Embedding and Coloring Designs

by

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Abstract

This dissertation focuses on two problems in design theory. Techniques from graph theory are frequently utilized and therefore the proofs may also be of interest to graph theorists.

The first problem focuses on completing partial latin squares with prescribed diagonals. Necessary and sufficient numerical conditions are known for the embedding of an incomplete latin square L of order n into a latin square T of order $t \geq 2n+1$ in which each symbol is prescribed to occur in a given number of cells on the diagonal of T outside of L. This includes the classic case where T is required to be idempotent. If $t < 2n$ then no such numerical sufficient conditions exist since it is known that the arrangement of symbols within the given incomplete latin square can determine the embeddability. All known examples where the arrangement is a factor share the common feature that one symbol is prescribed to appear exactly once in the diagonal of T outside of L. We show if the prescribed diagonal contains a symbol required to appear exactly once on the diagonal of T outside of L and $t \leq 2n$, then there always exists a incomplete latin square satisfying the known numerical necessary conditions that is non-embeddable. Also, we solve a conjecture made over 30 years ago stating it is only this feature that prevents numerical conditions sufficing for all $t \geq n$. Thus providing necessary and sufficient numerical conditions for the embedding of an incomplete latin square L of order n into a latin square T of order t for all $t \geq n$ in which the diagonal of T outside of L is prescribed in the case where no symbol is required to appear exactly once in the diagonal of T outside of L.

The second problem focuses on (not necessarily proper) s-edge-colorings of K_v in which, for all $u \in V(K_v)$, the edges incident with u are colored using exactly p colors. In the spirit of proper edge-colorings, such (s,p)-edge-colorings are required to be equitable: the edges at each vertex are shared evenly among p colors. First, results related to the existence of equitable (s,p)-edge-colorings of K_v and future directions related to equitable (s,p)-edge-colorings of λK_v are discussed. Then, the structure of equitable (s,p)-edge-colorings of K_v

is addressed, particularly, the number of vertices at which each color appears. Results are obtained determining how large and how small these numbers can be. Results concerning equitable (s,p)-C_4-colorings of $K_v - F$ follow as corollaries.

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Chapter 1

Introduction

Design theory is a branch of combinatorics concerned with mathematical structures, called designs, with properties of symmetry and balance. The name design theory comes from its applications to the design of experiments. However, design theory has applications in many other areas of mathematics including graph theory, recreational mathematics, geometry, coding theory, and algebra. Given a particular type of design, we typically ask two styles of questions: (1) Existence: When do such designs exist? (2) Enumeration: How many of such designs exist? We will focus on the first type of question for two designs: latin squares (See Chapter 2) and graph decompositions (See Chapter 3). Although many of the questions asked in both chapters are design theoretic, techniques from graph theory are used in their proofs and may be of interest to graph theorists.

Because the topics are sufficiently disjoint, introductions and definitions are included in each chapter and each of the chapters is self-contained. Some of the results from the two chapters can be found in published/accepted peer-reviewed articles [3, 11].

Chapter 2

Completing Partial Latin Squares with Prescribed Diagonals

2.1 History and Definitions

Historically, a *(partial) latin square* L of order n is an $n \times n$ array in which each cell contains (at most) one symbol in $S(n) = \{1, 2, \ldots, n\}$ and each of the symbols in $S(n)$ occurs (at most) once in each row and (at most) once in each column. Let $L(i, j)$ denote the symbol in cell (i, j) of L, and let $N_L(i)$ (or simply $N(i)$ if L is clear) be the number of cells that contain symbol i in L. A *(partial) incomplete latin square* of order n (also referred to as a *(partial) latin array* of order n) on the symbols in $S(t)$ is an $n \times n$ array in which each cell contains (at most) one symbol in $S(t)$ and each of the symbols in $S(t)$ occurs at most once in each row and at most once in each column. A partial or incomplete latin square L of order n is said to be *embedded* in the latin square T of order t if for each cell (i, j) of L that contains a symbol, $L(i, j) = T(i, j)$. The cells (i, i) for $n + 1 \leq i \leq t$ are said to be *the diagonal of* T *outside* L. A latin square of order n is said to be *idempotent* if $L(i, i) = i$ for $1 \leq i \leq n$, and is said to be *symmetric* if $L(i, j) = L(j, i)$ for $1 \leq i \leq j \leq n$.

There is a rich history of papers that consider the embedding of partial and incomplete latin squares; the following is a sample of such results. The classic result of Ryser [29] shows that an incomplete latin square L of order n on the symbols in $S(t)$ can be embedded in a latin square of order t if and only if $N_L(i) \geq 2n - t$ for $1 \leq i \leq t$. This condition is known as the *Ryser condition*. Evans [14] obtained a related result for partial latin squares, proving that any partial latin square of order n can be embedded in a latin square of order t for any $t \geq 2n$. This result is best possible in that there are partial latin squares of order n that cannot be embedded in a latin square of order t if $t < 2n$. Cruse [13] then found necessary and sufficient conditions for a partial latin square of order n to be embedded in a symmetric latin square of order t, and also to be embedded in an idempotent symmetric latin square of order t, where in both cases $t > n$ is arbitrary. It turns out that embedding partial and incomplete latin squares in an idempotent latin square is a very difficult problem. The Ryser conditions can naturally be extended to provide a necessary condition for an incomplete idempotent latin square L of order n with symbol set $S(t)$ to be embedded in an idempotent latin square of order t with symbol set $S(t)$, namely that $N_L(i) \geq 2n - t + f(i)$ for $1 \leq i \leq t$, where $f(i) = 0$ for $1 \leq i \leq n$ and $f(i) = 1$ for $n + 1 \leq i \leq t$. It was shown by Andersen et al. [1, 6] that for all $t < 2n$ these Ryser-type conditions are not sufficient: there exists an incomplete idempotent latin square of order n satisfying the Ryser-type conditions which cannot be embedded in an idempotent latin square of order t. In some cases, just swapping the placement of symbols in two cells results in one which does have an idempotent embedding. So, for the first time in these sorts of embedding problems, the arrangement of the symbols in L can determine its embeddability, thus making the idempotent setting quite special. The case where $t \geq 2n$ was finally settled after various results reduced the bound on t. Treash [30] showed that a finite embedding of a partial idempotent latin square was always possible, Lindner [23] reduced the bound to around $6n$, conjecturing that $2n + 1$ was the right lower bound (the Ryser-type conditions come into play when $t \leq 2n$), Andersen [2] further reduced it to $t \geq 4n$ and $t \neq 4n + 1$, and finally Andersen et al. [5] settled the Lindner conjecture which states that any partial idempotent latin square can be embedded in an idempotent latin square of order t, for any $t \geq 2n + 1$. The idempotent embedding for incomplete idempotent latin squares was then settled for all $t \geq 2n$ by Rodger [27].

A natural generalization to embedding an incomplete latin square L of order n with symbol set $S(t)$ into an idempotent latin square T of order t is to more generally prescribe what is to occur on the diagonal: suppose it is required that for $1 \leq i \leq t$ symbol i should occur $f(i)$ times in the diagonal cells of T outside L. Then the Ryser-type conditions, in this case $N_L(i) \geq 2n - t + f(i)$, are again necessary, and if $t \geq 2n + 1$ then Rodger [28] proved they, along with two other necessary conditions, are also sufficient. It is the case that if $f(\alpha) = 1$

for some symbol, α, then Andersen et al. [6] again showed that when $t < 2n$ the arrangement of symbols in L can determine if L can be embedded in T with the given prescribed diagonal of T outside L. In Section 2.2, we strengthen that result showing that if $t < 2n$, then for every function f such that $f(\alpha) = 1$ for some symbol, α, there exists an incomplete latin square satisfying the Ryser-type conditions that is non-embeddable. Rodger conjectured more than 30 years ago that if $f(i) \neq 1$ for $1 \leq i \leq t$ then, even when $t \leq 2n$, the Ryser-type conditions are sufficient to guarantee an embedding exists. It is this conjecture that we prove in Section 2.3. Finally, in Section 2.4, we discuss future research questions.

2.2 Non-embeddable Incomplete Latin Squares

Throughout Section 2.2, we will assume $t \geq n > 0$, L is an incomplete latin square of order n on the symbols in $S(t)$, and $f : S(t) \mapsto \mathbb{N}$ such that $\sum_{i=1}^{t} f(i) = t - n$ unless otherwise specified. We show that there always exists an incomplete latin square L of order n that satisfies the Ryser-type conditions and cannot be embedded into a latin square of order t if at least one symbol is prescribed to appear once on the diagonal outside of L and $t \leq 2n$.

The necessity of the Ryser condition, $N_L(i) \geq 2n - t + f(i)$, is well-known. We define a symbol i to be *marginal* if $N_L(i) = 2n - t + f(i)$. Define a symbol i to be *nearly marginal* if $N_L(i) = 2n - t + f(i) + 1$.

Observation 2.1. *To extend an incomplete latin square L of order n on the symbols in $S(t)$ with each symbol satisfying Ryser's condition to an incomplete latin square L' of order $n + 1$ on the symbols in $S(t)$ with each symbol satisfying Ryser's condition, each marginal symbol, i, must appear in both the added row and column (appearing in cell $(n + 1, n + 1)$ meets this condition if $f(i) \geq 1$). Similarly, a nearly marginal symbol, i, must appear at least once in the added row and column (appearing in cell $(n + 1, n + 1)$ meets this condition if $f(i) \geq 1$).*

Proof. Suppose symbol k is in cell $(n+1, n+1)$ of L'. Let $f'(i) = f(i)$ for $i \neq k$ and $1 \leq i \leq t$. Let $f'(k) = f(k)-1$. So for L' to meet Ryser's Condition, we need $N_{L'}(i) \geq 2(n+1)-t+f'(i)$ for $1 \leq i \leq t$. The symbol k satisfies Ryser's condition because $N_{L'}(k) = N_L(k) + 1 \geq 2n - t + f(k) + 1 = 2n - t + (f'(k) + 1) + 1 = 2(n + 1) - t + f'(k)$. For $1 \leq i \leq t$ and $i \neq k$, we need $N_{L'}(i) \geq 2(n+1) - t + f'(i) = 2n - t + f(i) + 2$. So, marginal symbols must appear both in the added row and column and nearly marginal symbols must appear at least once in the added row or column. $\square$

Given an incomplete latin square L of order n on the symbols in $S(t)$, define the graphs $G_\rho(L)$ and $G_c(L)$ as follows. Form a bipartite graph $G_\rho(L)$ with bipartition $R = \{\rho_1, \rho_2, \ldots, \rho_n\}$ and $S = \{\sigma_1, \sigma_2, \ldots, \sigma_t\}$ of the vertex set as follows. For $1 \leq j \leq n$ and $1 \leq i \leq t$, join ρ_j to σ_i if and only if symbol i is missing from row j of L. Similarly, form a bipartite graph $G_c(L)$ with bipartition $C = \{c_1, c_2, \ldots, c_n\}$ and $S' = \{\sigma'_1, \sigma'_2, \ldots, \sigma'_t\}$ of the vertex set as follows. For $1 \leq j \leq n$ and $1 \leq i \leq t$, join c_j to σ'_i if and only if symbol i is missing from column j of L. Andersen [1] defined the following family of incomplete latin squares and proved Theorem 2.3, showing that each member of this family is non-embeddable.

Definition 2.2. *Let $t \geq n > 0$. Let L be an incomplete latin square of order n on the symbols in $S(t)$. Let $f : S(t) \mapsto \mathbb{N}$ such that $\sum_{i=1}^{t} f(i) = t - n$. Define $\mathcal{F}(n, t, f)$ to be the family of incomplete latin squares such that $L \in \mathcal{F}(n, t, f)$ if and only if*

i) $N_L(i) \geq 2n - t + f(i)$ for $1 \leq i \leq t$ (Ryser's condition),

ii) $G_\rho(L)$ has a connected component, call it H, in which exactly one of the symbol vertices corresponds to a nearly marginal symbol, the remaining vertices all correspond to marginal symbols, and

iii) $G_c(L)$ has a connected component, call it H', and σ'_i is in H' and has degree less than $t - n$ if and only if σ_i is in H and has degree less than $t - n$. (So if σ'_i is in H' and σ_i is not in H, then $\deg_{G_c(L)}(\sigma'_i) = t - n$.)

Theorem 2.3. *[1] Let $t \geq n > 0$. Let L be an incomplete latin square of order n on the symbols in $S(t)$. Let $f : S(t) \mapsto \mathbb{N}$ such that $\sum_{i=1}^{t} f(i) = t - n$. If $L \in \mathcal{F}(n, t, f)$, then L cannot be embedded in a latin square T of order t on the same symbols in which each symbol i appears $f(i)$ times of the diagonal of T outside of L.*

We now show that if one symbol is prescribed to appear once on the diagonal then there aways exists one of these non-embeddable squares.

Theorem 2.4. *Let $4 \leq n+2 \leq t \leq 2n$. For all $f : S(t) \mapsto \mathbb{N}$ such that $\sum_{i=1}^{t} f(i) = t-n$ and $f(\alpha) = 1$ for some $1 \leq \alpha \leq t$, there exists a latin square $L \in \mathcal{F}(n,t,f)$.*

Proof. For convenience, suppose $f(i) = 0$ for $1 \leq i \leq n$, $\sum_{i=n+1}^{t-1} f(i) = t-n-1$, and $f(t) = 1$. We first construct a latin square L, then prove $L \in \mathcal{F}(n,t,f)$.

For $0 \leq j \leq n-1$, define $D_L(j)$ to be cells $(a, (a+j-1 \pmod n)+1)$ for $1 \leq a \leq n$ of L. Thus, $D_L(0)$ is the diagonal of L. We call $D_L(j)$ a generalized diagonal, and each generalized diagonal contains n cells with disjoint rows and columns. Define L as follows:

- Fill all cells in $D_L(0)$ with symbol t.
- Fill all cells in $D_L(j)$ with symbol j for $1 \leq j \leq t-n-2$
- Fill the remaining $D_L(j)$ for $t-n-1 \leq j \leq n-1$ with $2n-t+f(k)+1$ occurrences of a symbol k for some k such that $t-n-1 \leq k \leq t-1$ and $2n-t+f(i)$ occurrences of the symbol i for all $t-n-1 \leq i \leq t-1$ and $i \neq k$. There are exactly enough cells to do this because

$$\begin{aligned}
\textstyle\sum_{i=t-n-1}^{t-1}(2n-t+f(i)) &= (n+1)(2n-t)+\textstyle\sum_{i=t-n-1}^{t-1} f(i) \\
&= (n+1)(2n-t)+(t-n-1) \\
&= n(2n-t+1)-1.
\end{aligned}$$

  Also, as the following shows, it is possible to place these symbols in a way that satisfies the latin condition. Order the cells in these $D_L(j)$, $t-n-1 \leq j \leq n-1$, by row number within each generalized diagonal, and with cells in $D_L(j)$ occuring before cells in $D_L(j+1)$ for $t-n-1 \leq j \leq n-2$. Fill the cells one by one in this order, placing all occurences of one symbol before moving on to the next and ordering the symbols in non-increasing order according to how many cells they are to be placed in.

See Example 2.5. Thus, every symbol satisfies Ryser's condition (and therefore Definition 2.2 (i)). Symbol k is nearly marginal and all other symbols are marginal or appear n times in L.

Let H be the component of $G_\rho(L)$ containing σ_k. For $1 \leq i \leq t$ and $i \neq k$, vertex $\sigma_i \in S$ has degree 0 or corresponds to a marginal symbol. Thus, the symbol vertices in H correspond to one nearly marginal symbol and the remaining all correspond to marginal symbols. So, H satisfies Definition 2.2 (ii).

Let H' be the component of $G_c(L)$ containing σ'_k. If $\deg_{G_\rho(L)}(\sigma_k) = \deg_{G_c(L)}(\sigma'_k) = 0$, then H' satisfies Definition 2.2 (iii). So, assume $\deg_{G_\rho(L)}(\sigma_k) = \deg_{G_c(L)}(\sigma'_k) > 0$.

