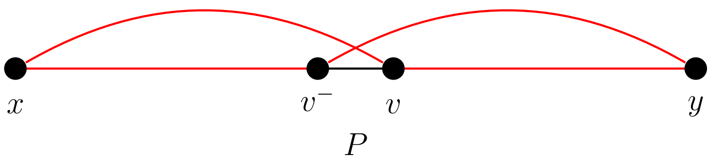


Figure 2.2: The Hamilton cycle described in the proof of Claim 2.2.1 given in red.

Claim 2.2.2. *The set Z is independent in G.*

Proof of Claim 2.2.2. Observe that Z is precisely the set of predecessors of neighbors of z on $\overrightarrow{C}$. If $u, v \in Z$ with $u \in N(v)$, then $uv\overrightarrow{C}u^-zv^-\overleftarrow{C}u$ is a Hamilton cycle in G, a contradiction. □

Claim 2.2.3. *We have $|R \cup (Z \setminus Y)| \leq t$ and $|Y \cap Z| \geq |R| + t$.*

Proof of Claim 2.2.3. By definition, we have $|X \cup Y \cup Z| \leq n - |R|$. Now, $|X| = \deg(x)$ and $|Y| = \deg(y)$ as $y \notin N(y)$. Claim 2.2.1 gives $|X \cup Y| = |X| + |Y|$ and Claim 2.2.2 gives $X \cap Z = \emptyset$. This gives

$$\begin{aligned} n - |R| \geq |X \cup Y \cup Z| &\geq |X| + |Y| + |Z| - |X \cap Z| - |Y \cap Z| \\ &\geq n - t + |Z| - |Y \cap Z| = n - t + |Z \setminus Y|. \end{aligned} \tag{2.1}$$

Rearranging (2.1) gives $t \geq |R| + |Z \setminus Y|$, which is to say $|R \cup (Z \setminus Y)| \leq t + 1$.

Recall that $|Z| = \deg(z) \geq 2t$. Thus, (2.1) gives $n - |R| \geq n - t + |Z| - |Z \cap Y| \geq n - t + 2t - |Z \cap Y|$. Rearranging gives $|Y \cap Z| \geq |R| + t$. □

We let $U = \{v \in V(P) : v^+ \in N(x), v^- \in N(y)\} = X^- \cap Y$. In the remainder of the proof, we show that U is an independent set of size at least $t - 1$, $|N(U)| \leq 2t + 2|U|$, and $c(G - N(U)) \geq |U| + 1$. Together, these properties contradict the assumption that G is 4-tough.

Claim 2.2.4. *We have $y \notin U$ and $y \notin N(U)$.*

Proof of Claim 2.2.4. Since $y \notin X^-$, we have $y \notin U$. Let $v \in U$. As $U = X^- \cap Y$, we have $v^+ \in X$. If $y \in N(v)$, then $v^+ \in Y$. This contradicts Claim 2.2.1. Therefore, $y \notin N(U)$. □

Claim 2.2.5. *We have* $|U| \geq t - 1$.

Proof of Claim 2.2.5. Claim 2.2.3 gives us $|Y \cap Z| \geq |R| + t$. We will show that $|(Y \cap Z) \setminus U| \leq |R| + 1$, which gives the desired bound on $|U|$. To do so, we show that with the possible exception of y, for every $v \in (Y \cap Z) \setminus U$ with $v \neq y$, we have $v^+ \in R$. Let $v \in ((Y \cap Z) \setminus U) \setminus \{y\}$.

First, suppose that $v \in V(x^+Py)$. As $v \in Y \cap Z$, we have $v^- \in N(y)$ and $v^+ \in N(z)$. Additionally, $v \notin U$, so $v^+ \notin N(x)$. By Claim 2.2.2, Z is independent in G. Thus, since $v, y \in Z$, $v \notin N(y)$ and $v^{++} \notin N(z)$. Therefore, $v^+ \notin Y \cup Z$. As $v^+ \notin X$, we have $v^+ \in R$.

Now, suppose that $v \in V(xPy^-)$. We have $v \neq x$ as $x \in Y$. Thus, $v \in N(y) \cap N(z)$. As $v \notin U$, $v^+ \notin X$. Again, Z is independent with $v, y \in Z$, so $v \notin N(y) \cup N(z)$. As stated above, $v^+ \notin X$, so $v^+ \in R$. Therefore,

$$\begin{aligned} |(Z \cap Y) \setminus U| &\leq 1 + \left| \Big(((Y \cap Z) \setminus \{y\}) \setminus U \Big)^+ \right| \\ &\leq 1 + |R|, \end{aligned}$$

as desired. □

Claim 2.2.6. *We have* $|N(U)| \leq 2t + 2|U|$.

Proof of Claim 2.2.6. Since $|N_P(U)| \leq 2|U|$, it is sufficient to show that $|N(U) \setminus N_P(U)| \leq 2t$. We do so by showing that for every vertex $v \in N(U) \setminus N_P(U)$, there exists a unique corresponding vertex $g(v) \in (Z \setminus Y) \cup R$ and that for every vertex $w \in (Z \setminus Y) \cup R$, there exist at most two distinct vertices $v, v' \in N(U) \setminus N_P(U)$ such that $g(v) = g(v') = w$. This suffices because $|(Z \setminus Y) \cup R| \leq t$ by Claim 2.2.3.

Let $v \in N(U) \setminus N_P(U)$ and $u \in U$ such that $v \in N(U)$. We proceed by cases to show that for every $v \in N(U) \setminus N_P(U)$ corresponds to a unique $g(v)$ in $(Z \setminus R) \cup R$.

Case 1: $u \in V(xPv)$**.**

In this case, we have $v^- \notin N(x)$. Otherwise, $xv^-PuvPyu^-Px$ is a Hamilton cycle in G. This gives $v^- \notin X$. Similarly, we have $v^- \notin N(y)$, as otherwise $yv^-Pu^+xPuvPy$ is a Hamilton cycle in G.

If $v^- \notin Y$, then we have $v^- \in (Z \setminus Y) \cup R$, in which case we let $g(v) = v^-$. If we assume $v^- \in Y$, we get $v^{--} \in N(y)$. Then as $v \notin N_P(U)$, we have $v^- \notin U$, so $v \notin N(x)$. Therefore, $v \notin X$. As $v \notin Y$ (because $v^- \notin N(y)$), we have $v \in (Z \setminus Y) \cup R$. Then we let $g(v) = v$.

Case 2: $u \in V(xPu)$**.**

In this case, we have $v^+ \notin N(x)$, as otherwise the cycle $xv^+Pu^-yPuvPx$ is Hamilton. This is to say, $v^+ \notin X$. Similarly, if $v^+ \in N(y)$, then $yv^+PuvPxu^+Py$ is a Hamilton cycle. Therefore, $v^+ \notin N(y)$.

In the case where $v^+ \notin Y$, we let $g(v) = v^+$ as $v^+ \in (Z \setminus Y) \cup R$. Thus, for the remainder of the proof we assume that $v^+ \in Y$. That is, $v \in N(y)$. Then $v^{++} \notin N(x)$ as $v \notin N_P(U)$ implies that $v^+ \notin U$. So, $v^{++} \notin X$. Since $v^+ \notin N(y)$ implies $v^{++} \notin Y$, we have $v^{++} \in (Z \setminus Y) \cup R$. In this case, we let $g(v) = v^{++}$.

It only remains to show that for any $w \in (Z \setminus Y) \cup R$, there are at most two distinct vertices $v_1, v_2 \in N(U) \setminus N_P(U)$ such that $g(v_1) = g(v_2) = w$. We proceed by contradiction. Assume that there exists $w \in (Z \setminus Y) \cup R$ such that there exists distinct vertices $v_1, v_2, v_3 \in N(U) \setminus N_P(U)$ for which $g(v_1) = g(v_2) = g(v_3) = w$. Then let $u_1, u_2, u_3 \in U$ such that $v_i \in N(u_i)$ for each $i \in [1, 3]$. Observe that it is not necessary that u_1, u_2, and u_3 be distinct. By the Pigeonhole Principle and without loss of generality, either $u_1, u_2 \in V(xPw)$ or $u_1, u_2 \in V(wPy)$, where $w = g(v_1) = g(v_2)$. By symmetry, we may assume that $u_1 \in V(xPu_2)$. In each case, we will show that G must have a Hamilton cycle, a contradiction.

First, assume that $u_1, u_2 \in V(xPw)$. By Case 1, for each $i \in \{1, 2\}$, we have either $g(v_i) = v_i^-$ or $g(v_i) = v_i$. Now, if $v_1 \in V(xPv_2)$, then $g(v_1) = g(v_2)$ implies that $g(v_1) = v_1$, $g(v_2) = v_2^-$, and $v_2^- = v_1$. However, this gives the Hamilton cycle $xu_2^+Pv_1u_1Pu_2v_2Pyu_1^-Px$. For reference, see Figure 2.3(a). If $v_2 \in V(xPv_1)$, then $g(v_1) = g(v_2)$ implies that $g(v_1) = v_1^-$, $g(v_2) = v_2$, and $v_1^- = v_2$. Then $xu_2^+Pv_2u_2Pu_1v_1Pyu_1^-Px$ is a Hamilton cycle.

Now, assume that $u_1, u_2 \in V(wPy)$. By Case 2, for each $i \in \{1, 2\}$, we have either $g(v_i) = v_i^+$ or $g(v_i) = v_i^{++}$. If $v_1 \in V(xPv_2)$, then $g(v_1) = g(v_2)$ implies that $g(v_1) = v_1^{++}$, $g(v_2) = v_2^+$, and $v_2 = v_1^+$. Then the cycle $xu_2^+Pyu_1^-Pv_2u_2Pu_1v_1Px$ is Hamilton. For reference, see Figure 2.3(b). If $v_2 \in V(xPv_1)$, then $g(v_1) = g(v_2)$ implies that $g(v_1) = v_1^+$, $g(v_2) = v_2^{++}$, and $v_2^+ = v_1$. In this case, $xu_2^+Pyu_1^-Pv_1u_1Pu_2v_2Px$ is a Hamilton cycle.

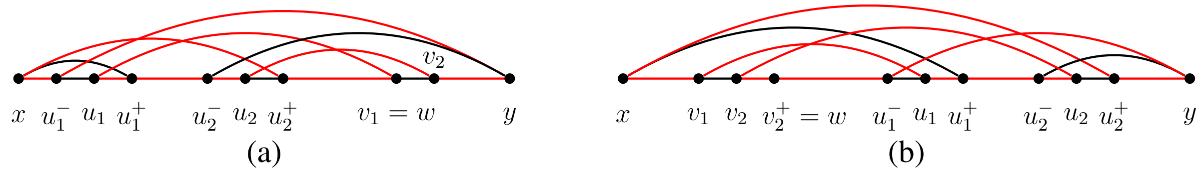


Figure 2.3: Hamilton cycles in G under the assumption that there exists $w \in (Z \setminus Y) \cup R$ such that there are three distinct vertices $v_1, v_2, v_3 \in N(U) \setminus N_P(U)$ for which $g(v_1) = g(v_2) = g(v_3) = w$.

Therefore, by the above arguments, we have shown that

$$|N(U)| = |N(U) \setminus N_P(U)| + |N_P(U)| \le 2|(Z \setminus Y) \cup R| + |U^-| + |U^+| \le 2t + 2|U|.$$

□

Now, set $S = N(U)$. Claim 2.2.6 gives $|S| \le 2t + 2|U|$ and Claim 2.2.4 gives that $c(G - S) \ge |U| + 1$. Claim 2.2.5 gives $|U| \ge t - 1$. Since $t \ge 4$, we have $2|U| < 4(|U| - 1) \le t(|U| - 1)$. Thus,

$$\frac{|S|}{c(G-S)} \le \frac{2t + 2|U|}{1 + |U|} < \frac{2t + t(|U| - 1)}{2 + |U| - 1} = t,$$

contradicting that $\tau(G) \ge t$. ■

2.3 Proof of Theorem 1.2.7

Proof of Theorem 1.2.7. Let G, t, x, and y be as described. Clearly, if G is Hamiltonian, then $G + xy$ is Hamiltonian as well. Therefore, we examine the converse statement. For the sake of contradiction, assume that $G + xy$ is Hamiltonian but G is not. Since G is not Hamiltonian, G is not complete. Additionally, $\delta(G) \ge 2t$.

Because $G + xy$ is Hamiltonian, there must exist some Hamiltonian path in G connecting x and y. Let $P = v_1, v_2, \ldots, v_n$ be such a path where $x = v_1$ and $y = v_n$. We orient P from x to y and write $u \preceq v$ for two vertices $u, v \in V(G)$ such that u is at least as close to x as v is along $\overrightarrow{P}$. For a vertex v_i, for $i \in [1, n-1]$ we let $v_i^+ = v_{i+1}$ and for $i \in [2, n]$ we let $v_i^- = v_{i-1}$. We will find a cutset S of G such that $|S| \le 2t$. This will contradict the toughness of G. To do so, we will examine how the neighbors of x and y are arranged along P, along with their adjacencies, heavily relying on the assumption that G is not Hamiltonian. We begin with a claim.

Claim 2.3.1. *Let distinct $i, j \in [2, n-1]$ and suppose $x \sim v_i$ and $y \sim v_j$. Then the following hold.*

(1) $y \not\sim v_i^-$ and $v_i^- \not\sim v_j^+$ if $i < j$.

(2) If $j < i$, then $v_i^+ \not\sim v_j^+$ and $v_i^- \not\sim v_j^-$.

(3) If $i \leq n-3$ and additionally $x \sim v_{i+2}$, then $v_{i+1} \not\sim v_k^+$ for any v_k with $v_k \sim y$.

(4) If $j \leq n-3$ and additionally $y \sim v_{j+2}$, then $v_{j+1} \not\sim v_k^-$ for any v_k with $v_k \sim x$.

Proof of Claim 2.3.1. To prove the statements, we find various Hamilton cycles which contradict our assumption of G being non-Hamiltonian. If $v_i^- \in N(y)$, then $v_i^- \overleftarrow{P} x v_i \overrightarrow{P} y v_i^-$ is Hamiltonian in G. If $v_i^- \in N(v_j^+)$ where $i < j$, then $v_i^- v_j^+ \overrightarrow{P} y v_j \overleftarrow{P} v_i x \overrightarrow{P} v_i^-$ is Hamilton in G. This proves (1).

Assume $j < i$. When $v_i^+ \in N(v_j^+)$, G has the Hamilton cycle $v_i^+ v_j^+ \overrightarrow{P} v_i x \overrightarrow{P} v_j y \overleftarrow{P} v_i^+$. If $v_i^- \in N(v_j^-)$, G has the Hamilton cycle $v_i^- v_j^- \overleftarrow{P} x v_i \overrightarrow{P} y v_j \overrightarrow{P} v_i^-$. This proves (2).

To prove (3), first suppose that $i < k$. Then (3) holds by (1). If $k < i$, then (3) follows from (2).

Observe that (4) holds by (1) when $k < j$ and by (2) when $j < k$. $\square$

Claim 2.3.1(1) tells us that there does not exist a vertex v_i on P such that $v_i, v_{i+1} \in N(x) \cap N(y)$. That is, x and y share no common neighbors that are consecutive on P. Because $\deg(x) + \deg(y) \geq n - t$, we expect there to be many sets of neighbors of x or of y appearing consecutively on P. To formalize this idea, we define neighbor intervals of consecutive neighbors on P for x and y. For $z \in \{x, y\}$ and v_i, v_j with $i, j \in [2, n-1]$ and $i < j$ such that $v_i, v_j \in N(z)$, we call $V(v_i P v_j)$ a *z-interval* and denote it by $I_z[v_i, v_j]$ if $V(v_i P v_j) \subseteq N(z)$ but $v_{i-1}, v_{j+1} \notin N(z)$.

Observe that for any intervals $I_x[v_i, v_j]$ and $I_y[v_k, v_l]$, by Claim 2.3.1(1), $|I_x[v_i, v_j] \cap I_y[v_k, v_l]| \leq 1$. In the case where $|I_x[v_i, v_j] \cap I_y[v_k, v_l]| = 1$, it must be the case that $v_j = v_k$. In this case, we let $I_{xy}[v_i, v_j, v_l] = I_x[v_i, v_j] \cup I_y[v_k, v_l]$ and call it a *joint-interval*. For

$i, j \in [3, n-2]$ with $i \leq j$, we define an *interval-gap* to be a set of consecutive vertices on P which are neighbors of neither x nor y. A *parallel-gap* is $J[v_i, v_j] := V(v_i P v_j)$ such that $V(v_i P v_j) \cap (N(x) \cup N(y)) = \emptyset$ and that $v_{i-1}, v_{j+1} \in N(x)$ or $v_{i-1}, v_{j+1} \in N(y)$ or $v_{i-1} \in N(x)$ but $v_{j+1} \in N(y)$. A *crossing-gap* is $J[v_i, v_j] := V(v_i P v_j)$ such that $V(v_i P v_j) \cap (N(x) \cup N(y)) = \emptyset$ and $v_{i-1} \in N(y)$ and $v_{j+1} \in N(x)$. Note that, as defined, since $i, j \in [3, n-2]$, neither x nor y are contained in an interval-gap.

We let $\mathcal{I}_x$ be the set of x-intervals which are not joint-intervals, $\mathcal{I}_y$ be the set of y-intervals which are not joint-intervals, and $\mathcal{I}_{xy}$ the set of joint-intervals. We set $p := |\mathcal{I}_x \cup \mathcal{I}_y|$ and $q := |\mathcal{I}_{xy}|$.

Claim 2.3.2. *Each crossing-gap contains at least two vertices.*

Proof of Claim 2.3.2. Suppose that $\{v_i\}$ is a crossing-gap with a single vertex for some $i \in [2, n-1]$. Then $G - v_i$ has the Hamilton cycle $C = v_{i+1} x \overrightarrow{P} v_{i-1} y \overleftarrow{P} v_{i+1}$. However, $\deg(x) + \deg(y) \geq n - t$ contradicts Theorem 2.1.1 when v_i plays the role of z, with $v_{i-1}, v_{i+1} \in N(v_i)$ as $x^- = v_{i+1}$ and $y^- = v_{i-1}$ on $\overrightarrow{C}$. □

Claim 2.3.3. *When $q \geq 1$, there are at least $q - 1$ distinct crossing-gaps.*

Proof of Claim 2.3.3. When $q = 1$, there are certainly at least $1 - 1 = 0$ distinct crossing-gaps. Thus, we assume $q \geq 2$. Let the q common neighbors of x and y be $u_1, u_2, \ldots, u_q$ such that $u_1 \preceq u_2 \preceq \cdots \preceq u_q$. We claim that for every $i \in [1, q-1]$, there exists a crossing-gap between u_i and u_{i+1}. Certainly, $u_i \in N(y)$ and $u_{i+1} \in N(x)$. Let $w_1 \in V(u_i P u_{i+1}^-) \cap N(y)$ such that for every $v \in V(u_i P u_{i+1}^-) \cap N(y)$ it holds that $v \preceq w_1$. By this choice of w_1, we have $V(w_1^+ P u_{i+1}^-) \cap N(y) = \emptyset$. Similarly, let $w_2 \in V(w_1^+ P u_{i+1}) \cap N(x)$ such that $w_2 \preceq v$ for all $v \in V(w_1^+ P u_{i+1}) \cap N(x)$. By the choice of w_2, we must have $V(w_1^+ P w_2^-) \cap N(x) = \emptyset$. Note that w_1 must exist as $u_i \in N(y)$ and w_2 must exist as $u_{i+1} \in N(x)$. Therefore, $V(w_1^+ P w_2^-) \cap (N(x) \cup N(y)) = \emptyset$. Also, $w_2 \neq w_1$ as $w_2 \in V(w_1^+ P u_{i+1})$. By Claim 2.3.1(1), we have that $V(w_1^+ P w_2^-)$ is a crossing-gap. Because $V(u_i P u_{i+1})$ and $V(u_j P u_{j+1})$ whenever $i, j \in [1, q-1]$ are distinct, we find at least $q - 1$ crossing gaps. □

Let p^* be the total number of distinct parallel-gaps and q^* be the total number of distinct crossing-gaps. Let the set of p^* parallel-gaps be $\{J[u_i, w_i] : i \in [1, p^*], u_1 \preceq w_1 \preceq u_2 \preceq w_2 \preceq$

$\cdots \preceq u_{p^*} \preceq w_{p^*}\}$. For each $i \in [1, p^*]$, let $p_i := |J[u_i, w_i]|$. Let the set of q^* crossing-gaps be $\{J[r_i, s_i] : i \in [1, q^*], r_1 \preceq s_1 \preceq r_2 \preceq s_2 \preceq \cdots \preceq r_{p^*} \preceq s_{p^*}\}$. For each $i \in [1, p^*]$ and $j \in [1, q^*]$, let $p_i = |J[u_i, w_i]|$ and $q_j = |J[r_j, s_j]|$.

Claim 2.3.4. *We have* $|\mathcal{I}_x \cup \mathcal{I}_y \cup \mathcal{I}_{xy}| = p + q \leq t - \sum_{i-1}^{p^*}(p_i - 1) - \sum_{i=1}^{q^*}(q_i - 2)$.

Proof of Claim 2.3.4. Because $\mathcal{I}_x, \mathcal{I}_y$, and $\mathcal{I}_{xy}$ are disjoint, it is clear that $|\mathcal{I}_x \cup \mathcal{I}_y \cup \mathcal{I}_{xy}| = p+q$. By definition, $|N(x) \cap N(y)| = |\mathcal{I}_{xy}| = q$. Thus, $|N(x) \cup N(y)| \geq n - t - q$. Since $|\mathcal{I}_x \cup \mathcal{I}_y \cup \mathcal{I}_{xy}| = p + q$ and v_2 and v_{n-1} are contained in an x-interval, a y-interval, or a joint-interval, we have exactly $p + q - 1$ interval-gaps. By definition, $p^* + q^* = p + q - 1$. By Claim 2.3.3, $q^* \geq q - 1$. Recall that we assume $x \notin N(y)$. That is, $x, y \notin N(x) \cup N(y)$. As neither x nor y is contained in any interval-gaps, we have

$$\begin{aligned} t + q &\geq |V(G) \setminus (N(x) \cup N(y))| \geq 2 + \sum_{i=1}^{p^*} p_i + \sum_{i=1}^{q^*} q_i \\ &\geq 2 + p^* + \sum_{i=1}^{p^*}(p_i - 1) + 2q^* + \sum_{i=1}^{q^*}(q_i - 2). \end{aligned}$$

Because $p + q - 1 = p^* + q^*$ and $q^* \geq q - 1$, we get $p + q \leq t - \sum_{i=1}^{p^*}(p_i - 1) - \sum_{i=1}^{q^*}(q_i - 2)$. Combining these facts gives

$$|\mathcal{I}_x \cup \mathcal{I}_y \cup \mathcal{I}_{xy}| = p + q \leq t - \sum_{i=1}^{p}(p_i - 1) - \sum_{i=1}^{q-1}(q_i - 2),$$

as desired. ☐

Claim 2.3.5. *For any* $i \in [2, n-2]$, *if* $\{v_i, v_{i+1}\}$ *is a crossing-gap of size two, then* $v_i \notin N(v_j)$ *for any* $j \in [3, n-2]$ *such that* $v_{j-1}, v_{j+1} \in N(y)$.

Proof of Claim 2.3.5. Let i and j be as described, and, for the sake of contradiction, suppose that $v_i \in N(v_j)$. We will show that $|N(v_{i+1})| < 2t$. Since $N(v_{i+1})$ is a cutset, this contradicts the toughness of G.

Let $v_k \in N(y)$. By Claim 2.3.1, if $v_k \preceq v_i$ we have $v_{k-1} \notin N(v_{i+1})$ and if $v_i \preceq v_k$ we have $v_{k+1} \notin N(v_{i+1})$. Thus, $N(v_{i+1}) \cap [(N(y) \cap V(v_2 P v_i))^-] = \emptyset$ and $N(v_{i+1}) \cap [(N(y) \cap$

$V(v_{i+2}Pv_{n-1}))^+] = \emptyset$. We define C to be

$$C := \begin{cases} v_j v_i \overleftarrow{P} x v_{i+2} \overrightarrow{P} v_{j-1} y \overleftarrow{P} v_j & \text{if } i < j \text{ (see Figure 2.4)}, \\ v_j v_i \overleftarrow{P} v_{j+1} y \overleftarrow{P} v_{i+2} x \overrightarrow{P} v_j & \text{if } i > j. \end{cases}$$

Observe that C is a Hamilton cycle of $G - v_{i+1}$. Fix an orientation $\overrightarrow{C}$. All predecessors and successors of vertices in the following discussion are taken to be with respect to $\overrightarrow{C}$. As G is not Hamilton, both $N(v_{i+1})^-$ and $N(v_{i+1})^+$ are independent in G. When $i < j$, since $v_{i+2} \in N(v_{i+1})$ and $x = v_{i+2}^-$, it follows that $z^+ \notin N(v_{i+1})$ for any $z \in N(x)$. Therefore, $N(x)^+ \cap N(v_{i+1}) = \emptyset$. When $j < i$, since $v_{i+2} \in N(v_{i+1})$ and $x = v_{i+2}^+$, it follows that $z^- \notin N(v_{i+1})$ for any $z \in N(x)$. Therefore, $N(x)^- \cap N(v_{i+1}) = \emptyset$.