Let $H_R \subseteq R$ and $H_S \subseteq S$ be the vertices in the connected component H from R and S respectively. Because $\deg_{G_\rho(L)}(\sigma_k) > 0$, there is at least one vertex in H_R. All vertices in H_R have degree $t-n$. The vertices in H_S correspond to 1 nearly marginal symbol and the rest are marginal symbols. So, counting edges in H,

$$\begin{aligned}
|H_R|(t-n) &= \Sigma_{\rho_j \in H_R} \deg(\rho_j) \\
&= \Sigma_{\sigma_i \in H_S} \deg(\sigma_i) \\
&= \Sigma_{\sigma_i \in H_S} (n - (2n - t + f(i)) - 1 \\
&= \Sigma_{\sigma_i \in H_S} (t - n - f(i)) - 1 \\
&= |H_S|(t-n) - 1 - \Sigma_{\sigma_i \in H_S} f(i).
\end{aligned}$$

By definition of f, $0 \le \Sigma_{\sigma_i \in H_S} f(i) \le t-n$. Since $t-n$ must divide both sides, it follows that $\Sigma_{\sigma_i \in H_S} f(i) = t - n - 1$. Because $N_L(t) = n$, $\deg_{G_\rho(L)}(\sigma_t) = 0$. So, for $n+1 \le i \le t-1$ such that $f(i) > 0$, $\sigma_i \in H_S$. For symbols i such that $f(i) = 0$ and $i \ne k$, $\deg_{G_\rho(L)}(\sigma_i) = n - N_L(i) = n - n = 0$ or $\deg_{G_\rho(L)}(\sigma_i) = n - N_L(i) = n - (2n - t + f(i)) = n - (2n - t + 0) = t - n$.

Let $H'_C \subseteq C$ and $H'_{S'} \subseteq S'$ be the vertices from C and S' in H' respectively. For symbols i such that $n+1 \le i \le t-1$ and $f(i) > 0$, it can be proved in a similar manner that $\sigma'_i \in H_{S'}$. Because $N_L(t) = n$, $\deg_{G_c(L)}(\sigma'_t) = 0$. For symbols i such that $f(i) = 0$ and $i \ne k$, $\deg_{G_c(L)}(\sigma'_i) = n - N_L(i) = n - n = 0$ or $\deg_{G_c(L)}(\sigma'_i) = n - N_L(i) = n - (2n - t + f(i)) = n - (2n - t + 0) = t - n$. So, H' satisfies Definition 2.2 (iii). $\square$

The following example shows a nonembeddable incomplete latin square constructed using the method described in the previous proof.

Example 2.5. *Suppose $f(9) = 1$, $f(10) = 2$, $f(11) = 1$, and $f(i) = 0$ for $1 \le i \le 8$. Then, Figure 2.2 shows an example of an $L \in \mathcal{F}(7, 11, f)$ constructed as described in Theorem 2.4. Symbol 6 is serving as the symbol k described in the proof. Figure 2.2 shows the associated graphs $G_\rho(L)$ and $G_c(L)$.*

11	1	2	10	6	3	5
7	11	1	2	10	6	3
4	7	11	1	2	10	9
9	4	7	11	1	2	10
10	9	4	8	11	1	2
2	6	9	5	8	11	1
1	2	6	3	5	8	11

Figure 2.1: $L \in \mathcal{F}(7, 11, f)$

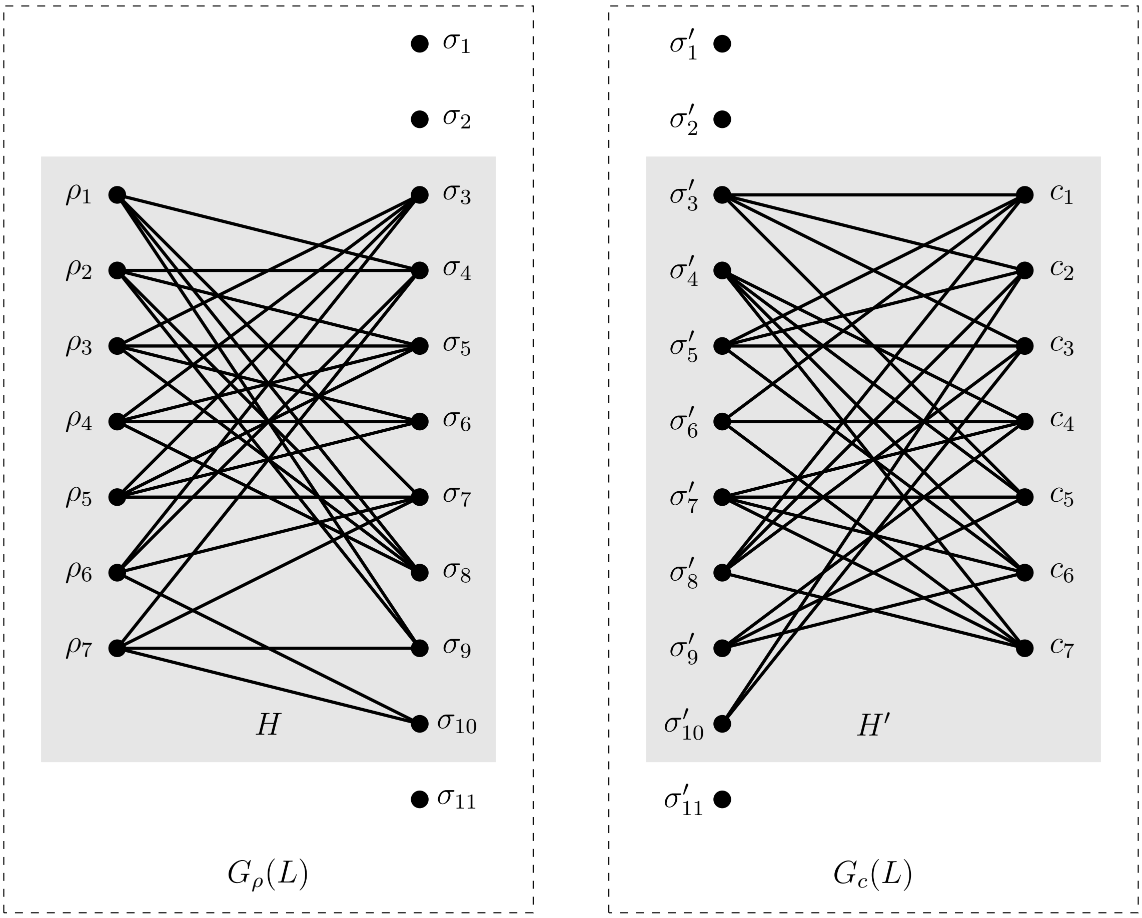

Figure 2.2: $G_\rho(L)$ and $G_c(L)$

2.3 Embeddable Incomplete Latin Squares

In this section, we prove a 30-year old conjecture by showing that any incomplete latin square satisfying the Ryser-type conditions can be embedded in a latin square with prescribed diagonal if no symbol is prescribed to appear exactly once on the diagonal. The result is joint work with Lars Døvling Andersen, Anthony J.W. Hilton, and Chris Rodger and is published [3].

2.3.1 Previous Results

Before proving the main result, Theorem 2.9, we note the following three results.

Andersen et al. [4] proved Theorem 2.6, which completely settles the embedding problem provided not all of the diagonal is prescribed.

Theorem 2.6 ([4]). *Let $t \geq n > 0$. Let L be an incomplete latin square of order n on the symbols in $S(t)$. Let $f : \{1, 2, \ldots, t\} \mapsto \mathbb{N}$ satisfy $\sum_{i=1}^{n} f(i) \leq t - n - 1$. Then L can be embedded in a latin square T of order t on the same symbols in which each symbol i appears at least $f(i)$ times on diagonal of T outside L if and only if $N_L(i) \geq 2n - t + f(i)$ for $1 \leq i \leq t$.*

The following classic theorem, proven by Ryser [29], will be used in Step 1 of the proof of Theorem 2.9.

Theorem 2.7 ([29]). *An incomplete latin square L of order n on the symbols in $S(t)$ can be embedded in a latin square of order t on the same symbols if and only if $N_L(i) \geq 2n - t$ for $1 \leq i \leq t$.*

A family $\mathcal{L}$ of sets is said to be a *laminar* set if $X, Y \in \mathcal{L}$ implies that $X \subseteq Y$, $Y \subseteq X$, or $X \cap Y = \emptyset$. Nash-Williams [26] proved the following result which will play a critical role in Step 3 of the proof of Theorem 2.9.

Theorem 2.8 ([26]). *If $\mathcal{L}_1$ and $\mathcal{L}_2$ are laminar sets of subsets of a finite set M, then for each integer $h > 0$ there exists $J \subseteq M$ such that*

$$\left\lfloor \frac{|Z|}{h} \right\rfloor \leq |J \cap Z| \leq \left\lceil \frac{|Z|}{h} \right\rceil$$

for every $Z \in \mathcal{L}_1 \cup \mathcal{L}_2$.

2.3.2 Main Result

We now proceed with the main proof. In the following proof, $f(i)$ will be modified in various ways. With this in mind, the incomplete latin square L of order n is said to be (f,t)-*satisfied* if $N_L(i) \geq 2n - t + f(i)$ for $1 \leq i \leq t$. We say a symbol i satisfies *Ryser's condition* if $N_L(i) \geq 2n - t + f(i)$.

Theorem 2.9. *Let $t \geq n > 0$. Let L be an incomplete latin square of order n on the symbols in $S(t)$. Let $f : S(t) \mapsto \mathbb{N}$ such that $\sum_{i=1}^{n} f(i) = t - n$ and $f(i) \neq 1$ for $1 \leq i \leq t$. Then L can be embedded in a latin square T of order t on the same symbols in which each symbol i appears $f(i)$ times on the diagonal of T outside L if and only if $N_L(i) \geq 2n - t + f(i)$ for $1 \leq i \leq t$.*

Proof. The necessity is well known, so assume that $N_L(i) \geq 2n - t + f(i)$ for $1 \leq i \leq t$.

Suppose there exists a symbol α for which $f(\alpha) \geq 3$. Let $f'(\alpha) = f(\alpha) - 1$ and $f'(i) = f(i)$ for $1 \leq i \leq t$, $i \neq \alpha$. Thus $\sum_{i=1}^{t} f'(i) = t - n - 1$. Then by Theorem 2.6, L can be embedded in a latin square T' of order t in which for $1 \leq i \leq t$, symbol i occurs at least $f'(i)$ times on the diagonal of T' outside L. By a permutation of rows and columns if needed, assume $T'(n+1, n+1) = \alpha$. Define the incomplete latin square L' of order $n+1$ by $L'(a,b) = T'(a,b)$ for $1 \leq a, b \leq n+1$. We now show that L' is (f',t)-satisfied. Because $T'(n+1, n+1) = \alpha$,

$$\begin{aligned} N_{L'}(\alpha) &= N_L(\alpha) + 1 \\ &\geq 2n - t + f(\alpha) + 1 \\ &= 2(n+1) - t + f'(\alpha). \end{aligned}$$

Also, since L' is embedded in T', by the necessary condition in Theorem 2.6, $N_{L'}(i) \geq 2(n+1) - t + f'(i)$ for $1 \leq i \leq t$, $i \neq \alpha$. Thus L' is an incomplete latin square of order $n+1$ satisfying the conditions of the theorem. Therefore, by repeating this process, we can assume that $f(i) \in \{0, 2\}$ for $1 \leq i \leq t$; so $t - n$ is even.

The remainder of the proof is completed in three steps. In each step, two rows and columns are added so that the resulting incomplete latin square satisfies the necessary condition after appropriately modifying f to allow for the symbol placed in both the added diagonal cells.

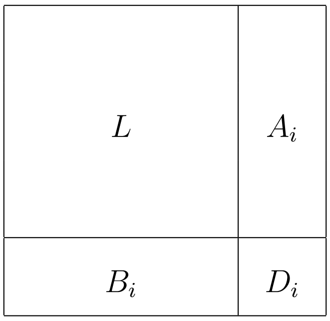

Figure 2.3: L_i

Step 1. Suppose $t - n = 2$. Then $f(\alpha) = 2$ for exactly one symbol α, and $f(i) = 0$ for all symbols $i \neq \alpha$. By assumption, $N_L(\alpha) \geq 2n - t + f(\alpha) = 2n - (n+2) + 2 = n$. Because L is of order n, $N_L(\alpha) = n$. Use Theorem 2.7 to embed L in a latin square T of order t. Because $N_L(\alpha) = n$, symbol α must appear twice in the 2×2 square formed with rows and columns $t-1$ and t of T. If α is on the diagonal, we are done. If not, then permute columns $t-1$ and t to obtain the required embedding. Thus we can assume $t - n \geq 4$.

Step 2. Suppose $t - n \geq 8$. Let $s = (t-n)/2$. By renaming symbols, we can assume that $f(i) = 2$ for $1 \leq i \leq s$ and $f(i) = 0$ for $s+1 \leq i \leq t$. We wish to extend L by 2 rows and 2 columns embedding L in a latin square of order $n+2$ that satisfies the conditions of the theorem. Define $f'(i) = 2$ for $1 \leq i \leq s-1$, $f'(s) = 1$, and $f'(i) = 0$ for $s+1 \leq i \leq t$. So, $\sum_{i=1}^{t} f'(i) = t - n - 1$. Thus by Theorem 2.6 and a permutation of rows and columns if needed, we can embed L in a latin square T' of order t with $T'(n+2i-1, n+2i-1) = i = T'(n+2i, n+2i)$ for $1 \leq i \leq s-1$ and $T'(n+2s-1, n+2s-1) = s$. (So at this stage we do not know what symbol appears in cell (t,t).) Define the sets of cells $\mathcal{A}_i$, $\mathcal{B}_i$ and $\mathcal{D}_i$ for $1 \leq i \leq s-1$ as follows:

$$\mathcal{A}_i := \{(a,b) : 1 \leq a \leq n, n+2i-1 \leq b \leq n+2i\},$$

$$\mathcal{B}_i := \{(a,b) : n+2i-1 \leq a \leq n+2i, 1 \leq b \leq n\}, \text{ and}$$

$$\mathcal{D}_i := \{(a,b) : n+2i-1 \leq a, b \leq n+2i\}.$$

Let A_i, B_i, and D_i be the $n \times 2$, $2 \times n$, and 2×2 latin subrectangles of T' formed by the cells in $\mathcal{A}_i$, $\mathcal{B}_i$, and $\mathcal{D}_i$ respectively. Similarly, let $A_i \cup B_i \cup D_i$ be the array formed by the cells in $\mathcal{A}_i$, $\mathcal{B}_i$, and $\mathcal{D}_i$. For $1 \leq i \leq s-1$, let L_i be the incomplete latin square of order $n+2$ depicted in Figure 2.3. We now have $s-1$ candidates for extending L by two rows and two columns, namely $L_1, \ldots, L_{s-1}$. We now show that at least one of them must satisfy the necessary conditions of the theorem. (It is only symbol s that is potentially problematic because $f'(s) \neq f(s)$. However, we show for at least one value of i, $1 \leq i \leq s-1$, s appears the necessary number of times in $A_i \cup B_i \cup D_i$, so L_i meets the necessary conditions of the theorem.)