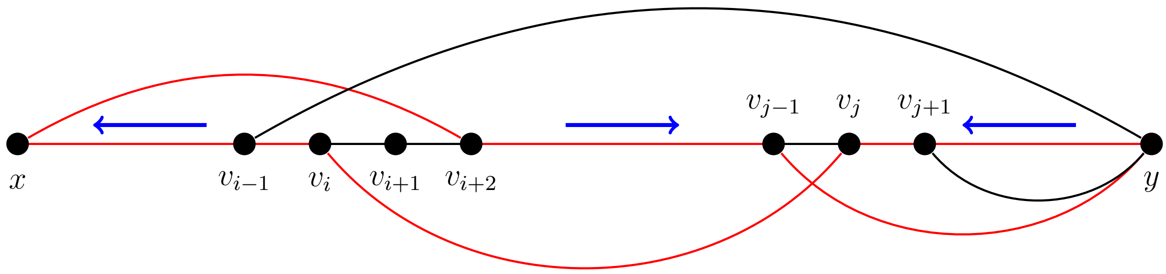


Figure 2.4: Construction of C when $i < j$, drawn in red. The blue arrows indicate the orientation of the corresponding segments of P on $\overrightarrow{C}$.

We claim that v_{i+1} can have at most one neighbor from each set in $\mathcal{I}_x \cup \mathcal{I}_y \cup \mathcal{I}_{xy}$. To see this, observe that when $i < j$, by the construction of C and the arguments given above, we have $z^+ \notin N(v_{i+1})$ when $z \in (N(x) \cup N(y)) \cap (V(xPv_i) \cup V(v_{i+2}Pv_{j-1})$ or when $z \in N(x) \cap V(v_jPy)$, and $z^- \notin N(v_{i+1})$ when $z \in N(y) \cap V(v_jPy)$. As any joint-interval contained in $V(v_jPy)$ has the related x-interval preceding the related y-interval along $\overrightarrow{C}$, there is no joint-interval with some vertices in $V(xPv_i) \cup V(v_{i+2}Pv_{j-1})$ and some other vertices in $V(v_jPy)$. Thus, v_{i+1} can have at most one neighbor from each set in $\mathcal{I}_x \cup \mathcal{I}_y \cup \mathcal{I}_{xy}$. The same holds when $i > j$ by a similar argument.

For each interval-gap, say $\{w\}$ of size one, we claim that $w \notin N(v_{i+1})$. First, suppose $w^-, w^+ \in V(xPv_i) \cup V(v_{i+2}Pv_{j-1})$. By the construction of C, we have $w \notin N(v_{i+1})$ as $w^- \in (N(x) \cup N(y)) \cap (V(xPv_i) \cup V(v_{i+2}Pv_{j-1}))$. The case where $j < i$ is similar.

Next, suppose $w^- \in V(v_{i+2}Pv_{j-1})$ and $w^+ \in V(v_jPy)$. Then $w^- = v_{j-1}$, $w^+ = v_{n-1}$, and $w = y$. But, v_{i+1} is a vertex from an interval-gap, and so definitionally $w = y \notin N(v_{i+1})$.

Finally, suppose $w^-, w^+ \in V(v_jPy)$. In this case, $z^+ \notin N(v_{i+1})$ when $z \in N(x) \cap V(v_jPy)$ and $z^- \notin N(v_{i+1})$ when $z \in N(y) \cap V(v_jPy)$. Therefore, $w \notin N(v_{i+1})$ when $w^- \in N(x)$. Thus, in what follows, we assume $w^- \in N(y)$. Also, when $w^+ \in N(y)$, then $w \notin N(v_{i+1})$. Thus, in what follows, we also assume $w^+ \in N(x)$. This implies that w^+ is the only possible neighbor of v_{i+1} from vertices in the x-interval containing w^+ and w^- is the only possible neighbor of v_{i+1} in the y-interval containing w^-. If $\{w^-, w^+\} \cap N(v_{i+1}) \neq \emptyset$, then $w \notin N(v_{i+1})$ as v_{i+1} has no consecutive neighbors on C, for otherwise we could find a Hamilton cycle of G. Therefore, assume that $\{w^-, w^+\} \cap N(v_{i+1}) = \emptyset$. This means that v_{i+1} has no neighbors in the x-interval containing w^+ or the y-interval containing w^-. By this argument, in the case where $w \in N(v_{i+1})$, we reallocate the contribution that w makes to $\deg(v_{i+1})$ by assuming that v_{i+1} has a neighbor in the x-interval containing w^+. Therefore, when we count $\deg(v_{i+1})$, we assume that $w \notin N(v_{i+1})$.

As v_{i+1} has no consecutive neighbors along C, our argument above indicates that v_{i+1} has at most $\frac{1}{2}(n-1-|N(x) \cup N(y)|)$ neighbors from $V(G-v_{i+1}) \setminus (N(x) \cup N(y))$. As Claim 2.3.4 gives $|\mathcal{I}_x \cup \mathcal{I}_y \cup \mathcal{I}_{xy}| = p + q \leq t$ and $|N(x) \cup N(y)| \geq n - (t+q)$, we have

$$\begin{aligned} \deg(v_{i+1}) &\leq |\mathcal{I}_x \cup \mathcal{I}_y \cup \mathcal{I}_{xy}| + \frac{1}{2}\big(n - 1 - |N(x) \cup N(y)|\big) \\ &\leq t + \frac{1}{2}(t + q - 1) \\ &< 2t. \end{aligned}$$

This contradicts the toughness of G as $N(v_{i+1})$ is a cutset. □

We are now ready to construct a cutset S of G with size less than $2t$. To do so, we need the following sets which define which vertices from various neighbor intervals and interval gaps we add to S.

Let

$$
\begin{aligned}
S_x &= \{v_j, v_{j+1} : v_j \text{ is the right endvertex of an } x\text{-interval that is not a joint-interval}\}, \\
S_y &= \{v_i, v_j : I_y[v_i, v_j] \text{ is a } y\text{-interval that is not a joint-interval}\}, \\
S_{xy} &= \{v_j, v_\ell : I_{xy}[v_i, v_j, v_\ell] \text{ is a joint-interval}\}, \\
T_1 &= \bigcup_{J[v_i,v_j] \text{ is a parallel-gap of size at least 2}} J[v_i, v_j], \\
T_2 &= \bigcup_{J[v_i,v_j] \text{ is a crossing-gap of size 3}} \left(J[v_i, v_j] \setminus \{v_j\}\right), \\
T_3 &= \bigcup_{J[v_i,v_j] \text{ is a crossing-gap of size at least 4}} J[v_i, v_j]
\end{aligned}
$$

and

$$
S = \begin{cases} S_x \cup S_y \cup S_{xy} \cup T_1 \cup T_2 \cup T_3 & \text{if } \{v_{n-1}\} \text{ is a } y\text{-interval,} \\ (S_x \cup S_y \cup S_{xy} \cup T_1 \cup T_2 \cup T_3) \setminus \{v_{n-1}\} & \text{otherwise.} \end{cases}
$$

Claim 2.3.6. *Let $v_i \in V(G) \setminus S$ for some $i \in [2, n-1]$. Then $v_i, v_{i+1} \in N(x)$ or $v_{i-1}, v_{i+1} \in N(y)$ or v_i is contained in a crossing-gap of size two or v_i is the right endvertex of a crossing-gap of size three.*

Proof of Claim 2.3.6. By the definition of S, we know that v_i is a nieghbor of x or y, or is contained in a parallel-gap of size one, or a crossing-gap of size two or three. If $v_i \in N(x)$, then the definition of S_x gives that $v_{i+1} \in N(x)$. If $v_i \in N(y)$, then the definiton of S_y gives that $v_{i-1}, v_{i+1} \in N(y)$. If v_i is in a parallel-gap of size one, by the definition of S_x we have $v_{i-1} \in N(y)$. As $\{v_i\}$ is a parallel-gap, this must mean $v_{i+1} \in N(y)$. If v_i is in a crossing-gap of size three, then T_3 gives that v_i is the right endvertex of that crossing gap. Otherwise, v_i is in a crossing-gap of size two. □

Claim 2.3.7. *We have $|S| \leq 2t - 1$.*

Proof of Claim 2.3.7. For each crossing-gap $J[r_i, s_i]$ of size q_i, we let

$$q_i^* := \begin{cases} q_i & \text{if } q_i \geq 4, \\ q_i - 1 & \text{if } q_i = 3, \\ 0 & \text{if } q_i = 2. \end{cases}$$

Note that by the definition, only one vertex of the y-interval containing v_{n-1} is contained in S. Now, by the definition of S and Claim 2.3.4, we have

$$\begin{aligned} |S| &\leq 2(p+q) - 1 + \sum_{i=1, p_i \geq 2}^{p^*} p_i + \sum_{i=1}^{q^*} q_i^* \\ &\leq 2\left(t - \sum_{i=1}^{p^*}(p_i - 1) - \sum_{i=1}^{q^*}(q_i - 2)\right) - 1 + \sum_{i=1, p_i \geq 2}^{p^*} p_i + \sum_{i=1}^{q^*} q_i^* \\ &= 2t - 1 + \sum_{i=1, p_i \geq 2}^{p^*} (p_i - 2(p_i - 1)) + \sum_{i=1}^{q^*} (q_i^* - 2(q_i - 2)) \\ &\leq 2t - 1, \end{aligned}$$

where the final inequality follows because $p_i - 2(p_i - 1) \leq 0$ when $p_i \geq 2$ and $q_i^* - 2(q_i - 2) \leq 0$ by the definition of q_i^* and the fact that $q_i \geq 2$ for all $i \in [1, q^*]$ by Claim 2.3.2. □

Claim 2.3.8. *We have* $c(G - S) \geq 2$.

Proof of Claim 2.3.8. For the sake of contradiction, assume that $G' = G - S$ is connected. Let $X' = N_{G'}(x) \cup \{x\}$ and $Y' = N_{G'}(y) \cup \{y\}$. As G' is connected, there must exist a path P' in G' which connects vertices in X' and Y'. Suppose $P' = uu_1u_2 \dots u_hv$ for some $u \in X'$ and $v \in Y'$. By Claim 2.3.6, we know that $v = y$ or $v^-, v^+ \in N(y)$ or $v = v_{n-1}$ when the y-interval containing v_{n-1} has size at least two and that $u^+ \in N(x)$. By Claim 2.3.1(4), we know $P' \neq uv$. Thus, $V(P') \geq 3$. As P' is internally disjoint with $X' \cup Y'$, we know that $\{u_i : i \in [1, h]\}$ is a set of vertices from interval-gaps of P.

Recall again that $v = y$ or $v^-, v^+ \in N(y)$ or $v = v_{n-1}$ when the y-interval containing v_{n-1} has size at least two. Since $u_h \in N(v)$, Claim 2.3.1(4) implies that $u_h^+ \notin N(x)$. Therefore, u_h is not the right endvertex of any crossing-gaps. By Claim 2.3.5, u_h is not the left endvertex of any crossing-gap of size two. Thus, by Claim 2.3.6, $\{u_h\}$ is a parallel-gap of size one such that $u_h^-, u_h^+ \in N(y)$. By a similar argument, for any $i \in [2, h-1]$, we can show that $\{u_i\}$ is a parallel-gap of size one with $u_i^-, u_i^+ \in N(y)$. Because $u_1 \in N(u)$ and $u^+ \in N(x)$, we get a contradiction of Claim 2.3.1. □

Claims 2.3.7 and 2.3.8 combine to contradict the toughness of G, completing our proof of Theorem 1.2.7. ■

2.4 Proofs of Theorems 1.2.8 and 1.2.9

We conclude this chapter by recalling and proving Theorems 1.2.8 and 1.2.9.