Suppose $1 \leq i \leq s-1$. Permute the rows and columns of T' to produce a latin square T_i such that L_i is embedded in T_i and for $1 \leq j \leq t$, $j \neq i$, symbol j appears in at least $f'(j)$ diagonal cells of T_i outside L_i. Define $f_i(j) = f(j)$ for $1 \leq j \leq t$, $j \neq i$, and define $f_i(i) = f(i) - 2 = 0$. Because i appears 2 more times on the diagonal of L_i than it did in L,

$$\begin{aligned} N_{L_i}(i) &= N_L(i) + 2 \\ &\geq 2n - t + f(i) + 2 \\ &= 2n - t + (f_i(i) + 2) + 2 \\ &= 2(n+2) - t + f_i(i). \end{aligned}$$

Since L_i is embedded in T_i, by the necessity of Theorem 2.6, for $1 \leq j \leq t, j \notin \{i, s\}$,

$$\begin{aligned} N_{L_i}(j) &\geq 2(n+2) - t + f'(j) \\ &= 2(n+2) - t + f(j) \\ &= 2(n+2) - t + f_i(j). \end{aligned}$$

Also, by the necessity of Theorem 2.6,

$$\begin{aligned} N_{L_i}(s) &\geq 2(n+2) - t + f'(s) \\ &= 2(n+2) - t + (f(s) - 1) \\ &= 2(n+2) - t + f_i(s) - 1. \end{aligned}$$

We claim that for some i, $1 \leq i \leq s-1$, s satisfies Ryser's condition in L_i, so in actuality $N_{L_i}(s) \geq 2(n+2) - t + f_i(s)$. Assume otherwise; so for all i, $1 \leq i \leq s-1$, assume that $N_{L_i}(s) = 2(n+2) - t + f_i(s) - 1 = 2(n+2) - t + 1$. But then,

$$\begin{aligned}\sum_{i=1}^{s-1} N_{L_i}(s) &= (s-1)(2(n+2) - t + 1) \\ &= (s-1)(2n - t + 5).\end{aligned}$$

Symbol s appears n times in the first n rows of T_i (by the definition of a latin square), but does not appear in the $(t-1)^{th}$ column of the first n rows because symbol s appears on the diagonal in that column. Symbol s could possibly appear in the t^{th} column of the first n rows. Thus $N_L(s) + \sum_{i=1}^{s-1} N_{A_i}(s) \geq n-1$. Similarly, $N_L(s) + \sum_{i=1}^{s-1} N_{B_i}(s) \geq n-1$. Therefore,

$$\begin{aligned}\sum_{i=1}^{s-1} N_{L_i}(s) &= \sum_{i=1}^{s-1} (N_L(s) + N_{A_i}(s) + N_{B_i}(s) + N_{D_i}(s)) \\ &\geq (s-3)N_L(s) + (n-1) + (n-1) + \sum_{i=1}^{s-1} N_{D_i}(s) \\ &\geq (s-3)N_L(s) + (n-1) + (n-1),\end{aligned}$$

implying

$$\begin{aligned}(s-3)N_L(s) &\leq (s-1)(2n-t+5) - 2n + 2 \\ &= (s-3)(2n-t+5) + 4n - 2t + 10 - 2n + 2 \\ &= (s-3)(2n-t+5) - 4s + 12 \\ &= (s-3)(2n-t+1).\end{aligned}$$

So, because $s \geq 4$, $N_L(s) \leq (2n-t+1)$, contradicting our original assumption. Therefore, for some value of i, $1 \leq i \leq s-1$, say $i = \alpha$, $N_{L_\alpha}(s) \geq 2(n+2) - t + f_\alpha(s)$. Also, as already stated, $N_{L_\alpha}(j) \geq 2(n+2) - t + f_\alpha(j)$ for $1 \leq j \leq t, j \neq s$. Thus L_α is an incomplete latin square of order $n+2$ that is (f_α, t)-satisfied and thus satisfies the conditions of the theorem. By repeating this process, we may now assume $t - n \leq 6$.

Step 3. Suppose $t - n \in \{4, 6\}$. Similar to what was defined in Section 2.2, form a bipartite multigraph G_c^* with bipartition $C = \{c_1, c_2, \ldots, c_n, c^*\}$ and $S = \{\sigma_1, \sigma_2, \ldots, \sigma_t\}$ of the vertex set as follows. For $1 \leq i \leq n$ and $1 \leq j \leq t$, join c_i to σ_j if and only if symbol j is missing from column i of L and join c^* to σ_j with $f(j)$ edges. Similarly, form a bipartite multigraph G_ρ^* with bipartition $R = \{\rho_1, \rho_2, \ldots, \rho_n, \rho^*\}$ and $S' = \{\sigma_1', \sigma_2', \ldots, \sigma_t'\}$ of the vertex set as follows. For $1 \leq i \leq n$ and $1 \leq j \leq t$, join ρ_i to σ_j' if and only if symbol j is missing from row i of L, and join ρ^* to σ_j' with $f(j)$ edges. Because each column and row of L contains n symbols, for $1 \leq i \leq n$,

$$\deg_{G_c^*}(c_i) = \deg_{G_\rho^*}(\rho_i) = t - n. \tag{2.1}$$

Because $\sum_{i=1}^{n} f(i) = t - n$,

$$\deg_{G_c^*}(c^*) = \deg_{G_\rho^*}(\rho^*) = t - n. \tag{2.2}$$

For $1 \leq j \leq t$, symbol j is missing from $n - N_L(j)$ rows of L and $n - N_L(j)$ columns of L, so

$$\deg_{G_c^*}(\sigma_j) = \deg_{G_\rho^*}(\sigma_j') = n - N_L(j) + f(j). \tag{2.3}$$

For $1 \leq j \leq t$, let $z(j) = N_L(j) - (2n - t + f(j))$. So $0 \leq z(j) \leq n - (2n - t + f(j)) = t - n - f(j)$. Thus, by (2.3),

$$\begin{aligned} \deg_{G_c^*}(\sigma_j) = \deg_{G_{\rho^*}}(\sigma_j') &= n - N_L(j) + f(j) \\ &= n - (2n - t + f(j) + z(j)) + f(j) \\ &= t - n - z(j) \\ &\leq t - n. \end{aligned} \tag{2.4}$$

So, $\Delta(G_{c^*}) = \Delta(G_{\rho^*}) = t - n$ and $z(j)$ measures how far σ_j and σ_j' are from this maximum degree.

Define a laminar set $\mathcal{L}_1$ of subsets of $E(G_c^*) \cup E(G_\rho^*)$ as follows. For $1 \leq i \leq n$, let $C_i \in \mathcal{L}_1$ be the set of edges incident to c_i. Let $C^* \in \mathcal{L}_1$ be the set of edges incident to c^*. For $1 \leq j \leq t$ such that $f(j) > 0$, let $C_j^* \in \mathcal{L}_1$ be the subset of C^* given by the two element set of the pair of edges joining c^* and σ_j. Similarly, for $1 \leq i \leq n$, let $R_i \in \mathcal{L}_1$ be the set of edges incident to ρ_i. Let $R^* \in \mathcal{L}_1$ be the set of edges incident to ρ^*. For $1 \leq j \leq t$ such that $f(j) > 0$, let $R_j^* \in \mathcal{L}_1$ be the subset of R^* given by the two element set of the pair of edges joining ρ^* and σ'_j. Define a second laminar set $\mathcal{L}_2$ of subsets of $E(G_c^*) \cup E(G_\rho^*)$ as follows. For $1 \leq j \leq t$, let $S_j \in \mathcal{L}_2$ be the set of edges incident to σ_j, $S'_j \in \mathcal{L}_2$ be the set of edges incident to σ'_j, and $\Sigma_j \in \mathcal{L}_2$ be the set of all edges incident to either σ_j or σ'_j. By Theorem 2.8, there exists a set $J \subseteq (E(G_c^*) \cup E(G_\rho^*))$ for which

$$\left\lfloor \frac{|Z|}{(t-n)/2} \right\rfloor \leq |J \cap Z| \leq \left\lceil \frac{|Z|}{(t-n)/2} \right\rceil$$

for every $Z \in (\mathcal{L}_1 \cup \mathcal{L}_2)$.

Let G_J be the graph induced by the edges of G_c^* and G_ρ^* in J. Later, a modified version of G_J will be colored with 2 colors and be used to fill rows and columns $n+1$ and $n+2$ to embed L in an incomplete latin square of order $n+2$. But first we explore G_J to see what modifications are needed.

By (2.1), for $1 \leq i \leq n$, $\deg_{G_c^*}(c_i) = \deg_{G_\rho^*}(\rho_i) = t - n$; so, because $C_i, R_i \in \mathcal{L}_1$, $\deg_{G_J}(c_i) = \deg_{G_J}(\rho_i) = 2$. By (2.2), $\deg_{G_c^*}(c^*) = \deg_{G_\rho^*}(\rho^*) = t - n$; so, because $C^*, R^* \in \mathcal{L}_1$, $\deg_{G_J}(c^*) = \deg_{G_J}(\rho^*) = 2$. By (2.4), $\deg_{G_c^*}(\sigma_j) = \deg_{G_\rho^*}(\sigma'_j) = t-n-z(j)$; so, because $S_j, S'_j \in \mathcal{L}_2$, $\deg_{G_J}(\sigma_j) \leq \lceil 2 - \frac{2z(j)}{t-n} \rceil \leq 2$ and $\deg_{G_J}(\sigma'_j) \leq \lceil 2 - \frac{2z(j)}{t-n} \rceil \leq 2$. Also, because $\Sigma_j \in \mathcal{L}_2$, $\deg_{G_J}(\sigma_j) + \deg_{G_J}(\sigma'_j) \geq \lfloor 4 - \frac{4z(j)}{t-n} \rfloor \geq 4 - z(j)$. So, for $1 \leq j \leq t$,

$$\begin{aligned} N_L(j) + \deg_{G_J}(\sigma_j) + \deg_{G_J}(\sigma'_j) &\geq (2n - t + f(j) + z(j)) + (4 - z(j)) \\ &= 2(n+2) - t + f(j). \end{aligned} \tag{2.5}$$

Recall, $\deg_{G_J}(c^*) = 2$. Because $C_j^* \in \mathcal{L}_1$ at most one edge $\{c^*, \sigma_j\} \in C_j^*$ is in J. So, the two edges in J incident to c^* are incident to two different vertices in S. Similarly, there are exactly two edges in J incident to ρ^*, each of which is incident to two different vertices in

S'. Because c^* and ρ^* are incident to σ_j and σ'_j respectively for the same two (if $t-n=4$) or three (if $t-n=6$) values of j, there must exist an α such that $1 \le \alpha \le t$, $\{c^*, \sigma_\alpha\} \in J$ and $\{\rho^*, \sigma'_\alpha\} \in J$.

In what follows we construct another set of edges J' through a modest modification of J so both edges in C^*_α and both edges in R^*_α will be in J'. The graph G_J is a bipartite graph with maximum degree 2. Thus, the edges of G_J can be properly colored with 2 colors, say 1 and 2. Consider the graphs $G^*_c - J$ and $G^*_\rho - J$. They are bipartite graphs with maximum degree $t-n-2$. Thus, the edges of $G^*_c - J$ and $G^*_\rho - J$ can be properly colored with $t-n-2$ colors, say $3, \ldots, t-n$. These two edge-colorings naturally induce a proper $(t-n)$-edge-coloring of $G^*_c \cup G^*_\rho$, $X : E(G^*_c \cup G^*_\rho) \to \{1, 2, \ldots, t-n\}$, in which all edges in J are colored 1 or 2.

In what follows we construct an edge-coloring $X' : E(G^*_c \cup G^*_\rho) \to \{1, 2, \ldots, t-n\}$ by interchanging colors on two 2-colored trails, T_1 and T_2, in X. In X' the edges in C^*_α will be colored 1 and 2 and the edges in R^*_α will be colored 1 and 2. Suppose the edges in C^*_α are colored 1 and 3 by X. Consider the maximal trail, T_1, containing the edge $\{c^*, \sigma_\alpha\}$ colored 3, in which the edges are alternately colored 2 and 3 by X. Because the edge-coloring is proper, T_1 is either a cycle or a path. Interchange the colors on T_1 and let this new edge-coloring be X' on the edges in G^*_c. The edges in C^*_α are now colored 1 and 2 by X'. If T_1 is a cycle, interchanging colors did not impact the number of edges of each color incident to each vertex. Suppose T_1 is a path. Interchanging colors did not impact the number of edges of each color incident to each vertex in the interior of T_1, but did impact the endpoints. For each $c \in C$, by (2.1) and (2.2), $\deg(c) = t-n$. So there is exactly one edge colored 2 and one edge colored 3 by X incident to vertex c. Thus c is not an endpoint of T_1, so the endpoints of T_1 must be in S. Because G^*_c is bipartite and both ends of T_1 are in S, exactly one of the ends was incident to an edge colored 2 by X. This end cannot be σ_α because σ_α was incident to an edge colored 3 by X. So one end of T_1 is a vertex $\sigma_u \in S \setminus \{\sigma_\alpha\}$ that now does not have an edge colored 2 by X' incident to it. The other end of T_1 was incident to an edge colored 3 by X. So this vertex now is incident to an edge colored 2 by X'. Similarly, we can use a trail T_2 to modify the proper edge-coloring, X, of G^*_ρ and define X' on the edges of G^*_ρ so the edges in R^*_α are now colored 1 and 2 in X'. After recoloring, at most one vertex in $S' \setminus \{\sigma'_\alpha\}$ has lost an edge colored 2 incident to it in X'. If such a vertex exists, name it σ'_v. All other vertices in $G_c^* \cup G_\rho^*$ have an equal or greater number of edges colored 2 incident to them. Thus, we have the revised edge-coloring $X' : E(G_c^* \cup G_\rho^*) \to \{1, 2, \ldots, t-n\}$. Define J' to be the set of edges colored 1 and 2 by X', and let $G_{J'}$ be the graph induced by the edges in J'.

It is important to note a property that will be used later in the proof if σ_u and/or σ'_v have been defined. In X', σ_u and σ'_v do not have an edge colored 2 incident to them, so $\deg_{G_c^*}(\sigma_u) = \deg_{G_\rho^*}(\sigma'_u) < t - n$ and $\deg_{G_\rho^*}(\sigma'_v) = \deg_{G_c^*}(\sigma_v) < t - n$. So, for each $j \in \{u, v\}$, by (2.3),

$$N_L(j) \geq 2n - t + f(j) + 1. \tag{2.6}$$

To finally arrive at the desired edge-coloring, $X'' : E(G_c^* \cup G_\rho^*) \to \{1, 2, \ldots, t-n\}$, X' is to be modified in the situation where $\deg_{G_{J'}}(\sigma_u) = \deg_{G_{J'}}(\sigma_v) = 1$ and/or $\deg_{G_{J'}}(\sigma'_u) = \deg_{G_{J'}}(\sigma'_v) = 1$ (as in Case 3 below). To do this we construct the edge-coloring X'' by interchanging colors on up to two trails, T_3 and T_4, whose edges are colored 1 and 2 in X' as follows.

We first define X'' on $E(G_c^*)$. Suppose $\deg_{G_{J'}}(\sigma_u) = \deg_{G_{J'}}(\sigma_v) = 1$. Let e_u and e_v be the edges incident to σ_u and σ_v in $G_{J'}$ respectively. In X'', we make sure that one of e_u and e_v is colored 1 and the other is colored 2. If $X'(e_u) \neq X'(e_v)$, then we already have the desired property, so define $X''(e) = X'(e)$ for all $e \in E(G_c^*)$. Otherwise, $X'(e_u) = X'(e_v)$. Take a maximal trail, T_3, in $G_{J'}$ that begins with e_u. Because $\Delta(G_{J'}) = 2$, the trail T_3 is necessarily a path. For each $c \in C$, $\deg_{G_{J'}}(c) = 2$. Thus c is not an endpoint of T_3, so the endpoint of T_3 must be in S. The path T_3 cannot end with e_v because $X'(e_u) = X'(e_v)$ and $G_{J'}$ is a bipartite graph. So, because $\deg_{G_{J'}}(\sigma_v) = 1$, T_3 does not include e_v. Interchange colors along T_3. All interior vertices of T_3 still have exactly one incident edge colored 1 and exactly one incident edge colored 2. The endpoints of T_3 now have the opposite color incident to them. Define X'' on the edges in G_c^* to be this new edge-coloring. Thus, in any case, if e_u and e_v exists, we can assume

$$X''(e_u) \neq X''(e_v). \tag{2.7}$$

For convenience, if $\deg_{G_{J'}}(\sigma_u) \neq 1$ or $\deg_{G_{J'}}(\sigma_v) \neq 1$, so we are not in the above case, then define $X''(e) = X'(e)$ for all $e \in E(G_c^*)$.