Theorem 1.2.8 ([27, Theorem 7]). *For any integer $n \geq 7$, there exists a graph G on n vertices with the following properties:*

1. *There exist distinct, non-adjacent vertices $x, y \in V(G)$ such that $\deg(x) + \deg(y) = n - 1$,*
2. *$G + xy$ is Hamiltonian but G is not Hamiltonian, and*
3. *$\tau(G) = 1$.*

Theorem 1.2.9 ([27, Theorem 8]). *For any integer $n \in [3, 6]$, if G is a 1-tough, n-vertex graph with distinct, non-adjacent $x, y \in V(G)$ for which $\deg(x) + \deg(y) \geq n - 1$, then G is Hamiltonian if and only if $G + xy$ is Hamiltonian.*

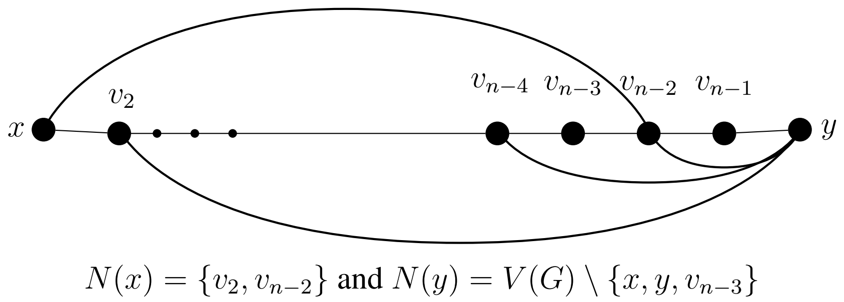


Figure 2.5: A depiction of G as described in the proof of Theorem 1.2.8

Proof of Theorem 1.2.8. Let $n \geq 7$. We form the graph G with vertex set $V(G) = \{x, y\} \cup \{v_i : i \in [2, n-1]\}$ and edge set $E(G) = \{xv_2, xv_{n-2}\} \cup \{v_iv_{i+1} : i \in [2, n-2]\} \cup \{yv_i :$

$i \in [2, n-1] \setminus \{n-3\}\}$. The graph G is depicted in Figure 2.5. We claim that G is as described in the statement of Theorem 1.2.8. Observe that G has the Hamilton path $P = xv_2v_3 \dots v_{n-1}y$. Thus, $G + xy$ is Hamiltonian. It is clear that x and y are not adjacent. Observe that $\deg(x) + \deg(y) = 2 + (n-3) = n-1$. To show that G is not Hamiltonian, we suppose that it is. Suppose C is a Hamilton cycle in G. Because $\deg(x) = \deg(v_{n-3}) = \deg(v_{n-1}) = 2$, we must have $xv_2, xv_{n-2}, v_{n-4}v_{n-3}, v_{n-3}v_{n-2}, v_{n-2}v_{n-1}, v_{n-1}y \in E(C)$. However, this means $\deg_C(v_{n-2}) \geq 3$, a contradiction.

Next, we show that $\tau(G) = 1$. Because $|N(x)| = 2$ and $c(G - N(x)) = 2$, we have $\tau(G) \leq 1$. Let S be any cutset of G. We will show that $c(G - S) \leq |S|$, which proves $\tau(G) = 1$. To do so, we first note that $C := xv_{n-2}v_{n-1}yv_{n-4}Px$ is a Hamilton cycle of $G - v_{n-3}$, which gives that $G - v_{n-3}$ is 1-tough as all cycles are 1-tough.

First, suppose $y \notin S$. If v_{n-3} is not a component of $G - S$, then because $v_{n-2}, v_{n-4} \in N(y)$, we know that $c(G - S) = c(G - v_{n-3} - (S \setminus \{v_{n-3}\}))$. Because $G - v_{n-3}$ is 1-tough, we have $c(G-S) = c(G - v_{n-3} - (S \setminus \{v_{n-3}\})) \leq |S \setminus \{v_{n-3}\}| \leq |S|$. Therefore, we assume v_{n-3} is in a component of $G - S$ containing no other vertices. This implies $v_{n-2}, v_{n-4} \in S$. Let Q be the path $xPv_{n-5}yv_{n-1}$. Then $c(Q - (S \cap V(Q))) \leq |S \cap V(Q)| + 1$. Therefore, $|S| = |S \cap V(Q)| + 2 \geq c(Q - (S \cap V(Q))) + 1 = c(G - S)$.

Next, suppose $y \in S$. Then $Q := P - y$ is a path. Again, we have $c(Q - (S \cap V(Q))) \leq |S \cap V(Q)| + 1$, which gives $|S| = |S \cap V(Q)| + 1 \geq c(Q - (S \cap V(Q))) + 1 = c(G - S)$. ■

Proof of Theorem 1.2.9. Let n, G, x, and y be as described. Clearly, $G + xy$ is Hamiltonian if G is Hamiltonian. Suppose $G + xy$ is Hamiltonian but G is not Hamiltonian. Then G has a Hamilton path $P = v_1v_2 \dots v_n$ where $x = v_1$ and $y = v_n$.

Claim 2.4.1. *Let $i, j \in [2, n-1]$ be distinct and suppose $v_i \in N(x)$ and $v_j \in N(y)$. Then the following hold.*

(1) $v_i^- \notin N(y)$ and $v_j^+ \notin N(x)$,

(2) If $i < j$, then $v_i^- \notin N(v_j^+)$,

(3) If $j < i$ then $v_i^+ \notin N(v_j^+)$ and $v_i^- \notin N(v_j^-)$, and

(4) If $v_k^- \in N(y)$ and $v_k^+ \in N(x)$ for some $k \in [3, n-2]$, then for any $v_\ell \in N(x) \cup N(y)$, we have $v_k \notin N(v_\ell^+)$ if $k < \ell$ and $v_k \notin N(v_\ell^-)$ if $\ell < k$.

Proof of Claim 2.4.1. We prove the first three statements by assuming to the contrary and finding a Hamilton cycle in G, a contradiction. The fourth statement is a corollary of the first three. If $v_i^- \in N(y)$, then $xv_i\overrightarrow{P}yv_i^-\overleftarrow{P}x$ is a Hamilton cycle and if $v_j^+ \in N(x)$, the same cycle is Hamilton replacing v_j^+ with v_i. This proves (1). When $i < j$, if $v_i^- \in N(v_j^+)$, then G has Hamilton cycle $xv_i\overrightarrow{P}v_jy\overleftarrow{P}v_j^+v_i^-\overleftarrow{P}x$, proving (2). When $j < i$, if $v_i^+ \in N(v_j^+)$, then the cycle $xv_i\overleftarrow{P}v_j^+v_i^+\overrightarrow{P}yv_j\overleftarrow{P}x$ is Hamilton in G and if $v_i^- \in N(v_i^+)$, G has Hamilton cycle $xv_i\overrightarrow{P}yv_j\overrightarrow{P}v_i^-v_j^-\overleftarrow{P}x$, which proves (3).

For Statement (4), first assume $k < \ell$. If $v_\ell \in N(y)$, then $v_k \notin N(v_\ell^+)$ by (2) where v_k^+ plays the role of v_i and v_ℓ plays the role of v_j. If $v_\ell \in N(x)$, then $v_k \notin N(v_\ell^+)$ by (3) where v_k^- plays the role of v_j and v_ℓ plays the role of v_i. Now, assume $\ell < k$. If $v_\ell \in N(y)$, by (3) we have $v_k \notin N(v_\ell^-)$ where v_ℓ plays the role of v_j and v_k^+ plays the role of v_i. If $v_\ell \in N(x)$, then (2) gives $v_k \notin N(v_\ell^-)$ where v_ℓ plays the role of v_i and v_k^- plays the role of v_j. □

Let $R = N(x) \cap N(y)$ and $S = V(G) \setminus (N(x) \cup N(y) \cup \{x, y\})$. Clearly then

$$n - 1 = d(x) + d(y) = |N(x) \cup N(y)| + |N(x) \cap N(y)| = n - 2 - |S| + |R|. \tag{2.2}$$

Rearranging gives $|S| = |R| - 1$.

Claim 2.4.2. *We have $|R| \geq 2$.*

Proof of Claim 2.4.2. Suppose $|R| \leq 1$. Then as $|S| = |R| - 1$, we must have $|R| = 1$, as otherwise $|S| < 0$. This, combined with equation 2.2 gives $|N(x) \cup N(y)| = n - 2$. Therefore, $N(x) \cup N(y) = V(G) \setminus \{x, y\}$. Claim 2.4.1(1) gives that for $i \in [2, n-1]$ if $v_i \in N(x)$, then $v_i^- \notin N(y)$. Thus, we must have $v_i^- \in N(x)$. Therefore, there must exist $k \in [2, n-1]$ such that $N(x) = \{v_2, v_3, \ldots, v_k\}$ and $N(y) = \{v_{k+1}, v_{k+2}, \ldots, v_{n-1}\}$. By Claim 2.4.1(2), we have $c(G - v_k) \geq 2$. However, this contradicts that G is 1-tough. □

Observe that

$$6 \geq n \geq |R| + |S| + |\{x, y\}| = |R| + (|R| - 1) + 2 = 2|R| + 1.$$

Therefore, we have $|R| \leq 2$. By Claim 2.4.2, we have $|R| = 2$ and $|S| = 1$. Let u be the lone vertex in S. If $u^-, u^+ \in R$, by Claim 2.4.1(2) and (4), we have $c(G - \{u^-, u^+\}) \geq 3$, contradicting the toughness of G. Therefore, $|\{u^-, u^+\} \cap R| \leq 1$.

Assume that $R = \{v_i, v_j\}$ with $i < j$ and $i, j \in [2, 5]$. As $u \in V(v_i P v_j)$, by Claim 2.4.1(1) and the fact that $|\{u^-, u^+\} \cap R| \leq 1$, we must have $n = 6$ and $R = \{v_2, v_5\}$. By symmetry, we may assume that $u = v_3$. Then $v_4 \in N(x)$ by Claim 2.4.1(1). However, $C = x v_5 y v_2 v_3 v_4 x$ is a Hamilton cycle of G, contradicting our assumption that G is not Hamiltonian. ■

Chapter 3

Applications of the Toughness Closure Lemma

3.1 Introduction

In this chapter, we use the Toughness Closure Lemma to prove Theorems 1.2.14 and 1.2.15. Both results support conjectures of Hoàng [14]. Before starting the proofs, we recall the predicate, $P(t)$, defined for non-negative integers t and a degree sequence $d_1, d_2, \ldots, d_n$:

$$P(t): \text{ For all } i, \text{ if } i < \frac{n}{2}, d_i \le i \text{ implies } d_{n-i+t} \ge n - i.$$

Observe that $P(0)$ recovers the degree sequence condition of Chvátal's Theorem. We now state Hoàng's conjectures.

We now recall our results.

Theorem 1.2.14 ([28, Theorem 3])**.** *Let $t \ge 4$ be a positive integer and G be a t-tough graph on $n \ge 3$ vertices with degree sequence $d_1, d_2, \ldots, d_n$. If G satisfies $P(t)$, then G is Hamiltonian.*

Theorem 1.2.15 ([27, Theorem 4])**.** *Let $t \ge 4$ be a positive integer and G be a t-tough graph on $n \ge 3$ vertices with degree sequence $d_1, d_2, \ldots, d_n$. Suppose for each $i \in [1, \lfloor \frac{n-1}{2} \rfloor]$, if $d_i \le i$ and $d_{n-i+t} < n - i$ implies $d_j + d_{n-j+t} \ge n$ for all $j \in [i+1, \lfloor \frac{n-1}{2} \rfloor]$, then G is Hamiltonian.*

3.2 Proof of Theorem 1.2.14

Before we begin our proofs of Theorems 1.2.14 and 1.2.15, we need an additional theorem.

Theorem 3.2.1 ([3, Corollary 12]). *Let $t \geq 0$ be a real number and G be a t-tough graph on $n \geq 3$ vertices. If $\delta(G) > \frac{n}{t+1} - 1$, then G is Hamiltonian.*

Proof of Theorem 1.2.14. The argument that follows is a standard one, roughly following that given by Hoàng and Robin in [15]. Observe that by Theorem 1.2.7, G is Hamiltonian if and only if its t-closure is Hamiltonian. As adding edges to G results in a graph that satisfies $P(t)$, it suffices to show that the t-closure of G is Hamiltonian. For simplicity of notation, we will assume that G is its own t-closure. We further assume that G is not Hamiltonian, and thus is not complete. Additionally, as G is 4-tough, we have $\delta(G) \geq 2t \geq 8$.

We label the vertices of G as $v_1, v_2, \ldots, v_n$ such that $\deg(v_i) = d_i$ for all $i \in [1, n]$. By the assumption of the Theorem, if $d_i + d_j \geq n - t$, we have $v_iv_j \in E(G)$. By Theorem 1.2.4, if $d_i > i$ for all $i < \frac{n}{2}$, then G is Hamiltonian. Thus, we assume that there exists some positive integer $k < \frac{n}{2}$ such that $d_k \leq k$. We choose k to be the minimum integer such that $d_k \leq k$. As $\delta(G) \geq 8$, we have $k \geq 8$. By the choice of k, $d_i > i$ for all $i \in [1, k-1]$. In particular, $d_{k-1} > k - 1$. As $d_{k-1} \leq d_k \leq k$, this gives $d_{k-1} = d_k = k$.