Similarly, we can define X'' on the edges in G_ρ^* by interchanging colors on a trail T_4 in X' if needed. So, if $\deg_{G_{J'}}(\sigma'_u) = 1$ and $\deg_{G_{J'}}(\sigma'_v) = 1$ and we let e'_u and e'_v be the edges incident to σ'_u and σ'_v in $G_{J'}$ respectively, then

$$X''(e'_u) \neq X''(e'_v). \tag{2.8}$$

For convenience, if $\deg_{G_{J'}}(\sigma'_u) \neq 1$ or $\deg_{G_{J'}}(\sigma'_v) \neq 1$, so we are not in the above case, then define $X''(e) = X'(e)$ for all $e \in E(G_\rho^*)$.

Thus, we have the revised edge-coloring $X'' : E(G_c^* \cup G_\rho^*) \to \{1, 2, \ldots, t - n\}$. Define J'' to be the set of edges colored 1 and 2 by X'', and let $G_{J''}$ be the graph induced by the edges in J''.

We will use $G_{J''}$ to fill in rows and columns $n+1$ and $n+2$ and extend L to an incomplete latin square of order $n+2$. For $1 \leq i \leq t$, $\deg_{G_{J''}}(c_i) = \deg_{G_{J''}}(\rho_i) = 2$. For $1 \leq j \leq t$, $\deg_{G_{J''}}(\sigma_j) \leq 2$ and $\deg_{G_{J''}}(\sigma'_j) \leq 2$. Also, for $1 \leq j \leq t$, by (2.5),

$$\begin{aligned} &N_L(j) + \deg_{G_{J''}}(\sigma_j) + \deg_{G_{J''}}(\sigma'_j) \\ &\qquad \geq N_L(j) + \deg_{G_J}(\sigma_j) + \deg_{G_J}(\sigma'_j) - \epsilon_j \\ &\qquad \geq 2(n+2) - t + f(j) - \epsilon_j, \end{aligned} \tag{2.9}$$

where $\epsilon_j = 0$ if $j \notin \{u, v\}$, $\epsilon_j = 1$ if $j \in \{u, v\}$ and $u \neq v$, and $\epsilon_j = 2$ if $j = u = v$.

Form a partial incomplete latin square L_α of order $n+2$ by adding two new rows and columns to L using J'' as follows. Let $L_\alpha(a, b) = L(a, b)$ for $1 \leq a \leq n$ and $1 \leq b \leq n$. Define the sets of cells $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$ as follows:

$$\begin{aligned} &\mathcal{A} := \{(a, b) : 1 \leq a \leq n, n+1 \leq b \leq n+2\}, \\ &\mathcal{B} := \{(a, b) : n+1 \leq a \leq n+2, 1 \leq b \leq n\}, \text{ and} \\ &\mathcal{D} := \{(a, b) : n+1 \leq a, b \leq n+2\}. \end{aligned}$$

Let A, B, and D be the $n \times 2$, $2 \times n$, and 2×2 latin subrectangles of L_α formed by the cells in $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{D}$ respectively. Similarly, let $A \cup B$ be the array formed by the cells in $\mathcal{A}$ and $\mathcal{B}$. We now fill A and B using $G_{J''}$. For $1 \le k \le 2$ and $1 \le i \le n$, let $L_\alpha(n+k, i) = j$ if and only if $\{c_i, \sigma_j\}$ is colored k in $G_{J''}$. Similarly, for $1 \le k \le 2$ and $1 \le i \le n$, let $L_\alpha(i, n+k) = j$ if and only if $\{\rho_i, \sigma'_j\}$ is colored k in $G_{J''}$. Every cell in $A \cup B$ is filled because $\deg_{G_{J''}}(c_i) = \deg_{G_{J''}}(\rho_i) = 2$ for $1 \le i \le n$. So, for $1 \le j \le t$, by (2.9),

$$N_{L_\alpha}(j) \ge 2(n+2) - t + f(j) - \epsilon_j. \tag{2.10}$$

In a later modification of L_α we will place α in the two new diagonal cells, so define $f'(\alpha) = f(\alpha) - 2$ and $f'(j) = f(j)$ for $1 \le j \le t$ and $j \ne \alpha$. For $1 \le j \le t$ and $j \notin \{u, v, \alpha\}$, by (2.10),

$$N_{L_\alpha}(j) \ge 2(n+2) - t + f'(j). \tag{2.11}$$

Thus, L_α is a partial incomplete latin square of order $n+2$ with all cells except those in D filled and all symbols satisfying Ryser's condition except possibly α (which will satisfy Ryser's condition once placed twice on the diagonal in D) and possibly u and v if they exist. The aim is to construct L'_α through a modest modification of L_α to form a partial incomplete latin square of order $n+2$ on $S(t)$ which is (f', t)-satisfied.

By (2.10), $N_{L_\alpha}(u) \ge 2(n+2) - t + f'(u) - \epsilon_u$ and $N_{L_\alpha}(v) \ge 2(n+2) - t + f'(v) - \epsilon_v$. We will now modify L_α to form L'_α so that if $N_{L_\alpha}(j) < 2(n+2) - t + f'(j)$ for any $j \in \{u, v\}$, u and/or v will be placed in cells $(n+1, n+2)$ and/or $(n+2, n+1)$ of L'_α as needed to ensure that $N_{L'_\alpha}(u) \ge 2(n+2) - t + f'(u)$ and $N_{L'_\alpha}(v) \ge 2(n+2) - t + f'(v)$.

Let $j \in \{u, v\}$. If $N_{L_\alpha}(j) < 2(n+2) - t + f'(j)$, by (2.10), $N_{L_\alpha}(j) = 2n - t + f'(j) + 2$ or $N_{L_\alpha}(j) = 2n - t + f'(j) + 3$. These two cases of $N_{L_\alpha}(j)$ correspond to L'_α needing 2 or 1 more occurrence of j respectively. To reveal more about these potentially problematic cases, consider the following properties. Recall, by (2.6), $N_L(j) \ge 2n - t + f(j) + 1$.

(i) Suppose $N_{L_\alpha}(u) < 2(n+2) - t + f'(u)$ and $N_L(u) = 2n - t + f(u) + 1$. By (2.3), $\deg_{G_c^*}(\sigma_u) = n - N_L(u) + f(u) = t - n - 1$. In the edge-coloring X'', σ_u is missing exactly one of the colors 1 or 2 (because T_1 ended on this vertex). So, since σ_u is incident to an edge of every color in G_c^* except one, σ_u must be incident to the other color (1 or 2). Therefore, $\deg_{G_{J''}}(\sigma_u) = 1$, and so because u appears at most 2 times (because $N_{L_\alpha}(u) - N_L(u) \leq 2$) in $A \cup B$, $\deg_{G_{J''}}(\sigma'_u) \leq 1$.

(ii) Similarly, if $N_{L_\alpha}(v) < 2(n+2) - t + f'(v)$ and $N_L(v) = 2n - t + f(v) + 1$, then $\deg_{G_{J''}}(\sigma'_v) = 1$ and $\deg_{G_{J''}}(\sigma_v) \leq 1$.

(iii) If $N_L(u) = 2n - t + f(u) + 1$, $N_{L_\alpha}(u) < 2(n+2) - t + f'(u)$, and $u = v$, then by (i-ii), $\deg_{G_{J''}}(\sigma_u) = 1 = \deg_{G_{J''}}(\sigma'_u)$. So, $N_{L_\alpha}(u) = 2n - t + f'(u) + 3$.

(iv) If $j \in \{u, v\}$, $N_{L_\alpha}(j) < 2(n+2) - t + f'(j)$, and $2n - t + f(j) + 2 \leq N_L(j) \leq 2n - t + f(j) + 3$, then j appears at most 1 time (because $N_{L_\alpha}(u) - N_L(u) \leq 1$) in $A \cup B$.

If $N_{L_\alpha}(j) \geq 2(n+2) - t + f'(j)$ for $j \in \{u, v\}$, then let $L'_\alpha(a,b) = L_\alpha(a,b)$ for $1 \leq a, b \leq n+2$. Otherwise we will make use of (i-iv) to modify L_α to define L'_α and place u and/or v in cells $(n+1, n+2)$ and/or $(n+2, n+1)$ of L'_α as needed to ensure that, for $j \in \{u, v\}$, $N_{L'_\alpha}(j) \geq 2(n+2) - t + f'(j)$. The following three cases are considered. The first two cases consider if exactly one of u or v, say u, does not meet Ryser's condition. So, by (2.10), $N_{L_\alpha}(u) = 2(n+2) - t + f'(u) - 2$ or $N_{L_\alpha}(u) = 2(n+2) - t + f'(u) - 1$. The third case considers when both u and v do not meet Ryser's Condition, so, by (2.10), $N_{L_\alpha}(u) = 2(n+2) - t + f'(u) - 1$ and $N_{L_\alpha}(v) = 2(n+2) - t + f'(v) - 1$.

Case 1: Suppose $N_{L_\alpha}(u) = 2(n+2) - t + f'(u) - 2$. Thus, by (2.10), $u = v$. By (iii), $N_L(u) = 2n - t + f(u) + 2$. Then u does not appear in A nor in B. Define $L'_\alpha(a,b) = L_\alpha(a,b)$ for $(a,b) \in A \cup B$ and for $1 \leq a, b \leq n$. Also, define $L'_\alpha(n+1, n+2) = L'_\alpha(n+2, n+1) = u$. Thus $N_{L'_\alpha}(u) = N_{L_\alpha}(u) + 2 = 2(n+2) - t + f'(u)$.

Case 2: Suppose that for exactly one of u or v, say u, $N_{L_\alpha}(u) = 2(n+2) - t + f'(u) - 1$ and for the other, say v, $N_{L_\alpha}(v) \geq 2(n+2) - t + f'(v)$, $u = v$, or v does not exist. By (i,iii,iv), u is in at most one row of B, say $n+2$, and at most one column of A, say $n+1$ (permuting the columns and/or rows of A and/or B respectively if need be). Define $L'_\alpha(a,b) = L_\alpha(a,b)$ for $(a,b) \in A \cup B$ or $1 \leq a,b \leq n$. Also, define $L'_\alpha(n+1,n+2) = u$. Thus $N_{L'_\alpha}(u) = N_{L_\alpha}(u) + 1 = 2(n+2) - t + f'(u)$.

Case 3: Suppose $u \neq v$, $N_{L_\alpha}(u) = 2(n+2) - t + f'(u) - 1$, and $N_{L_\alpha}(v) = 2(n+2) - t + f'(v) - 1$. By (i-ii, iv), u and v each appear at most once in A and at most once in B. By (2.7) and (2.8), we can assume u and v appear in different rows of B and different columns of A. Thus, permuting rows and/or columns if necessary we can assume u does not appear in row $n+1$ nor in column $n+2$ of L_α and v does not appear in row $n+2$ nor in column $n+1$ of L_α. Define $L'_\alpha(a,b) = L_\alpha(a,b)$ for $(a,b) \in A \cup B$ or $1 \leq a,b \leq n$. Also, define $L'_\alpha(n+1,n+2) = u$ and $L'_\alpha(n+2,n+1) = v$. So $N_{L'_\alpha}(u) = N_{L_\alpha}(u) + 1 = 2(n+2) - t + f'(u)$ and $N_{L'_\alpha}(v) = N_{L_\alpha}(v) + 1 = 2(n+2) - t + f'(v)$.

Thus, in any case, we can place u and/or v in cells $(n+1,n+2)$ and/or $(n+2,n+1)$ if needed so that for $j \in \{u,v\}$,

$$N_{L'_\alpha}(j) \geq 2(n+2) - t + f'(j).$$

Define $L'_\alpha(n+1,n+1) = L'_\alpha(n+2,n+2) = \alpha$. So

$$N_{L_\alpha}(\alpha) = N_L(\alpha) + 2 \geq 2(n+2) - t + f'(j).$$

Also for $1 \leq j \leq t$ and $j \notin \{u,v,\alpha\}$, $N_{L'_\alpha}(j) = N_{L_\alpha}(j)$, so by (2.11),

$$\begin{aligned} N_{L'_\alpha}(j) &= N_{L_\alpha}(j) \\ &\geq 2(n+2) - t + f'(j). \end{aligned}$$

Thus, L'_α is a partial incomplete latin square of order $n+2$ with all symbols satisfying Ryser's condition and all cells filled except possibly cells $(n+1,n+2)$ and $(n+2,n+1)$.

We now define L''_α through a modest modification of L'_α to fill cells $(n+1,n+2)$ and/or $(n+2,n+1)$ if needed to form an (f',t)-satisfied incomplete latin square. Suppose cell $(n+1,n+2)$ of L'_α is empty. Form the bipartite graph B with bipartition $C' = \{c_i \mid 1 \leq i \leq$

$n+2\}$ and $S = \{\sigma_j \mid 1 \le j \le t\}$ of the vertex set as follows. For $1 \le i \le n+2$ and $1 \le j \le t$, join c_i to σ_j if and only if symbol j is missing from column i of L'_α or $L'_\alpha(n+1, i) = j$. For $c_i \in C' \setminus \{c_{n+1}\}$, $\deg_B(c_i) = t - n - 1$. Because j appears at most once in row $n+1$, for $\sigma_j \in S$,

$$\begin{aligned} \deg_B(\sigma_j) &\le n + 2 - (N_{L'_\alpha}(j) - 1) \\ &\le n + 2 - (2(n+2) - t + f'(j) - 1) \\ &= t - n - f'(j) - 1 \\ &\le t - n - 1. \end{aligned} \tag{2.12}$$

Define the matching M by letting $\{c_i, \sigma_j\} \in E(B)$ be in M if and only if $L'_\alpha(n+1, i) = j$. Because symbol α appears in cell $(n+1, n+1)$, $\{c_{n+1}, \sigma_\alpha\} \in M$. Let B' be the induced subgraph of B formed by removing vertices c_{n+1} and σ_α. We wish to find an M-augmenting path in B' starting at c_{n+2}. For a contradiction, suppose there does not exist an M-augmenting path in B' starting at c_{n+2}. Let W be the subgraph of B' induced by the set of vertices that can be reached by an M-alternating path starting at c_{n+2}. All maximal M-alternating paths starting at c_{n+2} end at an M-saturated vertex in $C' \setminus \{c_{n+1}\}$. So $V(W)$ contains say x vertices from $S \setminus \{\sigma_\alpha\}$ and $V(W)$ contains $x+1$ vertices from $C' \setminus \{c_{n+1}\}$, namely c_{n+2} and the M-neighbors of the x vertices from $S \setminus \{\sigma_\alpha\}$. Let $C'_W = C' \cap V(W)$ denote the set of these $x+1$ vertices. By the definition of W, every edge in B' incident to a vertex in C'_W must be an edge in W (which implies the equality in the relations below). Because $\deg_B(\sigma_\alpha) \le t - n - 1$ (by (2.12)) and σ_α is adjacent to c_{n+1} in B, at most $t - n - 2$ vertices in C'_W have degree $t - n - 2$ in B' and all other vertices in C'_W have degree $t - n - 1$ in B'. So,

$$\begin{aligned} (t - n - 1)x &\ge \sum_{\sigma_j \in V(W)} \deg_{B'}(\sigma_j) \\ &\ge e(W) \\ &= \sum_{c_i \in V(W)} \deg_{B'}(c_i) \\ &\ge (t - n - 1)(x + 1) - (t - n - 2). \end{aligned}$$

This is a contradiction. Thus, there exists an M-augmenting path in B' starting at c_{n+2}. Form the matching M' from M by interchanging edges in M with edges not in M along this path. Now replace row $n+1$ of L'_α to form L''_α using M' by letting $L''_\alpha(n+1, i) = j$ if and only if $\{c_i, \sigma_j\}$ is in M' for $1 \leq i \leq n+2$ and $1 \leq j \leq t$ and letting $L''_\alpha(a, b) = L'_\alpha(a, b)$ for $1 \leq a \leq n$ or $a = n+2$ and $1 \leq b \leq n+2$. Thus, L''_α contains the same symbols as L'_α with the addition of one more symbol in row $n+1$. Now, all cells in row $n+1$ are filled. Similarly, if cell $(n+2, n+1)$ is empty, we can modify L''_α using the same approach. So we can assume all cells of L''_α are filled and all symbols satisfy Ryser's condition. Thus, L''_α is an incompete latin square of order $n+2$ that is (f', t)-satisfied and thus satisfies the conditions of the theorem. $\square$

2.4 Future Directions

This section summarizes possible future directions. The first question highlights an interesting difference in the idempotent and generally prescribed diagonal problem.

Question 2.10. *Suppose $t = 2n$. Let L be an incomplete latin square of order n on the symbols in $\{1, 2, \ldots, t\}$. Let $f : \{1, 2, \ldots t\} \mapsto \mathbb{N}$ such that $\sum_{i=1}^{n} f(i) = t - n$. What are the necessary and sufficient conditions for L to be embedded in a latin square T of order t on the same symbols in which each symbol i appears $f(i)$ times on the diagonal of T outside L?*

When $t = 2n$, Rodger found the necessary and sufficient conditions for the embedding of an incomplete idempotent latin square of order n in an idempotent latin square of order t [27]. However, this question is still open for the generally prescribed diagonal problem. There are incomplete latin squares of order n and functions $f : \{1, 2, \ldots t\} \mapsto \mathbb{N}$ meeting the Ryser-type conditions described in Rodger's result for $t \geq 2n+1$ [28] that are not embeddable in a latin square of order t with each symbol i appearing $f(i)$ times on the diagonal of T outside of L. We suggest investigating this interesting case to try and find necessary and sufficient conditions (even when $f(i) = 1$ for some $1 \leq i \leq t$).

The next question suggests looking at a higher dimension. Although there is not a consensus among mathematicians for the definition of a latin cube, we will focus on one variation. Let L be an $n \times n \times n$ array. A *layer* of L is an $n \times n$ array formed by fixing one coordinate of L. More formally, a layer is of the form $L_{i,*,*} = \{(i,j,k)|1 \leq j,k \leq n\}$, $L_{*,j,*} = \{(i,j,k)|1 \leq i,k \leq n\}$, or $L_{*,*,k} = \{(i,j,k)|1 \leq i,j \leq n\}$. We call L a *layer-rainbow latin cube of order* n if L contains the symbols in $S(n^2)$ such that each layer of L uses each symbol exactly once. In the past year, results analogous to many of the classic results mentioned in Section 2.1 have been proven for layer-rainbow latin cubes [7, 8, 9]. Therefore, we suggest the following question.

Question 2.11. *Let L be an $n \times n \times n$ array filled with the symbols from $S(t^2)$ such that each layer of L contains each of the symbols at most once. What are the necessary and sufficient conditions for embedding L in a layer-rainbow latin cube of order t in which the cells (i,i,i) for $n+1 \leq i \leq t$ are prescribed?*

Chapter 3

Equitable (s,p)-Edge-Colorings of Complete Graphs

3.1 History and Definitions

Although this chapter focuses on edge-colorings, historically, the problem is stated in terms of design theory. We will start with that history. An *H-decomposition* of a graph G is an ordered pair (V,B) where V is the vertex set of G and B is a partition of the edges of G into sets, each of which induces a copy of H. The graphs induced by the elements of B are known as the *blocks* of the decomposition. (V,B) is said to have an *(s,p)-block-coloring* (also referred to as a *(s,p)-H-coloring*) $E : B \mapsto C = \{1, 2, \dots, s\}$ if:

1. the blocks in B are colored with exactly s colors, and

2. for each vertex $u \in V(G)$ the blocks containing u are colored using exactly p colors.

The (s,p)-block-coloring, E, is said to be *equitable* if

3. for each vertex $u \in V(G)$ and for each $\{i,j\} \subset C(E,u)$, $|b(E,u,i) - b(E,u,j)| \leq 1$,

where $C(E,u) = \{i \mid E \text{ colors some block incident with } u \text{ with color } i\}$, and $b(E,u,i)$ is the number of blocks in B containing u that are colored i by E. Note that this definition of equitable generalizes the definition used in the usual edge-coloring situation where the blocks are copies of K_2 and $s = p$.

Such colorings were first introduced by Colbourn and Rosa who considered a more general notion in regards to Steiner Triple Systems (K_3-decompositions of K_v) in [12]. M. Gionfriddo et al. were the first to consider the equitable block-colorings as defined above, studying equitable block-colorings of Steiner Triple Systems in [16].

L. Gionfriddo, M. Gionfriddo, and Ragusa considered the existence of equitable block-colorings of 4-cycle systems of K_v in [15]. M. Gionfriddo and Ragusa extended their work in [17]. In [22], Li and Rodger considered the existence of equitable (s,p)-C_4-colorings of $K_{v'} - F$, where v' is even and F is a 1-factor of $K_{v'}$, allowing them to consider complete graphs on an even number of vertices previously excluded by necessary conditions. As we will see in the next paragraph, they showed this problem can be reduced to considering equitable (s,p)-edge-coloring (K_2-decompositions) of K_v for $v = \frac{v'}{2}$. In this chapter, we focus on these edge-colorings. We first summarize known results in this area and then discuss extension of results to multigraphs λK_v where λK_v is the graph formed by replacing every edge in K_v with λ edges. Finally, we consider the structure of these colorings as introduced and studied by Li, Matson, and Rodger in [21] and by Matson and Rodger in [25].

While studying equitable (s,p)-C_4-colorings of $K_{v'} - F$ where v' is even and F is a 1-factor of $K_{v'}$, the authors in [22] proved the following lemma which reduces the work to finding equitable (s,p)-edge-colorings of $K_{v'/2}$. Define $G \times 2$ to be the graph with vertex set $\{(u,1),(u,2) | u \in V(G)\}$ and edge set $\{\{(u,i),(w,j)\} | 1 \le i,j \le 2, \{u,w\} \in E(G)\}$. Thus, $K_{v'/2} \times 2 \cong K_{v'} - F$ where $F = \{\{(u,1),(u,2)\} | u \in V(K_{v'/2})\}$ is a 1-factor of $K_{v'}$.

Lemma 3.1 ([22]). *If there exists an equitable (s,p)-edge-coloring E of G, then there exists an equitable (s,p)-C_4-coloring E' of $G \times 2$.*

Therefore, the focus of this chapter is on equitable (s,p)-edge-colorings of K_v. In Section 3.2, we fix p and investigate the smallest value of s such that there exists an equitable (s,p)-edge-coloring of K_v and λK_v. In Section 3.3, we investigate the structure of these colorings. Finally, in Section 3.4, we discuss open questions in this area.

It is also worth noting that (s,p)-edge-colorings have the flavor of list edge-colorings. Typically in this area, allowable color lists are assigned to edges, and proper edge-colorings are sought so that each edge receives an allowable color. One could consider a variation where vertices receive allowable color lists, and incident edges must receive a color allowed by the lists of both incident vertices. This can be reformulated as a list edge-coloring problem by assigning the intersection of the lists associated with the two vertices incident with edge e to the allowable list for edge e for each edge e in the graph. This variation does not work well in the sense that finding the smallest p such that any list assignment of size p to each vertex results in a proper s-edge-coloring already requires p to be quite large: p must be greater than $s/2$ just to assure that the resulting intersections assigned to the edges are non-empty. This unpleasing feature can be overcome by adjusting the definition of this proposed variation of list edge-colorings so that, instead of assigning a list of size p to each vertex u from which incident edges must be colored, simply restrict the number of colors on edges incident with u to have size p. These are precisely (s, p)-edge-colorings, and then imposing the equitable requirement simply generalizes the expectation of a proper edge-coloring.

These colorings are also related to those considered by Hilton in [18] and [19]. There, he defined an $(r, r + 1)$-factor to be a spanning subgraph of a graph G in which each vertex has degree r or $r + 1$. Fixing r, he looked for $(r, r + 1)$-factorizations expressing G as the union of edge disjoint $(r, r+1)$-factors. Assigning each $(r, r+1)$-factor a color, an $(r, r+1)$-factorization is exactly an equitable (p, p)-edge-coloring, say E, of G with $b(E, u, i) \in \{r, r + 1\}$ for all $u \in V(G)$ and for all colors i for $1 \leq i \leq p$. He considered a more general problem, considering graphs with degrees within a set of consecutive values. In an effort to do this, he found the possible p for which a regular simple graph, G, can have an equitable (p, p)-edge-coloring, E, with $b(E, u, i) \in \{r, r + 1\}$ for all $u \in V(G)$ and for each color i for $1 \leq i \leq p$. In relation to this chapter, Hilton fixed v and $b(E, u, i)$ and found the possible p for which an equitable (p, p)-edge-coloring of K_v exists. Here, we fix p and v investigate the possible values of s. In Section 3.3 we focus on the interesting case requiring innovative proof techniques where s must be larger than p in order for an equitable (s, p)-edge-coloring of K_v to exist.

3.2 Lower p-Chromatic Index

A logical question may be to ask: Given a value of p, what are the possible values of s for an equitable (s, p)-H-coloring of a graph G? Particularly, what is the smallest possible value of s? First, in Subsection 3.2.1 we compile known results for equitable (s, p)-edge-colorings (K_2-decompositions) of K_v and therefore, equitable (s, p)-C_4-coloring of $K_{v'} - F$ where $v' = 2v$

and F is a 1-factor of $K_{v'}$. Then, in Subsection 3.2.2 we compile the known results for equitable (s,p)-edge-colorings of λK_v and state a new lemma that suggests some future work.

We begin with some definitions. For any H-decomposition $\Sigma = (V, B)$ of G, the authors in [22] defined the *spectrum* of Σ as $\Omega_p(\Sigma) = \{s \mid \text{there exists an equitable } (s,p)\text{-block-coloring of } \Sigma\}$. They defined the *p-color-spectrum* $\Omega_p(H,G) = \cup\Omega_p(\Sigma)$, the union being taken over all H-decompositions, Σ, of G. Two values in $\Omega_p(H,G)$ are of particular interest: the *lower p-chromatic index* is defined to be $\chi'_p(H,G) = \min\Omega_p(H,G)$ and the *upper p-chromatic index* is defined to be $\overline{\chi'_p(H,G)} = \max\Omega_p(H,G)$.

3.2.1 Equitable (s,p)-edge-colorings of K_v

In this subsection, we consider the lower p-chromatic index of edge-colorings (K_2-decompositions) of K_v and C_4-decompositions of $K_{v'} - F$ where $v' = 2v$ is even and F is a 1-factor of $K_{v'}$. Because results in this area are in several different papers and they use different notation, we list known results and then combine them for Theorem 3.10.

Observation 3.2. *Because p is bounded by the degree of every vertex,*

- *for all equitable (s,p)-edge-colorings of K_v, $1 \le p \le v-1$, and*
- *for all equitable (s,p)-blocking colorings of a C_4-decomposition of $K_{v'} - F$, $1 \le p \le (v'-2)/2$.*

Thus, for the remainder of this subsection, we assume $1 \le p \le v-1$ and $1 \le p \le (v'-2)/2$. We also note that an equitable (p,p)-edge-coloring of K_v is equivalent to what has previously been defined as an equitable edge-coloring of K_v with p colors. However, we will continue to call this an equitable (p,p)-edge-coloring of K_v.

The study of (p,p)-edge-colorings of K_v has a long history. The following theorem shows a sufficient condition for their existence.

Theorem 3.3. *[20] Let G be a simple graph and let $p \ge 2$. If $p \nmid d(v)$ (for all $v \in V(G)$), then G has an equitable (p,p)-edge-coloring.*

The following foundational theorem can be proved with the well-known Walecki's construction and will be used to prove another sufficent condition for equitable (p, p)-edge-colorings of K_v in Lemma 3.5. This theorem will also be used throughout the remainder of this section.

Theorem 3.4. *[24] λK_v (or $\lambda K_v - F$ where F is a 1-factor of λK_v) has a Hamilton decomposition if and only if $\lambda(v - 1)$ is even (or odd, respectively).*

Lemma 3.5. *Suppose v is odd, $p \mid (v-1)$, and $(v-1)/p$ is even. Then there exists an equitable (p, p)-edge-coloring of K_v.*

Proof. By Theorem 3.4, there exists a Hamilton decomposition of K_v. To create each color class, color the edges of $(v - 1)/(2p)$ Hamilton cycles the given color. $\square$

The authors in [22], prove the following theorem in their proof that there exists an equitable (p, p)-C_4-coloring of $K_{v'} - F$ for $v'/2$ even and F a 1-factor of $K_{v'}$. This shows yet another sufficient condition for (p, p)-edge-colorings of K_v.

Theorem 3.6. *[22] Let v be even. There exists an equitable (p, p)-edge-coloring of K_v.*

The next three results focus on an interesting case where equitable (p, p)-C_4-decompositions of $K_{v'} - F$ and equitable (p, p)-edge-colorings of K_v do not exist.

Lemma 3.7. *[22] Let $v' \equiv 4t + 2 \pmod{8t}$. Then there is no C_4-decomposition of $K_{v'} - F$ for which there exists an equitable $(2t, 2t)$-C_4-coloring.*

The next lemma is similar to the previous, and is proved in a similar manner as the authors in [22] proved Lemma 3.7.

Lemma 3.8. *[22] Let $v \equiv 2t + 1 \pmod{4t}$. Then there is no equitable $(2t, 2t)$-edge-coloring of K_v.*

Proof. Let $v = 4tx+2t+1$ for some integer x. Thus, $b(E, u, i) = (v-1)/p = (4tx+2t)/2t = 2x + 1$ is odd for all edge colorings E, vertices $u \in V(K_v)$, and colors i. Thus, a color cannot appear at every vertex, else the graph induced by that color class would be regular with odd degree on an odd number of vertices. Therefore, $s > p$. $\square$

As seen in the following theorem, the authors in [21] found the lower p-chromatic index in the previous case, showing $p+1$ colors suffice.

Theorem 3.9. *[21] Let* $v \equiv 2t+1 \pmod{4t}$ *and* $v' \equiv 4t+2 \pmod{8t}$. *Then,*

- *there exists a equitable* $(2t+1, 2t)$*-edge-coloring of* K_v *and*
- *there exists an equitable* $(2t+1, 2t)$*-*C_4*-coloring of* $K_{v'} - F$.

Finally, we combine all of these results to obtain the following theorem and corollary.

Theorem 3.10. *[20, 21, 22] For* $1 \leq p \leq v-1$,

$$\chi'_p(K_2, K_v) = \begin{cases} p+1 & \text{if } p \text{ even and } v \equiv p+1 \pmod{2p} \\ p & \text{otherwise.} \end{cases}$$

Proof. If v is even the result follows by Theorem 3.6. If v is odd and $p \nmid (v-1)$, the result follows by Theorem 3.3. If v is odd, $p \mid (v-1)$, and $(v-1)/p$ is even, the result follows by Theorem 3.5.

If v is odd and $p \mid (v-1)$ and $(v-1)/p$ is odd, then p must be even. Thus, $b(E, u, i) = (v-1)/p = 2x+1$ for some integer x for all edge colorings E, vertices $u \in V(K_v)$, and colors i. Also, $p = 2t$ for some integer t. Therefore, $v-1 = (2t)(2x+1)$. So, $v = 4tx+2t+1 \equiv 2t+1 \pmod{4t}$. Thus, by Theorem 3.8 and 3.9, $\chi'_p(K_2, K_v) = p+1$. □

Corollary 3.11. *[20, 21, 22] For* v' *even and* $1 \leq p \leq (v'-2)/2$,

$$\chi'_p(C_4, K_{v'} - F) = \begin{cases} p+1 & \text{if } p \text{ is even and } v \equiv 2p+2 \pmod{4p} \\ p & \text{otherwise.} \end{cases}$$

where F *is a* 1*-factor of* $K_{v'}$.

Proof. This follows directly from Lemmas 3.1 and 3.7 and Theorem 3.10. □

3.2.2 Equitable (s, p)-edge-colorings of λK_v

We now discuss equitable (s, p)-edge-colorings of λK_v.

Observation 3.12. *Because p is bounded by the degree of every vertex, for all equitable (s, p)-edge-coloring of K_v, $1 \leq p \leq \lambda(v-1)$.*

So, for the remainder of this subsection assume $1 \leq p \leq \lambda(v-1)$. Similar to the last subsection, we first list some results about sufficient conditions for equitable (p, p)-edge-colorings of λK_v. Then we highlight a case where the lower p-chromatic index is greater than p, leading to a conjecture and avenue for future research.