Let $S, T \subseteq V(G)$. We say that S is *complete to* T if for all $u \in S$ and $v \in T$ such that $u \neq v$ we have $v \in N(u)$. If S is complete to $V(G)$, we call S a *universal clique* of G. Clearly, if u is in a universal clique of G, then $\deg(v) = n - 1$. Additionally, if S is a universal clique of G, then $\delta(G) \geq |S| - 1$. Our goal is to find a universal clique of G with size larger than $\frac{n}{t+1} - 1$, which would show $\delta(G) > \frac{n}{t+1} - 1$. By Theorem 3.2.1, this gives that G is Hamiltonian, contradicting our assumption that G is not Hamiltonian. Let

$$U^\alpha = \{v_i : d_i \geq n - \alpha, i \in [1, n]\} \text{ for any integer } \alpha \text{ with } 1 \leq \alpha < \frac{n}{2}.$$

Claim 3.2.1. *For every integer $\alpha \in [1, \lfloor \frac{n}{2} \rfloor - 1]$, U^α is a clique complete to the set $\{v_i : d_i \geq \alpha - t, i \in [1, n]\}$.*

Proof of Claim 3.2.1. If $v_j \in U^\alpha$ for some $j \in [1, n]$ and $v_\ell \in \{v_i : d_i \geq \alpha - t, i \in [1, n]\}$ for some $\ell \in [1, n]$, then $d_j + d_\ell \geq n - \alpha + \alpha - t = n - t$. By the assumption of the Theorem, if $j \neq \ell$, then $v_\ell \in N(v_j)$. Thus, U^α is complete to $\{v_i : d_i \geq \alpha - t, i \in [1, n]\}$. Because $U^\alpha \subseteq \{v_i : d_i \geq \alpha - t, i \in [1, n]\}$, this also gives that U^α is a clique. □

Claim 3.2.2. *Let $\alpha < \frac{n}{2}$ be any positive integer. If, for every $i \in [1, n]$, it holds that $d_i < \alpha - t$ implies $d_i \geq i - t + 1$, then U^α is a universal clique in G.*

Proof of Claim 3.2.2. Assume that there exists a positive integer α that satisfies the hypothesis but U^α is not a universal clique. Choose $p \in [1, n]$ to be maximum such that there exists $q \in [1, n] \backslash \{p\}$ with $v_q \in U^\alpha$ and $v_p \notin N(v_q)$. By Claim 3.2.1, $v_p \notin \{v_i : d_i \geq \alpha - t, i \in [1, n]\}$. By the hypothesis of this Claim, $d_p \geq n - p - 1$. By the maximality of p, we have $v_\ell \in N(v_q)$ for all $\ell \in [p + 1, n]$. However, this gives $d_p + d_q \geq p - t + 1 + n - p - 1 = n - t$, which implies $v_p \in N(v_q)$, a contradiction. □

Choose $\Omega \subseteq V(G)$ to be a universal clique of G of maximum size.

Claim 3.2.3. *We have $|\Omega| \leq k - 2$.*

Proof of Claim 3.2.3. Suppose that $|\Omega| \geq k - 1$. As Ω is a universal clique in G, we have $d_i \geq \delta(G) \geq |\Omega| \geq k - 1$ for all $i \in [1, n]$. If $|\Omega| > k$, then $d_1 > k$, which contradicts $d_1 \leq d_k = k$. Thus, $k - 1 \leq |\Omega| \leq k$. Observe that $v_i \notin \Omega$ for any $i \in [1, k]$ as every vertex of Ω has degree $n - 1$ and $n - 1 > \frac{n}{2} > k \geq d_i$. Set $S = [\bigcup_{i \in [1,k]} N(v_i)] \setminus \{v_j : j \in [1, k]\}$. Then $\Omega \subseteq S$ as Ω is a universal clique in G. Note that as $d_i \leq k$ for all $i \in [1, k]$, each v_i has at most $k - |\Omega|$ neighbors from $\{v_\ell : \ell \in [k + 1, n]\} \setminus \Omega$ in G. As $k - |\Omega| \in \{0, 1\}$, we have

$$|S| \leq \begin{cases} |\Omega| = k & \text{if } |\Omega| = k, \\ |\Omega| + k = 2k - 1 & \text{if } |\Omega| = k - 1. \end{cases}$$

Since $\Delta(G[\{v_j : j \in [1, k]\}]) \leq 1$, we have $c(G - S) \geq c(G[\{v_j : j \in [1, k]\}]) \geq \frac{k}{2} \geq 4$. But, this gives $\frac{|S|}{c(G-S)} < 4$, contradicting the toughness of G. This proves the Claim. □

Claim 3.2.4. *For all positive integers $\alpha < \frac{n}{2}$ such that $d_\alpha \leq \alpha$, we have $|U^\alpha| \geq \alpha - t + 1$.*

Proof of Claim 3.2.4. Suppose $\alpha < \frac{n}{2}$ is a positive integer with $d_\alpha \leq \alpha$. By the hypothesis of this Theorem, $d_{n-\alpha+t} \geq n - \alpha$. That is, there are at least $n - (n - \alpha + t) + 1 = \alpha - t + 1$ vertices of degree at least $n - \alpha$. As each of these vertices is, by definition, in U^α, the Claim holds. □

Claim 3.2.5. *Let $\alpha \in [k + t - 1, \lfloor \frac{n}{2} \rfloor]$ be an integer. Then $d_\alpha > \alpha$.*

Proof of Claim 3.2.5. Assume that there exists an integer $\alpha \in [k+t-1, \lfloor \frac{n}{2} \rfloor]$ such that $d_\alpha \leq \alpha$. Choose such an α to be minimum. We claim that a contradiction is achieved if U^α is a universal clique in G. This is because if U^α is a universal clique in G, Claims 3.2.3 and 3.2.4 give $k-2 \geq |\Omega| \geq |U^\alpha| \geq \alpha - t$. Rearranging gives $k+t-2 \geq \alpha$. But, $\alpha \geq k+t-1$, which is a contradiction. Therefore, to prove the Claim, it suffice to show that U^α is a universal clique in G. By Claim 3.2.2, we can prove that U^α is a universal clique in G by showing that for every $j \in [1, n]$, it holds that $d_j < \alpha - t$ implies $d_j \geq j - t + 1$.

We first show that for $j \in [\alpha, n]$, $d_j \geq \alpha - t$. First, consider the case when $j = \alpha$. If $\alpha > k+t-1$, then $\alpha - 1 \geq k+t-1$. By the minimality of α, we have $\alpha - 1 < d_{\alpha-1} \leq d_\alpha \leq \alpha$, and so $d_\alpha = \alpha \geq \alpha - t$, as desired. If $\alpha = k+t-1$, then $d_\alpha \geq d_k = k > \alpha - t$. Thus, in either case, $d_j \geq \alpha - t$ when $j = \alpha$. Now, when $j \in [\alpha+1, n]$, $d_j \geq d_\alpha \geq n - \alpha$.

For $j \in [1, \alpha - 1]$, suppose $d_j < \alpha - t$. By the minimality of k, if $j \in [1, k]$, we have $d_j \geq j \geq j - t + 1$. Additionally, if $j \in [k+1, k+t-2]$, $d_j \geq d_k = k \geq j - t + 1$. By the minimality of α, we have $d_j > j > j - t + 1$ when $j \in [k+t-1, \alpha - 1]$. Thus, for every $j \in [1, n]$, it holds that $d_j < \alpha - t$ implies $d_j \geq j - t + 1$, and so by Claim 3.2.2, U^α is a universal clique, drawing the desired contradiction. □

Claim 3.2.6. *We have* $k \geq \frac{n}{2} - t$.

Proof of Claim 3.2.6. Suppose to the contrary that $k < \frac{n}{2} - t$ and set $p = \lfloor \frac{n-1}{2} \rfloor$. Certainly $k - t + 1 \leq p < \frac{n}{2}$. By Claim 3.2.5, we have $d_p > p$. If $d_p = n-1$, then the set $\{v_i : i \in [p, n]\}$ forms a universal clique of G, which implies $|\Omega| > \frac{n}{2}$. But, by Claim 3.2.3, $k > |\Omega|$. Thus, $\frac{n}{2} - t > k > \frac{n}{2}$, a contradiction. Thus, there exists $i \in [1, n]$ such that $v_i \notin N(v_p)$. Choose i to be the maximum among all such integers. Since $v_i \notin N(v_p)$, we have $d_i + d_p < n - t$, which gives $d_i < n - t - d_p < n - t - (\frac{n-1}{2} - 1) = \frac{n+1}{2} - t + 1 \leq d_p$. Thus, $i < p$. We now show that $d_i \geq i - t + 1$. If $i \in [1, k]$, then by the choice of k, we have $d_i \geq i \geq i - t + 1$. If $i \in [k, k+t-2]$, then $d_i \geq d_k = k \geq i - t + 1$. If $i \in [k+t-1, p-1]$, then Claim 3.2.5 gives $d_i \geq i - t + 1$. Thus, $d_i \geq i - t + 1$, as desired. By the maximality of i, $v_j \in N(v_p)$ for all $j \in [i+1, n] \setminus \{p\}$, so $d_p \geq n - i - 1$. However, this gives $d_p + d_i \geq n - i - 1 + i - t + 1 = n - t$, which implies $d_i \in N(v_p)$, a contradiction. □

Claim 3.2.7. *We have $\delta(G) > \frac{n}{t+1} - 1$.*

Proof of Claim 3.2.7. Assume that $\delta(G) \le \frac{n}{t+1} - 1$. Then $2t \le \delta(G) \le \frac{n}{t+1} - 1$. Rearranging gives $(2t+1)(t+1) \le n$. By the choice of k, $d_i > i$ whenever $i \in [1, k-1]$. For $i \in [k, k+t-2]$, we have $d_i \ge d_k = k \ge i - t + 1$. When $i \in [k + t - 1, \lfloor \frac{n-1}{2} \rfloor]$, Claim 3.2.5 gives $d_i > i$. Therefore, by Claim 3.2.2, U^k is a universal clique in G. Combining Claims 3.2.4 and 3.2.6 gives $\delta(G) \ge |U^k| \ge k - t \ge \frac{n}{2} - 2t$. Now, if $t \ge 3$,

$$\begin{aligned} \frac{n}{2} - \frac{n}{t+1} &= \frac{n(t-1)}{2(t+1)} \ge \frac{(2t+1)(t+1)(t-1)}{2(t+1)} \\ &= (t+0.5)(t-1) > 2t - 1. \end{aligned}$$

Thus, $\frac{n}{2} - 2t > \frac{n}{t+1} - 1$. This gives $\delta(G) \ge k - t > \frac{n}{t+1} - 1$, a contradiction. □

With Claim 3.2.7, Theorem 3.2.1 implies that G is Hamiltonian, contradicting our assumption that G is not Hamiltonian. ■

3.3 Proof of Theorem 1.2.15

Proof of Theorem 1.2.15. As before, this argument is relatively standard. Let $t \ge 4$ be an integer and G a t-tough graph with a degree sequence $d_1, d_2, \ldots, d_n$ as described. For each $i \in [1, n]$, we let $v_i \in V(G)$ such that $\deg(v_i) = d_i$. We assume that G is not Hamiltonian. Therefore, G is not a complete graph. Additionally, by the toughness of G, $\delta(G) \ge 2t \ge 8$. By Theorem 1.2.14, there must exist an integer h such that

$$1 \le h < \frac{n}{2} \text{ where } d_h \le h \text{ and } d_{n-h+t} < n - h,$$

as otherwise G would be Hamiltonian. By the hypothesis of the theorem, we have

$$\text{For any } i < h \text{ with } d_i \le i, \text{ it holds that } d_{n-i+t} \ge n - i. \tag{3.1}$$

This is because if there were some $i < h$ with $d_i \le i$ and $d_{n-i+t} < n - i$, then as $h \in [i + 1, \lfloor \frac{n-1}{2} \rfloor]$, we must have $d_h + d_{n-h+t} \ge n$.