Lemma 3.13. *Let v be even. Then there exists an equitable (p, p)-edge-coloring of λK_v.*

Proof. Let $\{F_0, F_1, \ldots, F_{\lambda(v-1)-1}\}$ be a 1-factorization of λK_v. For $0 \leq i \leq \lambda(v-1)-1$ color the edges of F_i color j if and only if $j \equiv i \pmod{p}$. Thus, each vertex is incident to $\lceil \frac{\lambda(v-1)}{p} \rceil$ or $\lfloor \frac{\lambda(v-1)}{p} \rfloor$ edges of each of the p colors. Because $p \leq \lambda(v-1)$, there is at least 1 edge of each color appearing at each vertex. Thus, this is an equitable (p, p)-edge coloring of λK_v. $\square$

The following proof is similar to the proof of Lemma 3.5.

Theorem 3.14. *Suppose v is odd, $p \mid \lambda(v-1)$, and $\lambda(v-1)/p$ is even. Then there exists an equitable (p, p)-edge-coloring of λK_v.*

Proof. By Theorem 3.4 there exists a Hamilton decomposition of λK_v. To create each color class, color the edges of $(\lambda(v-1))/(2p)$ Hamilton cycles the given color. $\square$

The following lemma highlights a case where there does not exist an equitable (p, p)-edge-coloring of λK_v and gives a lower bound for s.

Lemma 3.15. *Suppose v is odd, $p \mid \lambda(v-1)$, and $\lambda(v-1)/p$ is odd. Then $s \geq p + \lceil \frac{p}{v-1} \rceil$.*

Proof. Each of the s colors can appear on at most $v-1$ vertices because v is odd and $\lambda(v-1)/p$ is odd. Also, each vertex has exactly p colors on the edges incident to it. So, $s(v-1) \geq vp$. Thus,

$$
\begin{aligned}
s &\geq \tfrac{vp}{v-1} \\
&= \tfrac{(v-1)p}{v-1} + \tfrac{p}{v-1} \\
&= p + \lceil \tfrac{p}{v-1} \rceil.
\end{aligned}
$$

□

This leads to the following conjecture. We predict it can be proved in a similar manner to Theorem 3.9 using the technique of amalgamations that will be described in more depth in Section 3.3.

Conjecture 3.16. *Suppose v is odd, $p \mid \lambda(v-1)$, and $\lambda(v-1)/p$ is odd. Then there exists an equitable $(p + \lceil \frac{p}{v-1} \rceil, p)$-edge-coloring of λK_v.*

If the previous conjecture is true, then we can obtain a more general version of Theorem 3.10 concerning equitable (s, p)-edge-colorings of λK_v by combining that result, a modification of Theorem 3.3, Lemma 3.13, Lemma 3.14, and Lemma 3.15.

3.3 Structure of Colorings

The results in this section are joint work with Chris Rodger and have been accepted for publication. See [11].

We begin by motivating the avenue of research in this section with a scheduling problem. Suppose v people, say $v = 25$, want to meet one-on-one with each other during one day of a convention. Through the $s = 9$ hours of the day, each person meets with others for $p = 8$ of the hours, taking the other hour for a break. Ideally, the meetings for each person should be spread out evenly throughout the day, so during each hour the aim is for each person not on a break to meet $(v-1)/p = 3$ people. Participants are requested to list the $s - p = 1$ hour when they would prefer to take a break (hours around lunch time or the first and last hours to arrive late or leave early being likely popular choices). This information is described by a vector $V = (c_1, c_2, \ldots, c_s)$, the i^{th} component being the number of participants wanting to meet during the i^{th} of the s hours (so $v - c_i$ is the number of participants wanting their break during the i^{th} hour).

This problem can be modeled by using the complete graph K_v, each vertex representing a participant, with the edge $\{i, j\}$ being colored k to indicate that participants i and j should meet during the k^{th} hour. To meet the requirements, at each vertex w, each of some choice of p colors should appear on exactly $(v-1)/p$ edges incident to w, the remaining $s-p$ colors appearing on no incident edges. Furthermore, to meet the choice of when to take a break, the number of vertices missing color k should be $v - c_k$ for $1 \leq k \leq s$ where c_k is the k^{th} component of V. This is an equitable (s, p)-edge-coloring of K_v, say E, with what we will define below as the associated color vector $V(E)$.

Completing the scheduling would then require, for $1 \leq k \leq s$, taking the subgraph $G(k)$ induced by the edges colored k and giving it an edge-coloring with $\max\{(v-1)/p, \chi'(G(k))\}$ colors to subdivide the k^{th} hour into meeting times. But this step is not the focus of the research in this section.

In this section we will focus on the particularly interesting case of Theorem 3.10 where the lower p-chromatic index is not equal to p; i.e., $\chi'_p(K_2, K_v) > p$. Thus, for the rest of this section we will assume $v \equiv 2t+1 \pmod{4t}$, $p = 2t$, and $s = 2t+1$. It's interest is generated for edge-colorings (i.e., K_2-decompositions) by the observation that, because at each vertex a color is prohibited from appearing on an incident edge, the usual edge-coloring technique of interchanging colors along a 2-edge-colored path does not work. So completely new techniques are needed.

In [21], the authors began to focus on the structure of equitable (s, p)-block-colorings using the following concepts. The *color vector* of an equitable (s, p)-block-coloring E of an H-decomposition $(V(G), B)$ of a graph G is the vector $V(E) = (c_1(E), c_2(E), \ldots, c_s(E))$ in which, for $1 \leq i \leq s$, $c_i(E)$ is the number of vertices in G that are incident with a block of color i; by renaming colors if necessary, it is always assumed that $c_1(E) \leq c_2(E) \leq \cdots \leq c_s(E)$. The following definitions help us explore the color vector. For any graphs G and H and for $1 \leq i \leq s$, define

(i) $\phi(H, G; s, p, i) = \{c_i(E) \mid E$ is an equitable (s, p)-block-coloring of an H-decomposition of $G\}$,

(ii) $\psi'(H, G; s, p, i) = \min \phi(H, G; s, p, i)$, and

(iii) $\overline{\psi'}(H, G; s, p, i) = \max \phi(H, G; s, p, i)$.

Because we are only considering $s = 2t + 1$ and $p = 2t$, we define $\psi'_i(H, G) = \psi'(H, G; 2t + 1, 2t, i)$ and $\overline{\psi'_i}(H, G) = \overline{\psi'}(H, G; 2t + 1, 2t, i)$.

As stated earlier, the motivation behind this work stems from design theory. Similar to the last section, assume $2v = v'$ and F is a 1-factor of $K_{v'}$. In [21], the values of $\overline{\psi'_1}(C_4, K_{v'} - F)$ and $\psi'_{2t+1}(C_4, K_{v'} - F)$ were found. In [25], $\psi'_1(C_4, K_{v'} - F)$ and $\overline{\psi'_{2t+1}}(C_4, K_{v'} - F)$ were found, along with $\overline{\psi'_i}(C_4, K_{v'} - F)$ for $2 \leq i \leq 2t$. In this chapter we find $\psi'_2(C_4, K_{v'} - F)$ (see Theorem 3.41), $\psi'_t(C_4, K_{v'} - F)$ (see Theorem 3.43), and $\psi'_{2t}(C_4, K_{v'} - F)$ (see Theorem 3.40).

We use the methods established in [21, 22, 25] to find $\psi'_i(C_4, K_{v'} - F)$ by considering $\psi'_i(K_2, K_v)$. Thus, most of the chapter is focused on equitable (s, p)-edge-colorings of K_v, finding $\psi'_{2t}(K_2, K_v)$ (see Theorem 3.24), $\psi'_2(K_2, K_v)$ (see Theorem 3.27), and $\psi'_t(K_2, K_v)$ (see Theorem 3.34). This focus on these three special values of $\psi_i(K_2, K_v)$, namely when $i \in \{2, t, 2t\}$, is driven by the fascinating observation that in rare cases such as these the color vector attaining this value is unique and contains only 2 different integers (see Propositions 3.25, 3.29, and 3.35). These values are found using the graph theory technique of amalgamations and may be of particular interest to graph theorists. The results concerning equitable (s, p)-C_4-decompositions of $K_{v'} - F$ then follow as corollaries (see Subsection 3.3.3).

3.3.1 Preliminary Results

Lemma 3.17 ([25]). *Let $v' = 2v = 8tx + 4t + 2$ for some integer x. Let E and E' be an equitable $(2t + 1, 2t)$-edge-coloring and an equitable $(2t + 1, 2t)$- C_4-coloring of K_v and $K_{v'} - F$, respectively. Then*

$$b(E, u, i) = b(E', u', i') = 2x + 1$$

for all $u \in V(K_v)$, $i \in C(E, u)$, $u' \in V(K_{v'} - F)$ and $i' \in C(E', u')$.

In view of Lemma 3.17, because $b(E,u,i)$ and $b(E'u',i')$ are independent of E, u, i, E', u', and i', for the remainder of this section we define $b(v) = b'(v') = b(E,u,i) = b(E'u',i') = 2x+1$. So for such an edge-coloring, at each vertex u, $p = 2t$ colors each occur on exactly $b(v) = 2x+1$ edges incident with u, and the remaining $s-p = 1$ color appears on no edges incident with u.

Lemma 3.18, proved in [21], will be used to establish lower bounds for $\psi_i'(K_2, K_v)$ in Lemma 3.19.

Lemma 3.18 ([21])**.** *Let $v \equiv p+1 \pmod{2p}$. In any equitable (s,p)-edge-coloring E of K_v, for $1 \leq i \leq s$,*

(i) $c_i(E)$ must be even,

(ii) $c_i(E) \geq b(v) + 1 = \frac{v-1}{p} + 1$, and

(iii) if v is odd then $c_i(E) \leq v-1$.

The following generalizes results in [21] and [25] that considered the $i = 2t+1$ and $i = 1$ cases respectively. Let $\lceil a \rceil_e$ denote the smallest even integer greater than or equal to a.

Lemma 3.19. *Let $v = 4tx + 2t + 1$ for some integer x. Then $\psi_i'(K_2, K_v) \geq \max\{b(v) + 1, \lceil \frac{v(s-1)-(s-i)(v-1)}{i} \rceil_e\}$ for $1 \leq i \leq s$.*

Proof. Let E be an equitable $(2t+1, 2t)$-edge-coloring of K_v. Because $p = 2t$ and $s = 2t+1$, we know each vertex must be missing exactly one color; so $s-1$ colors appear at each vertex. Counting in two ways, we have $v(s-1) = \sum_{i=1}^{s} c_i(E)$. By the definition of the color vector, $c_1(E) \leq c_2(E) \leq \cdots \leq c_i(E)$. Also, because v is odd, by Lemma 3.18 (iii), $c_j(E) \leq v-1$; in particular this holds for the $s-i$ values $(i+1) \leq j \leq s$. Thus, $\sum_{i=1}^{s} c_i(E) \leq ic_i(E) + (s-i)(v-1)$. So, $v(s-1) \leq ic_i(E) + (s-i)(v-1)$. Therefore, $\lceil \frac{v(s-1)-(s-i)(v-1)}{i} \rceil \leq c_i(E)$. By Lemma 3.18 (i), $c_i(E)$ must be even. Thus, $\lceil \frac{v(s-1)-(s-i)(v-1)}{i} \rceil_e \leq c_i(E)$. By Lemma 3.18 (ii) $c_i(E) \geq b(v)+1$. Therefore, $\psi_i'(K_2, K_v) \geq \max\{b(v)+1, \lceil \frac{v(s-1)-(s-i)(v-1)}{i} \rceil_e\}$ for $1 \leq i \leq s$. $\square$

To this point we know of no case where ψ_i' is not equal to the lower bound in Lemma 3.19. We plan to pursue this avenue of research in the future, hoping to show no such example exists.

To prove the results in the next subsection we use the proof technique of amalgamations. In the following definitions and throughout the chapter, all sets are multisets, but are still denoted using set notation. The *amalgamation* of a graph G defined by the amalgamation function $\phi : V(G) \to V(H)$ is the graph H, possibly with multiple edges and loops, with vertex set $V(H)$ and the multiset of edges $E(H) = \{\{\phi(a), \phi(b)\} \mid \{a, b\} \in E(G)\}$, where $\{\phi(a), \phi(a)\}$ denotes a loop on vertex $\phi(a)$. For each $u \in V(H)$, let $\eta(u) = |\{\phi^{-1}(u)\}|$; η is said to be the number function associated with ϕ. So $\eta(u)$ is the number of vertices in G that were amalgamated to form u. Given H, we define G to be an η-*detachment* of H if there exists an amalgamation function $\phi : V(G) \to V(H)$ such that $|\phi^{-1}(u)| = \eta(u)$ for every $u \in V(H)$. Theorem 3.20 below is a special case of a theorem in [10]. Some notation is needed. Let $\ell(u)$ be the number of loops on vertex u where each loop contributes 2 to the degree of u, let $G(j)$ be the subgraph of G induced by the edges colored j, let $m(u, v)$ be the multiplicity (i.e. number of edges) between the vertices u and v, and let $x \approx y$ represent $\lfloor y \rfloor \leq x \leq \lceil y \rceil$.

Theorem 3.20 ([10]). *Let H be a k-edge-colored graph and let η be a function from $V(H)$ into $\mathbb{N}$ such that for each $w \in V(H)$, $\eta(w) = 1$ implies $\ell_H(w) = 0$. Then there exists a loopless η-detachment G of H with amalgamation function $\phi : V(G) \to V(H)$, η being the number function associated with ϕ, such that G satisfies the following conditions:*

(i) $d_{G(j)}(u) \approx d_{H(j)}(w)/\eta(w)$ *for each* $w \in V(H)$, *each* $u \in \phi^{-1}(w)$, *and for* $1 \leq j \leq k$;

(ii) $m_G(u, u') \approx \ell_H(w)/\binom{\eta(w)}{2}$ *for each* $w \in V(H)$ *with* $\eta(w) \geq 2$ *and every pair of distinct vertices* $u, u' \in \phi^{-1}(w)$; *and*

(iii) $m_G(u, v) \approx m_H(w, z)/(\eta(w)\eta(z))$ *for every pair of distinct vertices* $w, z \in V(H)$, *each* $u \in \phi^{-1}(w)$, *and each* $v \in \phi^{-1}(z)$.

The following two theorems will be used in the proofs of the main theorems in Subsection 3.3.2. Theorem 3.21, attributed to Petersen, can be easily proved using Euler circuits and 1-factorizations of bipartite graphs.

Theorem 3.21. *Every $2k$-regular graph can be partitioned into k edge-disjoint 2-factors.*

Theorem 3.22 ([31])**.** *For any $k \geq 1$ and any bipartite graph B, there exists an equitable (k,k)-edge-coloring of B.*

In the next Subsection we prove our main theorems, finding $\psi'_{2t}(K_2, K_v)$, $\psi'_2(K_2, K_v)$, and $\psi'_t(K_2, K_v)$. Each theorem is preceded by a lemma which uses Lemma 3.19 to establish a lower bound. Then, the proof of each theorem constructs an edge-coloring reaching this lower bound.

3.3.2 Main Results

In this subsection we focus on edge-colorings, finding $\psi'_{2t}(K_2, K_v)$, $\psi'_2(K_2, K_v)$, and $\psi'_t(K_2, K_v)$. These are proved in Theorems 3.24, 3.27, and 3.34 using the method of amalgamations, applying Theorem 3.20 to a suitably chosen amalgamation of K_v. Theorem 3.24 could be proved more directly, but in this simpler setting it will help the reader to see the amalgamation technique in action before moving to the two more complicated cases. The values of $\psi'_{2t}(C_4, K_{2v} - F)$, $\psi'_2(C_4, K_{2v} - F)$, and $\psi'_t(C_4, K_{2v} - F)$ will then be shown to be corollaries of these results in the next subsection.

Lemma 3.23. *Let $v = 4tx + 2t + 1$ for some integer x. Then, $\psi'_{2t}(K_2, K_v) \geq 4tx + 2t - 2x$.*

Proof. Using the postulated values of v, s, and i,

$$\begin{aligned}
&\lceil \tfrac{v(s-1)-(s-i)(v-1)}{i} \rceil_e \\
&\quad = \lceil \tfrac{(4tx+2t+1)((2t+1)-1)-((2t+1)-2t)((4tx+2t+1)-1)}{2t} \rceil_e \\
&\quad = \lceil \tfrac{8t^2x+4t^2-4tx}{2t} \rceil_e \\
&\quad = 4tx + 2t - 2x.
\end{aligned}$$

Thus, by Lemma 3.19, $\psi'_{2t}(K_2, K_v) \geq \max\{b(v)+1, \lceil \frac{v(s-1)-(s-i)(v-1)}{i} \rceil_e\} = 4tx+2t-2x.$ $\square$

Throughout Section 3.3.2, for any simple graph G, let λG be the multigraph in which each pair of vertices is joined by λ edges if they are adjacent in G and no edges otherwise. Also, at times it will cause no confusion to let c_i instead of $c_i(E)$ denote the i^{th} component of the color vector $V(E)$.

Theorem 3.24. *Let $v = 4tx + 2t + 1$ for some integer x. Then, $\psi'_{2t}(K_2, K_v) = 4tx + 2t - 2x$.*

Proof. Because $s = 2t + 1$ and $p = 2t$, we want every vertex in our final graph, $G = K_v$, to be missing exactly one color. This will be done so that color $2t + 1$ is missing from exactly one vertex and every other color is missing from exactly $2x + 1$ vertices. This will show that $\psi'_{2t}(K_2, K_v) \leq 4tx + 2t + 1 - (2x + 1) = 4tx + 2t - 2x$. Then it will follow from Lemma 3.23 that $\psi'_{2t}(K_2, K_v) = 4tx + 2t - 2x$.

Let $U = \{u_1, \ldots, u_{2t}\}$. We will form the edge-colored graphs G_1, G_2, and G_3 on the vertex set $U \cup \{w\}$ as follows. Once this is done, defining $G' = \bigcup_{i=1}^{3} G_i$, Theorem 3.20 will be used to detach G' to form $G = K_v$ with our desired coloring. To do so, $\phi : V(G) \to V(G')$ will be the amalgamation function with amalgamation numbers $\eta(w) = 1$ and $\eta(u_i) = 2x + 1$ for $1 \leq i \leq 2t$.

Define α to be a proper $(2t+1)$-edge-coloring of K_{2t+1} on the vertex set $U \cup \{w\}$. Then α is a near-one-factorization. Use colors $\{1, 2, \ldots, 2t + 1\}$ so that each vertex $u_i \in U$ is missing color i and w is missing color $2t + 1$.

(i) Define the $(2t+1)$-edge-colored graph $G_1 = (2x+1)K_{2t+1}$ on the vertex set $U \cup \{w\}$ as follows: for each $\{u, u'\} \subseteq U \cup \{w\}$ join u and u' with $2x + 1$ edges colored $\alpha(\{u, u'\})$.

(ii) Define the $(2t+1)$-edge-colored graph $G_2 = (4x^2+2x)K_{2t}$ on the vertex set U as follows: for each $\{u, u'\} \subseteq U$ join u and u' with $(4x^2 + 2x)$ edges colored $\alpha(\{u, u'\})$.

(iii) Define the $(2t + 1)$-edge-colored graph G_3 on the vertex set U by adding $(2x^2 + x)$ loops on u_i of color j if and only if $\alpha(\{w, u_i\}) = j$. (So G_3 has no edges; just loops, each contributing degree 2 to its incident vertex.)

As indicated earlier in the proof, we now define $G' = \bigcup_{i=1}^{3} G_i$. Apply Theorem 3.20 to G' using the number function η defined earlier to form G. We now show that $G = K_v$ and that the resulting edge-coloring of G has the desired properties.

We first show w is incident in G' to $b(v)$ edges of each color with one exception, a color which appears on no edge, namely color $2t + 1$. We also conclude what this means for G. For $1 \leq j \leq 2t$, by (i), $d_{G'(j)}(w) = 2x + 1 = b(v)$, so $d_{G(j)}(w) = d_{G'(j)}(w)/\eta(w) = b(v)/1 = b(v)$

by Theorem 3.20 (i). We will now check that each vertex detached from u_i with $1 \leq i \leq 2t$ is incident in G to $b(v)$ edges of each color, with one exception, a color which appears on no edge, namely color i. For $1 \leq i \leq 2t$, $1 \leq j \leq 2t+1$, and $i \neq j$, by (i) and either (ii) or (iii), $d_{G'(j)}(u_i) = (2x+1)+(4x^2+2x) = 4x^2+4x+1$ or $d_{G'(j)}(u_i) = (2x+1)+2(2x^2+x) = 4x^2+4x+1$ respectively, so $d_{G(j)}(a) = d_{G'(j)}(u_i)/\eta(u_i) = (4x^2+4x+1)/(2x+1) = 2x+1 = b(v)$ for each $a \in \phi^{-1}(u_i)$ by Theorem 3.20 (i).

We will now check that there is exactly one edge between each pair of vertices in G. For $1 \leq i \leq 2t$, by (iii), u_i has $2x^2+x$ loops, so $m_G(a,a') = \ell_{G'}(u_i)/\binom{\eta(u_i)}{2} = (2x^2+x)/(\frac{(2x+1)(2x)}{2}) = 1$ for each distinct $a, a' \in \phi^{-1}(u_i)$ by Theorem 3.20 (ii). For $1 \leq i, j \leq 2t$ and $i \neq j$, by (i) and (ii), $m_{G'}(u_i, u_j) = (2x+1)+(4x^2+2x) = 4x^2+4x+1$, so $m_G(a,a') = m_{G'}(u_i,u_j)/(\eta(u_i))(\eta(u_j)) = (4x^2+4x+1)/(2x+1)(2x+1) = 1$ for each $a \in \phi^{-1}(u_i)$ and $a' \in \phi^{-1}(u_j)$ by Theorem 3.20 (iii). For $1 \leq i \leq 2t$, by (i), $m_{G'}(w, u_i) = 2x+1$, so $m_G(a,a') = m_{G'}(w,u_i)/(\eta(w))(\eta(u_i)) = (2x+1)/(1)(2x+1) = 1$ for $a \in \phi^{-1}(w)$ and for each $a' \in \phi^{-1}(u_i)$ by Theorem 3.20 (iii). Therefore, every vertex in G is attached to each other vertex with exactly one edge. Thus, $G = K_v$. □

The following proposition highlights the interesting property that the color vector used in Theorem 3.24 to find $\psi_{2t}(K_2, K_v)$ is unique and contains only 2 different integers.

Proposition 3.25. *Let $v = 4tx+2t+1$ for some integer x. In any equitable (s,p)-edge-coloring of K_v with $c_{2t} = \psi'_{2t}(K_2, K_v)$, $c_1 = c_2 = \cdots = c_{2t} = \psi'_{2t}(K_2, K_v)$ and $c_s = v-1$.*

Proof. Since $\psi'_{2t}(K_2, K_v) = 4tx+2t-2x$, by definition of color vector and Lemma 3.18,

$$\sum_{i=1}^{s} c_i \leq 2t(4tx+2t-2x)+(v-1)$$

with equality if and only if

$$c_i = \begin{cases} 4tx+2t-2x & \text{for } 1 \leq i \leq 2t, \text{ and} \\ v-1 & \text{for } i = s. \end{cases}$$

Because $p = 2t$ and $s = 2t + 1$, we know each vertex must be missing exactly one color; so $s - 1$ colors appear at each vertex. Counting in two ways, we have $(s-1)v = \sum_{i=1}^{s} c_i(E)$. The result follows since,

$$\begin{aligned}\textstyle\sum_{i=1}^{s} c_i &= (s-1)v \\ &= 2t(4tx + 2t + 1) \\ &= 2t(4tx + 2t - 2x) + 4tx + 2t \\ &= 2t(4tx + 2t - 2x) + (v - 1).\end{aligned}$$

□

Lemma 3.26. *Let $v = 4tx + 2t + 1$ for some integer x. Then, $\psi_2'(K_2, K_v) \geq 2tx + 2t$.*

Proof. Using the postulated values of v, s, and i,

$$\begin{aligned}&\lceil \tfrac{v(s-1)-(s-i)(v-1)}{i} \rceil_e \\ &= \lceil \tfrac{(4tx+2t+1)((2t+1)-1)-((2t+1)-2)((4tx+2t+1)-1)}{2} \rceil_e \\ &= \lceil \tfrac{4tx+4t}{2} \rceil_e \\ &= 2tx + 2t.\end{aligned}$$

Thus, by Lemma 3.19, $\psi_2'(K_2, K_v) \geq \max\{b(v) + 1, \lceil \frac{v(s-1)-(s-i)(v-1)}{i} \rceil_e\} = 2tx + 2t$. □

Theorem 3.27. *Let $v = 4tx + 2t + 1$ for some integer x. Then $\psi_2'(K_2, K_v) = 2tx + 2t$.*

Proof. Because $s = 2t + 1$ and $p = 2t$, we want every vertex in our final graph, $G = K_v$, to be missing exactly one color. This will be done so that colors 1 and 2 are each missing from exactly $2tx + 1$ vertices and every other color is missing from exactly one vertex. This will show that $\psi_2'(K_2, K_v) \leq 4tx + 2t + 1 - (2tx + 1) = 2tx + 2t$. Then it will follow from Lemma 3.26 that $\psi_2'(K_2, K_v) = 2tx + 2t$.

Let $U = \{u_1, u_2\}$, $W = \{w_3, \ldots, w_{2t+1}\}$ and $V = \{v_1, v_2\}$. We will form the edge-colored graphs $G_1, \ldots, G_{10}$ on the vertex set $U \cup W \cup V$ as follows. A general approach is described below similar to the proof used for the next theorem, but also for this theorem a specific example of $G_1, \ldots G_{10}$ follows in Example 3.28 to potentially help the reader. In (ii) and (iii) below, the vertex $z \notin U \cup W \cup V$ is introduced simply to ensure that w_i is missing color i when G_j ($j \in \{2, 3\}$) is defined. Once this is done, defining $G' = \bigcup_{i=1}^{10} G_i$, Theorem 3.20 will be used to detach G' to form $G = K_v$ with our desired coloring. To do so, $\phi : V(G) \to V(G')$

will be the amalgamation function with amalgamation numbers $\eta(u_i) = 1$ for $1 \leq i \leq 2$, $\eta(w_i) = 1$ for $3 \leq i \leq 2t+1$, and $\eta(v_i) = 2tx$ for $1 \leq i \leq 2$.

i) Define the graph $G_1 = K_{2t+1}$ on the vertex set $U \cup W$. Use a near-one-factorization of G_1 to properly color the edges of G_1 with colors $\{1, 2, \ldots, 2t+1\}$ so that each vertex $u_i \in U$ is missing color i and each vertex $w_j \in W$ is missing color j.

ii) Consider $G_2' = (2t-1)K_{2t-1,1} \cup K_{2t-1,1}$ with parts W and $\{v_1\}$ and W and $\{z\}$ respectively. Using Theorem 3.22, equitably $(2t, 2t)$-edge-color G_2' with colors $2, \ldots, 2t+1$ such that the edge $\{w_i, z\}$ is colored i. Form the graph $G_2 = (2tx - x)K_{2t-1,1}$ from G_2' with parts W and $\{v_1\}$ by deleting z and replacing each remaining edge $\{u, v\}$ in G_2' colored j with x edges joining u and v colored j in G_2. (So v_1 is incident with $(2t-1)x$ edges colored 2 and $(2t-2)x$ edges of each color $3, 4, \ldots, 2t+1$.)

iii) Consider $G_3' = (2t-1)K_{2t-1,1} \cup K_{2t-1,1}$ with parts W and $\{v_2\}$ and W and $\{z\}$ respectively. Using Theorem 3.22, equitably $(2t, 2t)$-edge-color G_2' with colors $1, 3, 4, \ldots, 2t+1$ such that the edge $\{w_i, z\}$ is colored i. Form the graph $G_3 = (2tx - x)K_{2t-1,1}$ from G_3' with parts W and $\{v_2\}$ by deleting z and replacing each remaining edge $\{u, v\}$ in G_3' colored j with x edges joining u and v colored j in G_3. (So v_2 is incident with $(2t-1)x$ edges colored 1 and $(2t-2)x$ edges of each color $3, \ldots, 2t+1$.)

iv) Define the graph $G_4 = x(K_{2t-1,1})$ with parts W and $\{v_1\}$ with all edges colored 2.

v) Define the graph $G_5 = x(K_{2t-1,1})$ with parts W and $\{v_2\}$ with all edges colored 1.

vi) Define the graph G_6 to be the bipartite graph with parts U and V such that u_i and v_j are connected with $2tx - 2x$ edges if $i = j$ and $2tx$ edges otherwise. Using Theorem 3.22, equitably $(2t-1, 2t-1)$-edge-color G_6 with colors $3, \ldots, 2t+1$. (So, for $1 \leq i \leq 2$ and $3 \leq j \leq 2t+1$, v_i is incident to $((2tx - 2x) + 2tx)/(2t-1) = 2x$ edges of color j.)

vii) Let $G_7 = (2x)K_{1,1}$ with parts $\{u_1\}$ and $\{v_1\}$ with all edges colored 2.

viii) Let $G_8 = (2x)K_{1,1}$ with parts $\{u_2\}$ and $\{v_2\}$ with all edges colored 1.

ix) Let G_9 be formed from $2tx^2 - tx$ loops colored 2 on vertex v_1 and $2tx^2 - tx$ loops colored 1 on vertex v_2.

x) Let G_{10} be formed from $(4t^2x^2)K_2$ on the vertex set V by adding $2t^2x^2 - 2tx^2$ loops to each vertex. Pair the loops forming edge-disjoint 2-factors (each loop contributes 2 to the degree of its incident vertex) and pair the edges to partition $(4t^2x^2)K_2$ into 2-factors. Therefore, G_{10} is decomposed into $(4t^2x^2 - 2tx^2)$ 2-factors. By appropriately combining the 2-factors, decompose G_{10} into $2t-1$ $(4tx^2)$-factors, coloring their edges with $3, 4, \dots, 2t+1$ in turn.

As indicated earlier in the proof, we now define $G' = \cup_{i=1}^{10} G_i$. Apply Theorem 3.20 to G' using the number function η defined earlier to form G. We now show that $G = K_v$ and the the resulting edge-coloring of G has the desired properties.

We first show the vertices in the sets U and W are incident in G' to $b(v)$ edges of each color with one exception, a color which appears on no edges, namely color i for each $u_i \in U$ and color j for each $w_j \in W$. We also conclude what this means for G. For $1 \leq i \leq 2$ and $3 \leq j \leq 2t+1$, using (i) and (vi), $d_{G'(j)}(u_i) = 1 + 2x = b(v)$, so $d_{G(j)}(a) = d_{G'(j)}(u_i)/\eta(u_i) = b(v)/1 = b(v)$ for each $a \in \phi^{-1}(u_i)$ by Theorem 3.20 (i). For $1 \leq i \leq 2$, $1 \leq j \leq 2$, and $i \neq j$, using (i) and (vii) or (viii), $d_{G'(j)}(u_i) = 1 + 2x = b(v)$, so $d_{G(j)}(a) = d_{G'(j)}(u_i)/\eta(u_i) = b(v)/1 = b(v)$ for each $a \in \phi^{-1}(u_i)$ by Theorem 3.20 (i). For $3 \leq i \leq 2t+1$, $1 \leq j \leq 2t+1$, and $i \neq j$, using (i) and (ii) and (iii) if $3 \leq j \leq 2t+1$, (ii) and (iv) if $j = 2$, or (iii) and (v) if $j = 1$, $d_{G'(j)}(w_i) = 1 + x + x = b(v)$, so $d_{G(j)}(a) = d_{G'(j)}(w_i)/\eta(w_i) = b(v)/1 = b(v)$ for each $a \in \phi^{-1}(w_i)$ by Theorem 3.20 (i).