Observe that adding edges to G preserves the condition on the degree sequence of G. Therefore, by Theorem 1.2.7, it suffices to show that the t-closure of G is Hamiltonian. For simplicity of notation, we assume that G is its own t-closure. Thus, we assume

$$\text{for any two distinct vertices } u, v \in V(G),\ d(u) + d(v) \geq n - t \text{ implies } u \in N(v). \tag{3.2}$$

Claim 3.3.1. *For any $i \in [h+1, \lfloor \frac{n-1}{2} \rfloor]$, we have $d_i \geq i - t + 1$ and $d_{n-i+t} \geq n - i$.*

Proof of Claim 3.3.1. By the degree sequence condition of the hypothesis of this Theorem, $d_i + d_{n-i+t} \geq n$. Thus, $d_i + d_j \geq n$ for all $j \in [n-i+t, n]$ and $d_j + d_{n-i+t} \geq n$ for all $j \in [i+1, n] \setminus \{n-i+t\}$. Therefore, by 3.2, v_i has at least $n - (n-i+t) = i - t + 1$ neighbors and v_{n-i+t} has at least $n - 1 - (i-1) = n - i$ neighbors. □

We choose $k \in [1, \lfloor \frac{n-1}{2} \rfloor]$ to be the smallest integer such that $d_k \leq k$. Then $d_i > i$ for all $i \in [1, k-1]$. As $k \geq d_k \geq d_{k-1} > k-1$, we have $d_k = k$. As $d_h \leq h$, by the choice of k, we have $k \leq h$. Additionally, as $\delta(G) \geq 2t$, $k \geq 2t$.

We now aim to find a universal clique in G of size larger than $\frac{n}{t+1} - 1$, which is sufficient to prove the result by Theorem 3.2.1.

For any integer $\alpha \in [1, \lfloor \frac{n-1}{2} \rfloor]$, let

$$U^\alpha := \{v_i \in V(G) : d_i \geq n - \alpha, i \in [1, n]\}.$$

Claim 3.3.2. *Let $\alpha \in [1, 1, \lfloor \frac{n-1}{2} \rfloor]$. For every $i \in [1, n]$, if $d_i \geq \alpha - t$ or $d_i \geq i - t + 1$, then U^α is a universal clique in G.*

Proof of Claim 3.3.2. Assume to the contrary. As U^α is not a universal clique, there exists $v_p \in U^\alpha$ and $v_q \in V(G) \setminus \{v_p\}$ such that $v_q \notin N(v_p)$. By the definition of U^α and (3.2), we have $d_q < \alpha - t$. Choose $q \in [1, n]$ to be the maximum integer with the property that $v_q \notin N(v_p)$. By the hypothesis of this claim, $d_q \geq q - t + 1$. By the choice of q, $v_\ell \in N(v_p)$ for all $\ell \in [q+1, n] \setminus \{p\}$. Thus, $d_p \geq n - q - 1$. However, this gives $d_p + d_q \geq n - q - 1 + q - t + 1 = n - t$, implying $v_q \in N(v_p)$, a contradiction. □

Claim 3.3.3. *For every $\alpha \in [1, \lfloor \frac{n-1}{2} \rfloor]$, if $d_\alpha \leq \alpha$, then $|U^\alpha| \geq \alpha - t$.*

Proof of Claim 3.3.3. We first consider when $\alpha \neq h$. If $\alpha < h$, then (3.1) gives $d_{n-\alpha+t} \geq n-\alpha$. Then $|U^\alpha| \geq \alpha-t+1$ as there are at least $n-(n-\alpha+t)+1 = \alpha-t+1$ vertices of degree at least $n-\alpha$. If $\alpha > h$, the hypothesis of the theorem says $d_\alpha + d_{n-\alpha+t} \geq n$. Thus, $d_i + d_{n-\alpha+t} \geq n$ for any $i \geq \alpha$. Then by (3.2), $v_{n-\alpha+t}$ has at least $n-1-(\alpha-1) = n-\alpha$ neighbors, so $d_{n-\alpha+t} \geq n-\alpha$. This gives that there are at least $n-(n-\alpha+t)+1 = \alpha-t+1$ vertices of degree at least $n-\alpha$, again giving $|U^\alpha| \geq \alpha-t+1$.

We now consider when $\alpha = h$. If $d_{h-1} \leq h-1$, then by (3.1), $d_{n-(h-1)+t} \geq n-(h-1) = n-h+1$. That is, there are at least $n-(n-(\alpha-1)+t)+1 = \alpha-t$ vertices of degree at least $n-\alpha$, so $|U^\alpha| \geq \alpha-t$. Therefore, we assume $d_{h-1} > h-1$. This gives $h-1 < d_{h-1} \leq d_h \leq h$, so $d_{h-1} = d_h = h$. By Claim 3.3.1, $d_{n-(h-1)+t} \geq n-h-1$. So, $v_i \in N(v_{n-(h+1)+t})$ whenever $i \geq h-1$ by (3.2). That is, $d_{n-(h+1)+t} \geq (n-1)-(h-2)+1 = n-h+2$. So, there are at least $n-(n-(h+1)+t)+1 = h+2-t$ vertices of degree at least $n-h$, which gives $|U^\alpha| \geq \alpha-t+2$, as desired. □

■

3.4 A Further Strengthening of Theorem 1.2.14

We conclude this chapter by using the following theorem to prove a corollary which strengthens Theorem 1.2.14. The corollary first appeared in [28]

Theorem 3.4.1 ([14, Theorem 7]). *Let $t \geq 0$ be an integer and G be a t-tough graph on $n \geq 3$ vertices with degree sequence $d_1, d_2, \ldots, d_n$. If G satisfies $P(t)$, then G is either pancyclic or bipartite. That is, either there exists a cycle in G of length j for all $j \in [3, n]$, or G is bipartite.*

Corollary 3.4.2 ([28, Corollary 4]). *Let $t \geq 4$ be an integer and G be a t-tough graph on $n \geq 3$ vertices with degree sequence $d_1, d_2, \ldots, d_n$. If G satisfies $P(t)$, then G is pancyclic.*

Proof of Corollary 3.4.2. If $t \geq 4$ and G is a t-tough graph such that G satisfied $P(t)$, then by Theorem 1.2.14, G is Hamiltonian. By Theorem 3.4.1, G is either pancyclic or bipartite. Since all bipartite graphs have toughness at most one, it must be the case that G is pancyclic. ■

Chapter 4

Hamilton Cycles in Tough $(2P_2 \cup P_1)$-free graphs

4.1 Definitions and Useful Lemmas

We begin with some definitions. A *star graph* is a complete bipartite graph in which one of the parts contains only one vertex. A *star-matching* in a graph is the union of vertex-disjoint copies of stars. We call the vertices of degree greater than one in a star-matching the *centers* of the star-matching. For an integer $p \geq 1$, we say a star-matching M is a $K_{1,p}$-matching if every star in M is isomorphic to $K_{1,p}$. For a star-matching M, if $x, y \in V(M)$ and $xy \in E(M)$, we say that x is a *partner* of y under M. Given a graph G, we define the graph $\overline{G} = (V(G), \{uv : u, v \in V(G), uv \notin E(G)\})$.

We first cite some useful lemmas which provide sufficient conditions for the Hamiltonicity of a graph. With the first lemma, given a cycle C in graph G, we can extend C to include vertices with "many" neighbors in $V(C)$, where "many" is determined by a function involving the toughness of G.

Lemma 4.1.1 ([26, Lemma 2.16]). *Let $t > 0$ be a real number and G a t-tough graph on n vertices with non-Hamiltonian cycle C. For a connected subgraph H of $G - V(C)$, if $|N_G(V(H)) \cap V(C)| > \frac{n}{t+1} - 1$, then we can extend C to a cycle C^* such that $V(C) \subseteq V(C^*)$ and $V(C^*) \cap V(H) \neq \emptyset$.*

Let G be a graph. We say the edges $uv, wz \in E(G)$ are *independent* if $\{u, v\} \cap \{w, z\} = \emptyset$. If G is non-complete, the *connectivity* of G is the size of a minimum cutset of G. If G is complete, the connectivity of G is defined to be $|V(G)| - 1$. We use $\kappa(G)$ to denote the connectivity of G. A set of vertices $S \subseteq V(G)$ is said to be an *independent set* if for every pair of distinct vertices $u, v \in S$, $u \notin N(v)$. The size of a maximum independent set of G is called the *independence number* of G and is denoted $\alpha(G)$. The next lemma uses these definitions to describe when Hamilton cycles can be found through specific sets of independent edges.

Lemma 4.1.2 ([12, Theorem 2]). *Let G be a graph and $L \subseteq E(G)$ be a set of independent edges. If $\kappa(G) \geq |L| + \alpha(G)$, then G has a Hamilton cycle containing every edge of L.*

Let G be a graph and S a cutset of G. A component of $G - S$ is said to be *trivial* if it contains only one vertex; otherwise, it is called non-trivial. If G is a $(2P_2 \cup P_1)$-free and $G - S$ has a non-trivial component, then every other component of $G - S$ must be $(P_2 \cup P_1)$-free. We now prove some properties of $(P_2 \cup P_1)$-free graphs in a lemma first proved in [29].

Lemma 4.1.3 ([29, Lemma 6]). *Let G be a $(P_2 \cup P_1)$-free graph. Then the following properties hold.*

1. *If S is a cutset of G, then every component of $G - S$ is trivial.*
2. *If S is a minimal cutset of G, then $G[S, V(G) \setminus S]$ is a complete bipartite graph.*
3. *$\kappa(G) = \delta(G)$.*
4. *$\delta(G) \geq |V(G)| - \alpha(G)$.*

Proof of Lemma 4.1.3. Property (1) follows directly from the $(P_2 \cup P_1)$-freeness of G: if H is a non-trivial component of $G - S$ and J is any other component, we find an induced copy of $P_2 \cup P_1$ by taking the endvertices of any edge of H and any vertex of J.

To prove (2), we assume to the contrary that there exist $x \in S$ and $y \in V(G) \setminus S$ such that $y \notin N(x)$. By Property (1), y must form a trivial component of $G - S$. However, this means $S \setminus \{x\}$ is also a cutset of G, which contradicts that S is a minimal cutset of G.

For (3), we have that $\kappa(G) = \delta(G)$ if G is complete. We therefore assume that G is non-complete. Let W be a minimum cutset of G. Combining Properties (1) and (2) tells us that there exists a vertex $v \in V(G)$ with $\deg(v) = |W|$. Therefore, $\delta(G) \leq \deg(v) = |W|$. Observe that $\kappa(G) \leq \delta(G)$ as any neighborhood is a cutset of G. Thus, $|W| = \kappa(G) \leq \delta(G)$, which gives $\kappa(G) = \delta(G)$.

Property (4) holds if G is a complete graph, as $\delta(G) = n - 1$ and $\alpha(G) = 1$. -Assume that G is not complete. Let W be a minimum cutset of G. By Property (3), $|W| = \kappa(G) = \delta(G)$. Property (1) gives that each component of $G - W$ is trivial. Thus, $\alpha(G) \geq |V(G)| - |W| = |V(G)| - \delta(G)$. Rearranging gives $\delta(G) \geq |V(G)| - \alpha(G)$. ■

The next lemma concerns P_4-free graphs. This is relevant because P_4 contains an induced copy of $P_2 \cup P_1$, so all $(P_2 \cup P_1)$-free graphs must also be P_4-free. To state the lemma, we must define the scattering number of a graph.

Let G be a graph. The *scattering number* of G is

$$s(G) := \begin{cases} \max\{c(G-S) - |S| : S \subseteq V(G), c(G-S) \geq 2\} & \text{if } G \text{ is not complete and} \\ \infty & \text{if } G \text{ is complete.} \end{cases}$$

A cutset $S \subseteq V(G)$ such that $c(G - S) - |S| = s(G)$ is called a *scattering set* of G.

Lemma 4.1.4 ([16, Theorem 4.4(3)]). *Let G be a P_4-free graph. Then G is Hamilton-connected if and only if $s(G) < 0$.*

Finally, we need two results on the existence of star-matching in graphs. The first is a generalization of Hall's matching theorem. The second was first proved in [29]

Lemma 4.1.5 ([1, Theorem 2.10]). *Let G be a bipartite graph with partite sets X and Y, and let f be a function from X to the set of positive integers. For every $S \subseteq X$, if $|N_G(S)| \geq \sum_{v \in S} f(v)$, then G has a subgraph H such that $X \subseteq V(H)$, $\deg_H(v) = f(v)$ for every $v \in X$, and $\deg_H(u) = 1$ for every $u \in Y \cap V(H)$.*

Lemma 4.1.6 ([29, Lemma 8]). *Let $t \geq 1$ be a real number and G be a t-tough non-complete graph. Then for any independent set X of G, there is a $K_{1,\lfloor t \rfloor}$-matching with centers precisely as the vertices of X.*

Proof of Lemma 4.1.6. Let $H = G[X, V(G) \setminus X]$ and $S \subseteq X$ be non-empty. If $|S| = 1$, then we have $|N_H(S)| \geq \delta(G) \geq 2t$. Thus, we assume that $|S| \geq 2$. Then $N_H(S)$ is a cutset of G. Since G is t-tough, we have $|N_H(S)| = |N_G(S)| \geq t|S|$. By Lemma 4.1.5, G has a $K_{1,\lfloor t \rfloor}$-matching with centers precisely as the vertices of X. ■

4.2 Proof of Theorem 1.2.18

We restate our theorem on tough $(2P_2 \cup P_1)$-free graphs.

Theorem 1.2.18 ([29, Theorem 2])**.** *Every 11-tough $(2P_2 \cup P_1)$-free graph on at least three vertices is Hamiltonian.*

Proof of Theorem 1.2.18. Let G be an 11-tough $(2P_2 \cup P_1)$-free graph on $n \geq 3$ vertices. By Theorem 3.2.1, we may assume that $\delta(G) \leq \frac{n}{t+1} - 1$. Therefore, G is not complete and we have $\delta(G) \geq 2t \geq 22$. Additionally, we have $\alpha(G) \leq \frac{n}{t+1}$ because G is t-tough. We consider two cases in this proof. Roughly, in both cases we split the graph into two parts, R_1 and R_2 where R_1 consists of vertices of small degree in G. We "warp" vertices of R_1 using a union of $\mathcal{Q}$, a set of vertex-disjoint paths, with endvertices in R_2. We then construct R_2^* by adding a set of L independent edges whose endvertices are precisely the endvertices of the components of $\mathcal{Q}$. It will be shown that R_2^* has high connectivity, and so R^* has a Hamilton cycle C^* containing all edges of L by Lemma 4.1.2. Replacing each edge xy of L in C^* by the path of $\mathcal{Q}$ with endvertices x and y gives a Hamilton cycle of G.

Case 1: There exists $uv \in E(G)$ such that $|N_G(u) \cup N_G(v)| \leq \frac{5n}{12}$.

Let $S := (N_G(u) \cup N_G(v)) \setminus \{u, v\}$. By the assumption of this case, $|S| \leq \frac{5n}{12} - 2$. Furthermore, S is a cutset of G.

Claim 4.2.1. *We have $c(G - S) = 2$.*

Proof of Claim 4.2.1. Certainly $D_1 := G[\{u, v\}]$ is one component of $G - S$. Since $|S \cup \{u, v\}| \leq \frac{5n}{12}$, we have $c(G - S) \geq 2$. If $G - S$ has a non-trivial component distinct from D_1, then $c(G - S) = 2$ by the $(2P_2 \cup P_1)$-freeness of G. Therefore, we assume all components of $G - S$ excepting D_1 are trivial. This gives $c(G - S) = n - |S| - 1 \geq \frac{7n}{12} + 1$. However,

$$\frac{|S|}{c(G - S)} \leq \frac{\frac{5n}{12} - 2}{\frac{7n}{12} + 1} < 11,$$

which is a contradiction of the toughness of G. $\square$

Let $D_1 := G[\{u, v\}]$. By Claim 4.2.1, $G - S$ has only one other component, say D_2. Since G is $(2P_2 \cup P_1)$-free and D_1 is non-trivial, we have that D_2 is $(P_2 \cup P_1)$-free. Let $S_1 := \{x \in S : \deg_G(x, D_2) < \frac{2n}{t+1}\}$, $S_2 := S \setminus S_1$, $G_1 := G[S_1 \cup \{u, v\}]$, and $G_2 = G[S_2 \cup V(D_2)]$.

Claim 4.2.2. *The graph G_1 is $(P_2 \cup P_1)$-free.*

Proof of Claim 4.2.2. Assume the Claim is false. The, we may choose $x, y, z \in S_1 \cup \{u, v\}$ such that $G_1[\{x, y, z\}] = P_2 \cup P_1$. By the definition of S_1, we have $|N_G(\{x, y, x\}) \cap V(D_2)| < \frac{6n}{t+1}$. Since $|V(D_2)| \geq \frac{7n}{12}$, we must have $|V(D_2)| - |N_G(\{x, y, z\}) \cap V(D_2)| > \frac{n}{t+1}$. As $\alpha(D_2) \leq \alpha(G) \leq \frac{n}{t+1}$, $V(D_2) \setminus N_G(\{x, y, z\})$ is not independent in G. Thus, there exists $u, v \in V(D_2) \setminus N_G(\{x, y, z\})$ such that $uv \in E(G)$. But then $G[\{x, y, z, u, v\}] = 2P_2 \cup P_1$, a contradiction. □

We define a *path-cover* $\mathcal{Q}$ of G_1 to be a union of some vertex-disjoint paths such that $V(G_1) \subseteq V(\mathcal{Q})$. The path-cover $\mathcal{Q}$ is called *W-matched* for a set $W \subseteq V(G_2)$ if the two endvertices of each path of $\mathcal{Q}$ belong to W.

Claim 4.2.3. *The graph G_1 has a W-matched path-cover $\mathcal{Q}$ for some $W \subseteq V(G_2)$ such that the internal vertices of each path of $\mathcal{Q}$ are all from $V(G_1)$, and $\mathcal{Q}$ has exactly $\max\{1, s(G_1)\}$ components.*

Proof of Claim 4.2.3. We construct $\mathcal{Q}$ according to the scattering number of G_1.

First, suppose that $s(G_1) \leq -1$. By Claim 4.2.2, G_1 is $(P_2 \cup P_1)$-free. This implies G_1 is P_4-free, as $P_2 \cup P_1 \subseteq P_4$. By Lemma 4.1.4, G_1 is Hamilton-connected. Since G is 11-tough and $|V(G_1)| \geq 2$, we may choose distinct $x, y \in V(G_1)$ and distinct $z, w \in V(G_2)$ such that $x \in N(z)$ and $y \in N(w)$. We may also choose a Hamiltonian (x, y)-path P in G_1. Then $\{zxPyw\}$ is a W-matched path-cover of G_1 as desired with $W = \{z, w\}$.

Now, suppose that $s(G) \geq 0$. Let $T \subseteq V(G_1)$ be a minimum cutset of G_1. By Lemma 4.1.3(1) and (2), $c(G_1 - T) = |V(G_1) \setminus T|$ and $G_1[T, V(G_1) \setminus T]$ is a complete bipartite graph. Let $T^* \subseteq V(G_1)$ be a scattering set of G_1. We have that $c(G_1 - T) \leq s(G_1)$ and $c(G_1 - T^*) - s(G_1) = |T^*| \geq |T|$. This gives that $c(G_1 - T^*) \geq c(G_1 - T)$. Additionally, $c(G_1 - T^*) = |V(G_1) \setminus T^*|$ and $c(G_1 - T) = |V(G_1) \setminus T|$, so $|T^*| \leq |T|$. Thus, $|T| = |T^*|$, which is to say T is a scattering set of G_1. By our assumption that $s(G_1) \geq 0$, we have $c(G - T) \geq |T|$.

By Lemma 4.1.6 and the toughness of G, there exists a $K_{1,2}$-matching M with centers precisely the vertices of $V(G_1) \setminus T$. Observe that by our assumption that $s(G_1) \geq 0$, and because $c(G_1 - T) = |V(G_1) \setminus T|$, we have $|T| \leq |V(G) \setminus T|$. Therefore, at most $\lfloor \frac{1}{2}|V(G) \setminus T| \rfloor$ vertices of $V(G_1) \setminus T$ are matched to two vertices of T by M. Let $U \subseteq V(G_1) \setminus T$ such that every vertex of U is matched to at least one vertex of T by M. Clearly, $|U| \leq |T|$. If $|U| = |T|$, then at least one vertex of U is matched to a vertex in $V(G_2)$ by M. If $|T| < |V(G_1) \setminus T|$, then choose U^* to be a subset of $V(G_1) \setminus T$ such that $U \subseteq U^*$ and $|U^*| = |T| + 1$. By this choice of U^*, we can find two vertices $x, y \in U^*$ such that both x and y are matched to a vertex of $V(G_2)$ by M. Because $G_1[T, U^*]$ is a complete bipartite graph, we can choose a Hamiltonian (x, y)-path P. Let z and w respectively be the partners of x and y in $V(G_2)$ under M. Then we form the desired W-matched path-cover of G_1 with the path $zxPyw$ and all paths making up the matching M which do not contain any vertices from P. Here, W is the set of endvertices of the paths in the path-cover.

Therefore, we assume that $|T| = |V(G_1) \setminus T|$. First, assume $|V(G_1)| \leq 21$. As $\delta(G) \geq 22$, every vertex of G_1 must have at least two neighbors in $V(G_2)$. Thus, we may choose $x \in T$ and $y \in V(G_1) \setminus T$ and distinct $z, w \in V(G_2)$ such that $z \in N(x)$ and $w \in N(y)$. Because $G_1[T, V(G_1) \setminus T]$ is a complete bipartite graph, there exists a Hamilton (x, y)-path P in G_1. Then $\{zxPyw\}$ forms the desired W-matched path-cover of G_1 where $W = \{z, w\}$. Now, assume $|V(G_1)| \geq 22$. Then $|T| = |V(G_1) \setminus T| \geq 11$. Therefore, there are at least two distinct vertices, $x, y \in V(G_1) \setminus T$ that each have a partner in $V(G_2)$ under M. Let z and w, respectively, be the partners of x and y in $V(G_2)$. If there are $x_1, x_2 \in T$ such that $x_1 \in N(x_2)$,then G_1 has a Hamilton (x, y)-path P (where $x_1 x_2 \in E(P)$). Then $\{zxPyw\}$ forms the desired W-matched path-cover with $W = \{z, w\}$.

Thus, we assume that T is independent in G. Because G is 11-tough and $|T| = |V(G_1) \setminus T|$, there must exist $x^* \in T$ such that $x^* \in N(z^*)$ for some $z^* \in V(G_2)$ where $z^* \neq z$. We can choose a Hamilton (x, x^*)-path P in G_1. Then $\{zxPx^*z^*\}$ is the desired W-matched path-cover of G_1 where $W = \{z, z^*\}$. □

Choose $\mathcal{Q}$ to be a W-matched path-cover of G_1 for some $W \in V(G_2)$ as described in Claim 4.2.3. Let $L := \{xy : x \text{ and } y \text{ are two endvertices of a component of } \mathcal{Q}\}$ be the set of edges from the graph $G \cup \overline{G}$ with endvertices as endvertices of the components of $\mathcal{Q}$. We let $G_2^* = (V(G_2), E(G_2) \cup L)$.

Claim 4.2.4. *We have* $\kappa(G_2^*) \geq |L| + \alpha(G_2^*)$.

Proof of Claim 4.2.4. By Claim 4.2.3, we have $|L| = \max\{1, s(G_1)\} \leq \alpha(G_1) \leq \alpha(G) \leq \frac{n}{t+1}$. Observe that $\kappa(G_2^*) \geq \kappa(G_2)$ and $\alpha(G_2) \leq \alpha(G) \leq \frac{n}{t+1}$. Thus, it suffices to show that $\kappa(G_2) \geq \frac{2n}{t+1}$.

We may assume G_2 is not a complete graph, as $|V(G_2)| = |S_2| + |V(D_2)| \geq \frac{7n}{12}$. Let $W \subseteq V(G_2)$ be a minimum cutset of G_2. For the sake of contradiction, assume that $|W| < \frac{2n}{t+1}$. Because $G_2 \subseteq G$, it must be the case that G_2 is $(2P_2 \cup P_1)$-free. Therefore, $G_2 - W$ has at most two non-trivial components. If $G_2 - W$ has two non-trivial components, then $c(G_2 - W) = 2$. Additionally, as $|V(G_1) \cup W| < \frac{7n}{12}$, $|V(D_2) \setminus W| > \frac{5n}{12}$, and $c(G - (V(G_1) \cup W)) = c(G_2 - W)$, by the toughness of G, we must have at least one non-trivial component of $G_2 - W$ and at least one of the non-trivial components of $G_2 - W$ must contain an edge of D_2. We let Q_1 be a component of $G_2 - W$ which contains an edge of D_2. Because D_1 is non-trivial and G is $(2P_2 \cup P_1)$-free, D_2 must be $(P_2 \cup P_1)$-free. Thus, $V(G_2 - W) \setminus V(Q_1) \subseteq S_2$. Let Q_2 be a component of $G_2 - W$ distinct from Q_1. Let $x \in V(Q_2)$. We have that $x \in S_2$. By the definition of S_2, along with the fact that $V(Q_2) \cap V(D_2) = \emptyset$, we have $\deg_G(x, D_2) \geq \frac{2n}{t+1}$. It follows that $|W| \geq |N_G(x) \cap V(D_2)| \geq \frac{2n}{t+1}$. This contradicts the assumption that $|W| < \frac{2n}{t+1}$. □

By Claim 4.2.4 and Lemma 4.1.2, there exists a Hamilton cycle C^* of G_2^* such that $L \subseteq E(C^*)$. By replacing each edge xy of L in the cycle C^* with the path of $\mathcal{Q}$ with endvertices x and y gives a Hamilton cycle of G.

Case 2: For every $uv \in E(G)$, it holds that $|N_G(u) \cup N_G(v)| > \frac{5n}{12}$.

We let

$$S = \{v \in V(G) : \deg_G(v) < \frac{5n}{24}\} \text{ and } S_1 := \{x \in S : \deg_G(x) < \frac{n}{t+1}\}.$$

Recall that we assumed that $\deg(G) \leq \frac{n}{t+1} - 1$. Therefore, $S_1 \neq \emptyset$. By the assumption that defines Case 2, S must be independent in G. Since G is t-tough, we have $|S| \leq \alpha(G) \leq \frac{n}{t+1}$.

We apply Lemma 4.1.6 to find a $K_{1,2}$-matching M in G with centers precisely the vertices of S_1.

Let $G_2 = G - S$ and $L = \{xy : x \text{ and } y \text{ are endpoints of a path in } M\}$. Form G_2^* from G_2 by adding all edges of L to $E(G_2)$ which are not already contained in $E(G_2)$. We use Lemma 4.1.2 to show that G_2^* has a Hamilton cycle C^* such that $L \subseteq E(C^*)$. We will form a Hamilton cycle of G from C^*.

Claim 4.2.5. *We have $\kappa(G_2^*) \geq |L| + \alpha(G_2^*)$.*

Proof of Claim 4.2.5. As in Case 1, we have $\kappa(G_2^*) \geq \kappa(G_2)$ and $\alpha(G_2^*) \leq \alpha(G) \leq \frac{n}{t+1}$, so to prove the Claim it suffices to show that $\kappa(G_2) \geq |L| + \frac{n}{t+1}$. By the construction of G_2, we have $|V(G_2)| \geq \frac{11n}{12}$, so we assume that G_2 is not complete, as otherwise G_2 has the desired connectivity. Choose $W \subseteq V(G_2)$ to be a minimum cutset of G_2. For the sake of contradiction, assume that $|W| < |L| + \frac{n}{t+1}$.

First, suppose that $G_2 - W$ has a trivial component consisting of the single vertex y. Either $y \in N_G(S_1)$ or $y \notin N_G(S_1)$. Suppose $y \in N_G(S_1)$. Then by the assumption of Case 2, $\deg_G(y) > \frac{4n}{12}$. Certainly, $N_G(y) \subseteq S \cup W$. However, $|S \cup W| < \frac{3n}{t+1}$, contradicting $\deg_G(y) > \frac{4n}{12}$. Thus, we assume $y \notin N_G(S_1)$. Then $\deg_{G_2}(y) \geq \frac{5n}{24} - |S \setminus S_1| = \frac{5n}{24} - |S| + |L|$. Therefore, $|W| \geq \frac{5n}{24} - |S| + |L|$ as $N_{G_2}(y) \subseteq W$. However, as $|S| \leq \frac{n}{t+1}$, this contradicts our assumption that $|W| < |L| + \frac{n}{t+1}$.

Thus, we assume $G - W$ has no trivial component. Because G_2 is $(2P_2 \cup P_1)$-free, $G_2 - W$ must have exactly two non-trivial components. Let these components be Q_1 and Q_2. Let $i \in \{1, 2\}$. We have $|S \cup W| < \frac{3n}{t+1}$ and $|N_G(u) \cup N_G(v)| > \frac{5n}{12}$ for all $uv \in E(G)$, so $|V(Q_i)| \geq \frac{5n}{t+1} - \frac{3n}{t+1} \geq \frac{2n}{t+1}$. Because G is $(2P_2 \cup P_1)$-free and Q_i is a nontrivial component of $G - (S \cup W)$, Q_i must be $(P_2 \cup P_1)$-free. By Lemma 4.1.3(4), we have $\delta(Q_i) \geq |V(Q_i)| - \alpha(Q_i) \geq |V(Q_i)| - \frac{n}{t+1} > \frac{1}{2}|V(Q_i)|$. Thus, by Theorem 1.2.1, Q_i is Hamiltonian. This, combined with the fact that $|V(Q_i)| \geq \frac{2n}{t+1}$, gives that each Q_i has a set M_i of independent edges where $|M_i| \geq \frac{n}{t+1}$. Let $x \in S_1$. Take any edge $x_1y_1 \in M_1$ and any edge $x_2y_2 \in M_2$. Then By the $(2P_2 \cup P_1)$-freeness of G, $x \in N(x_1) \cup N(y_1) \cup N(x_2) \cup N(y)$. As we can form at least $\frac{n}{t+1}$ such unique pairs of edges with one edge from M_1 and one edge from M_2, it must follow that $\deg_G(x) \geq \frac{n}{t+1}$. However, this contradicts the definition of S_1 $\square$

By Claim 4.2.5 and Lemma 4.1.2, G_2^* contains a Hamilton cycle C^* such that $L \subseteq E(C^*)$. Form C^{**} from C^* by replacing every edge $xy \in L$ with the path from M with endvertices x and y. Then $V(C^{**}) \subseteq V(G_2) \cup S_1$. If $S \setminus S_1 = \emptyset$, then C^{**} is the desired Hamilton cyle of G. Assume $S \setminus S_1 \neq \emptyset$. For each vertex $x \in S \setminus S_1$, we have $\deg_G(x) = \deg_G(x, G_2) \geq \frac{n}{t+1}$ as S is independent in G. Lemma 4.1.1 allows us to extend C^{**} to a Hamilton cycle C of G.

Thus, Theorem 1.2.18 is proved in both cases. ■

Chapter 5

Open Problems

We conclude this dissertation by describing some remaining open problems and potential future work. Conjecture 1.2.12 remains open for $t < 3$. A new approach may be needed to settle it, as Theorem 1.2.7 is false when $t = 1$. Proving and using a statement analogous to Theorem 1.2.7 for $1 < t \leq 3$ appears challenging because the technique used to prove Theorem 2.1.1 fails for $t < 4$. Nevertheless, proving or disproving statements similar to Theorems 1.2.7 and 2.1.1 for small t would be fruitful.

Hoàng conjectured a further generalization of Conjecture 1.2.12 which is similar to Corollary 3.4.2.

Conjecture 5.0.1 ([14, Conjecture 5])**.** *Let $t \geq 1$ be a positive integer and G be a t-tough graph on $n \geq 3$ vertices with degree sequence $d_1, d_2, \ldots, d_n$. Suppose for each $i \in [1, \lfloor \frac{n-1}{2} \rfloor]$, if $d_i \leq i$ and $d_{n-i+t} < n - i$ implies $d_j + d_{n-j+t} \geq n$ for all $j \in [i+1, \lfloor \frac{n-1}{2} \rfloor]$. Then if G is Hamiltonian, G is either pancyclic or bipartite.*

Conjecture 1.2.17 remains open in general and appears to remain difficult. It would be of interest to prove it for P_5-free graphs, confirming the conjecture for all R-free graphs where R is a five-vertex linear forest. Additionally, it is not clear that $t = 11$ is the best-possible toughness for Theorem 1.2.18. More work needs to be done to investigate this.

References

[1] J. Akiyama and M. Kano. *Factors and factorizations of graphs*, volume 2031 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2011. Proof techniques in factor theory.

[2] D. Bauer, H. Broersma, and E. Schmeichel. Toughness in graphs – a survey. *Graphs and Combinatorics*, 22(1):1–35, 2006.

[3] D. Bauer, H. J. Broersma, J. van den Heuvel, and H. J. Veldman. Long cycles in graphs with prescribed toughness and minimum degree. *Discrete Mathematics*, 141(1-3):1–10, 1995.

[4] D. Bauer, H. J. Broersma, and H. J. Veldman. Not every 2-tough graph is Hamiltonian. *Discrete Applied Mathematics*, 99(1-3):317–321, 2000.

[5] J. A. Bondy and V. Chvátal. A method in graph theory. *Discrete Mathematics*, 15(2): 111-135, 1976.

[6] H. Broersma, V. Patel, and A. Pyatkin. On toughness and Hamiltonicity of $2K_2$-free graphs. *Journal of Graph Theory*, 75(3):244–255, 2014.

[7] V. Chvátal. On Hamilton's ideals. *Journal of Combinatorial Theory, Series B.*, 12(2):163–168, 1972.

[8] V. Chvátal. Tough graphs and Hamiltonian circuits. *Discrete Mathematics*, 5:215–228, 1973.

[9] J. S. Deogun, D. Kratsch, and G. Steiner. 1-tough cocomparability graphs are Hamiltonian. *Discrete Mathematics*, 170(1):99–106, 1997.

[10] G. A. Dirac. Some theorems on abstract graphs. *Proceedings of the London Mathematical Society*, s3-2(1):69–81, 1952.

[11] Y. Gao and S. Shan. Hamiltonian cycles in 7-tough $(P_3 \cup 2P_1)$-free graphs. *Discrete Mathematics*, 345(12):Paper No. 113069, 2022.

[12] R. Häggkvist and C. Thomassen. Circuits through specified edges. *Discrete Mathematics*, 41(1):29–34, 1982.

[13] A. Hatfield and E. Grimm. Hamiltonicity of 3-tough $(K_2 \cup 3K_1)$-free graphs. arXiv:2106.07083, 2021.

[14] C. T. Hoàng. Hamiltonian degree conditions for tough graphs. *Discrete Mathematics*, 142(1-3):121–139, 1995.

[15] C. T. Hoàng and C. Robin. A closure lemma for tough graphs and hamiltonian degree conditions. *Discrete Mathematics*, 347(4):113872, 2024.

[16] H. A. Jung. On a class of posets and the corresponding comparability graphs. *Journal of Combinatorial Theory, Series B*, 24(2):125–133, 1978.

[17] A. Kabela and T. Kaiser. 10-tough chordal graphs are Hamiltonian. *Journal of Combinatorial Theory, Series B*, 122:417–427, 2017.

[18] R. M. Karp. Reducibility among combinatorial problems. *Complexity of computer computations*, pages 85–103. Springer, New York, 1972.

[19] D. Kratsch, J. Lehel, and H. Müller. Toughness, Hamiltonicity, and split graphs. *Discrete Mathematics*, 150(1):231–235, 1996.

[20] B. Li, H. J. Broersma, and S. Zhang. Forbidden subgraphs for Hamiltonicity of 1-tough graphs. *Discussiones Mathematicae Graph Theory*, 36(4):915–929, 2016.

[21] Ø. Ore. Note on Hamilton circuits. *The American Mathematical Monthly*, 67(1):55, 1960.

[22] K. Ota and M. Sanka. Some conditions for Hamiltonian cycles in 1-tough $(K_2 \cup kK_1)$-free graphs. *Discrete Mathematics*, 347(3):Paper No. 113841, 2024.

[23] L. Pósa. A theorem concerning Hamilton lines. *A Magyar Tudomanyos Akademia Matematikai Kutató Intezetenek Közleményei*, 7(1-2):225–226, 1962.

[24] S. Shan. Hamiltonian cycles in 3-tough $2K_2$-free graphs. *Journal of Graph Theory*, 94(3):349–363, 2020.

[25] S. Shan. Hamiltonian cycles in tough $(P_2 \cup P_3)$-free graphs. *Electronic Journal of Combinatorics*, 28(1):Paper No. P1.36, 2021.

[26] S. Shan. Hamiltonian cycles in tough $(P_4 \cup P_1)$-free graphs. arXiv:2504.08936, 2025.

[27] S. Shan and A. Tanyel. A strengthening of a degree sequence condition for Hamiltonicity in tough graphs. *Discrete Mathematics*, 349(9):Paper No. 115134, 2026.

[28] S. Shan and A. Tanyel. Degree sequence condition for Hamiltonicity in tough graphs. arXiv:2405.04728, 2025.

[29] S. Shan and A. Tanyel. Hamilton cycles in tough $(2P_2 \cup P_1)$-free graphs. arXiv:2506.12684, 2025.

[30] L. Shi and S. Shan. A note on hamiltonian cycles in 4-tough $(P_2 \cup kP_1)$-free graphs. *Discrete Mathematics*, 345(12):Paper No. 113081, 2022.

[31] W. T. Tutte. A theorem on planar graphs. *Transactions of the American Mathematical Society*, 82:99–116, 1956.

[32] L. Xu, C. Li, and B. Zhou. Hamiltonicity of 1-tough $(P_2 \cup kP_1)$-free graphs. *Discrete Mathematics*, 347(2):Paper No. 113755, 2024.