We will now check that each vertex detached from v_i with $1 \leq i \leq 2$ is incident in G to $b(v)$ edges of each color with one exception, a color which appears on no edge, namely color i for v_i. For $3 \leq j \leq 2t+1$, using (ii), (vi), and (x), $d_{G'(j)}(v_1) = (2t-2)x + 2x + 4tx^2 = 2tx + 4tx^2$, so $d_{G(j)}(a) = d_{G'(j)}(v_1)/\eta(v_1) = (2tx + 4tx^2)/(2tx) = 1 + 2x = b(v)$ for each $a \in \phi^{-1}(v_1)$ by Theorem 3.20 (i). Using (ii), (iv), (vii), and (ix), $d_{G'(2)}(v_1) = (2t-1)x + (2t-1)x + 2x + 2(2tx^2 - tx) = 2tx + 4tx^2$, so $d_{G(2)}(a) = d_{G'(2)}(v_1)/\eta(v_1) = (2tx + 4tx^2)/(2tx) = 1 + 2x = b(v)$ for each $a \in \phi^{-1}(v_1)$ by Theorem 3.20 (i). Similarly, for $3 \leq j \leq 2t+1$, using (iii), (vi), and (x), $d_{G'(j)}(v_2) = (2t-2)x + 2x + 4tx^2 = 2tx + 4tx^2$, so $d_{G(j)}(a) =$

$d_{G'(j)}(v_2)/\eta(v_2) = (2tx + 4tx^2)/(2tx) = 1 + 2x = b(v)$ for each $a \in \phi^{-1}(v_2)$ by Theorem 3.20 (i). Using (iii), (v), (viii), and (ix), $d_{G'(1)}(v_2) = (2t-1)x+(2t-1)x+2x+2(2tx^2-tx) = 2tx + 4tx^2$, so $d_{G(1)}(a) = d_{G'(1)}(v_2)/\eta(v_2) = (2tx + 4tx^2)/(2tx) = 1 + 2x = b(v)$ for each $a \in \phi^{-1}(v_2)$ by Theorem 3.20 (i).

We will now check there is exactly one edge between each pair of vertices in G. For $b, b' \in (U \cup W)$, by (i), $m_{G'}(b, b') = 1$, so $m_G(a, a') = m_{G'}(b, b')/(\eta(b)\eta(b')) = 1/((1)(1)) = 1$ for each $a \in \phi^{-1}(b)$ and $a' \in \phi^{-1}(b')$ by Theorem 3.20 (iii). For each $u_i \in U$ and $v_j \in V$, by (vi) if $i \neq j$, (vi) and (vii) if $i = j = 1$, or (vi) and (viii) if $i = j = 2$, $m_{G'}(u_i, v_j) = 2tx$, $m_{G'}(u_i, v_j) = (2tx - 2x) + 2x = 2tx$, or $m_{G'}(u_i, v_j) = (2tx - 2x) + 2x = 2tx$ respectively. Therefore, $m_G(a, a') = m_{G'}(u, v)/(\eta(u)\eta(v)) = 2tx/((1)(2tx)) = 1$ for each $u \in U$, $v \in V$, $a \in \phi^{-1}(u)$ and $a' \in \phi^{-1}(v)$ by Theorem 3.20 (iii). For each $w \in W$ and $v \in V$, by either (ii) and (iv) or (iii) and (v), $m_{G'}(w, v) = x(2t - 1) + x = 2tx$, so $m_G(a, a') = m_{G'}(w, v)/(\eta(w)\eta(v)) = 2tx/((1)(2tx)) = 1$ for each $a \in \phi^{-1}(w)$ and $a' \in \phi^{-1}(v)$ by Theorem 3.20 (iii). For each distinct $v, v' \in V$, by (x), $m_{G'}(v, v') = 4t^2x^2$, so

$$m_G(a, a') = m_{G'}(v, v')/(\eta(v)\eta(v')) = 4t^2x^2/((2tx)(2tx)) = 1$$

for each $a \in \phi^{-1}(v)$ and $a' \in \phi^{-1}(v')$ by Theorem 3.20 (iii). Finally, for $1 \leq i \leq 2$, by (ix) and (x), $\ell(v_i) = (2tx^2 - tx) + (2t^2x^2 - 2tx^2) = 2t^2x^2 - tx$, so $m_G(a, a') = \ell_{G'}(v_i)/\binom{\eta(v_i)}{2} = (2t^2x^2 - tx)/(\frac{(2tx)(2tx-1)}{2}) = 1$ for each distinct $a, a' \in \phi^{-1}(v_i)$ by Theorem 3.20 (ii). Therefore, every vertex in G is attached to each other vertex with exactly one edge. Thus, $G = K_v$. $\square$

Example 3.28. The following is an example of a specific coloring and construction of G' which could emerge from the general construction described in Theorem 3.27.

(i) Use a near-one-factorization to properly color K_{2t+1} with vertex set $U \cup W$ with colors $\{1, 2, \ldots 2t + 1\}$ so that u_i is missing color i and w_j is missing color j.

(ii) Join w_i for $3 \leq i \leq 2t + 1$ with x edges of each color in $\{2, \ldots 2t + 1\} \setminus \{i\}$ to v_1.

(iii) Join w_i for $3 \leq i \leq 2t + 1$ with x edges of each color in $\{1, 3, 4, \ldots 2t + 1\} \setminus \{i\}$ to v_2.

(iv) Join w_i for $3 \leq i \leq 2t + 1$ with x edges of of color 2 to v_1.

(v) Join w_i for $3 \leq i \leq 2t+1$ with x edges of of color 1 to v_2.

(vi) Join u_i to v_j for $1 \leq i, j \leq 2$ with x edges of each color in $\{4, \ldots, 2t+1\}$. Additionally, join u_i and v_j with $2x$ edges of color 3 for $i \neq j$.

(vii) Join u_1 to v_1 with $2x$ edges colored 2.

(viii) Join u_2 to v_2 with $2x$ edges colored 1.

(ix) Attach $\frac{(2tx)(2tx-1)}{2} - (t-1)(2tx^2)$ loops of color 2 to v_1 and $\frac{(2tx)(2tx-1)}{2} - (t-1)(2tx^2)$ loops of color 1 to v_2.

(x) Attach $\frac{2t^2x^2}{t}$ loops of each color in $\{3, \ldots, t+1\}$ to each of v_1 and v_2. Join v_1 to v_2 using $\frac{4t^2x^2}{t}$ edges of each color in $\{t+2, \ldots 2t+1\}$.

Proposition 3.29. *Let $v = 4tx+2t+1$ for some integer x. In any equitable (s,p)-edge-coloring of K_v with $c_2 = \psi_2'(K_2, K_v)$, $c_1 = c_2 = \psi_2'(K_2, K_v)$ and $c_3 = c_4 = \cdots = c_s = v-1$.*

Proof. Since $\psi_2'(K_2, K_v) = 2tx + 2t$, by definition of color vector and Lemma 3.18,

$$\sum_{i=1}^{s} c_i \leq 2(2tx+2t) + (s-2)(v-1)$$

with equality if and only if

$$c_i = \begin{cases} 2tx + 2t & \text{for } 1 \leq i \leq 2, \text{ and} \\ v-1 & \text{for } 3 \leq i \leq s. \end{cases}$$

Because $p = 2t$ and $s = 2t+1$, we know each vertex must be missing exactly one color; so $s-1$ colors appear at each vertex. Counting in two ways, we have $v(s-1) = \sum_{i=1}^{s} c_i(E)$. The result follows since,

$$\begin{aligned} \textstyle\sum_{i=1}^{s} c_i &= (s-1)v \\ &= 2t(4tx+2t+1) \\ &= 2(2tx+2t) + 2t(4tx+2t) - 1(4xt+2t) \\ &= 2(2tx+2t) + (2t-1)(4tx+2t) \end{aligned}$$

$$= 2(2tx + 2t) + (s - 2)(v - 1). \qquad \square$$

Next, we will find $\psi_t'(K_2, K_v)$. We first note the case where $t = 1$ has been settled since $\psi_1'(K_2, K_v)$ was found in [25], as stated below.

Theorem 3.30 ([25]). *Let $v = 4tx + 2t + 1$ for some integer x. Then $\psi_1'(K_2, K_v) = \max\{b(v) + 1, 2t\}$.*

Lemma 3.31 and Lemma 3.32 use constructions similar to the well-known Walecki construction and will be used in the proof of Theorem 3.34.

Lemma 3.31. *There exists a $(t-1)$-cycle system of $2K_t$.*

Proof. Let $V(2K_t) = \{a_0, a_1, \ldots, a_{t-1}\}$. Define the cycles

$$C' = (a_0, a_1, \ldots, a_{t-2})$$

and

$$C + j = (h_{1,j}, h_{2,j}, \ldots h_{(t-2),j}, a_{t-1})$$

where $h_{i,j} = a_{((-1)^i \lfloor \frac{i}{2} \rfloor + j + 1) \pmod{t-1}}$. Thus, $\{C + j \mid 0 \le j \le t-2\} \cup \{C'\}$ is a $(t-1)$-cycle system on $2K_t$. $\square$

Lemma 3.32. *The graph $K_2^c \vee 2K_t$ with $\{b_1, b_2\}$ being the vertex set of K_2^C and $\{a_0, \ldots a_{t-1}\}$ the vertex set of K_t can be decomposed into a t-cycle on the vertices $a_0, \ldots a_{t-1}$ and t paths, each path being of length t with ends b_1 and b_2.*

Proof. Define the cycle $C = (a_0, \ldots, a_{t-1})$. Define the path

$$P + j = (b_1, h_{1,j}, h_{2,j}, \ldots h_{(t-1),j}, b_2)$$

where $h_{i,j} = a_{((-1)^i \lfloor \frac{i}{2} \rfloor + j + 1) \pmod{t}}$. Thus, $\{P + j \mid 0 \le j \le t-1\} \cup \{C\}$ is the desired decomposition. $\square$

Lemma 3.33. *Let $v = 4tx + 2t + 1$ for some integer x and $t \ge 2$. Then, $\psi_t'(K_2, K_v) \ge 4tx + 2t - 4x$.*

Proof. Using the postulated values of v, s, and i,

$$\begin{aligned}
&\lceil \tfrac{v(s-1)-(s-i)(v-1)}{i} \rceil_e \\
&= \lceil \tfrac{(4tx+2t+1)((2t+1)-1)-((2t+1)-t)((4tx+2t+1)-1)}{t} \rceil_e \\
&= \lceil \tfrac{4t^2x+2t^2-4tx}{t} \rceil_e \\
&= 4tx+2t-4x.
\end{aligned}$$

So, using Lemma 3.19, $\psi_t'(K_2, K_v) \geq \max\{b(v)+1, \lceil \frac{v(s-1)-(s-i)(v-1)}{i} \rceil_e\} \geq 4tx+2t-4x$. $\square$

Theorem 3.34. *Let $v = 4tx+2t+1$ for some integer x and $t \geq 2$. Then, $\psi_t'(K_2, K_v) = 4tx+2t-4x$.*

Proof. Because $s = 2t+1$ and $p = 2t$, we want every vertex in our final graph, $G = K_v$, to be missing exactly one color. This will be done so that colors $t+1, \ldots, 2t+1$ are each missing from exactly one vertex and colors $1, \ldots, t$ are each missing from exactly $4x+1$ vertices. This will show that $\psi_t'(K_2, K_v) \leq 4tx+2t+1-(4x+1) = 4tx+2t-4x$. Then it will follow from Lemma 3.33 that $\psi_t'(K_2, K_v) = 4tx+2t-4x$.

Let $U = \{u_1, \ldots, u_t\}$, $U' = \{u_{t+1}, \ldots u_{2t-1}\}$, $W = \{w_{2t}, w_{2t+1}\}$, and $V = \{v_1, \ldots, v_t\}$. We will form the edge-colored graphs $G_1, \ldots, G_7$ on the vertex set $U \cup U' \cup W \cup V$ as follows. In (ii), (iii), and (iv) below, z_1 and z_2 are introduced simply to ensure that v_j is missing color j and u_i is missing color i respectively when G_k is defined for $k = 2, 3, 4$. Once this is done, defining $G' = \bigcup_{i=1}^{7} G_i$, Theorem 3.20 will be used to detach G' to form $G = K_v$ with our desired coloring. To do so, $\phi : V(G) \to V(G')$ will be the amalgamation function with amalgamation numbers $\eta(u_i) = 1$ for $1 \leq i \leq 2t-1$, $\eta(w_i) = 1$ for $2t \leq i \leq 2t+1$, and $\eta(v_i) = 4x$ for $1 \leq i \leq t$.

(i) Define the graph $G_1 = K_{2t+1}$ with vertex set $U \cup U' \cup W$. Use a near-one-factorization of G_1 to properly color the edges of G_1 with colors $\{1, 2, \ldots, 2t+1\}$ so that each vertex $u_i \in U \cup U'$ is missing color i and each vertex $w_j \in W$ is missing color j.

(ii) Let $H = 2K_{t,t}$ with parts U and V, $H' = (t+1)K_2$ on the vertex set $\{z_1, z_2\}$, $H'' = K_{1,t}$ with parts $\{z_1\}$ and V, and $H''' = K_{1,t}$ with parts $\{z_2\}$ and U. Let $G_2' = H \cup H' \cup H'' \cup H'''$.

Properly edge-color G_2' with $2t+1$ colors such that the $t+1$ edges with endpoints z_1 and z_2 are colored $t+1, \ldots 2t+1$. Rename the vertices in $U \cup V$ so that color i is on edge $\{u_i, z_2\}$ and $\{v_i, z_1\}$. Form the graph $G_2 = (4x)K_{t,t}$ from G_2' with parts U and V by deleting z_1 and z_2 and then replacing each remaining edge $\{u, v\}$ in G_2' colored j with $2x$ edges joining u and v colored j in G_2.

(iii) Let $G_3' = K_{t+1,t+1}$ with parts $U' \cup W$ and $V \cup \{z_2\}$. Properly edge-color G_3' with colors $t+1, \ldots, 2t+1$ such that the edge $\{u_i, z_2\}$ is colored i and the edge $\{w_j, z_2\}$ is colored j for each $u_i \in U'$ and $w_j \in W$. Form the graph $G_3 = (2x)K_{t+1,t}$ from G_3' with parts $U' \cup W$ and V by deleting z_2 and then replacing each remaining edge $\{u, v\}$ in G_3' colored j with $2x$ edges joining u and v colored j in G_3.

(iv) Let $G_4' = K_{t,t}$ with parts $U' \cup \{z_1\}$ and V. Properly edge-color G_4' with colors $1, \ldots, t$ such that the edge $\{z_1, v_i\}$ is colored i. Form the graph $G_4 = (2x)K_{t-1,t}$ from G_4' with parts U' and V by deleting z_1 and then replacing each remaining edge $\{u, v\}$ in G_4' colored j with $2x$ edges joining u and v colored j in G_4.

(v) Let $G_5' = K_2^C \vee 2K_t$ with W being the vertex set of K_2^C and V being the vertex set of the $2K_t$. Use Lemma 3.32 to decompose G_5' into a t-cycle on the vertices $v_1, \ldots v_t$ and t paths, each path being of length t with ends w_{2t} and w_{2t+1}. Color the edges of the t-cycle with color $t+1$. Each path P_i, for $1 \leq i \leq t$, is missing exactly one vertex, namely v_i; color the edges of P_i with color i. Define the graph $G_5 = (2x)(K_2^C \vee 2K_t)$ with W being the vertex set of the K_2^C and V being the vertex set of the K_t by replacing each edge $\{u, v\}$ in G_5' colored j with $2x$ edges joining u and v colored j in G_5.

(vi) Let $G_6' = 2K_t$ with vertex set V. By Lemma 3.31, G_6' can be decomposed into $(t-1)$-cycles, $C_1, \ldots, C_t$, where, for $1 \leq i \leq t$, C_i is missing exactly one vertex, namely v_i. Color the edges of C_i with color i for $1 \leq i \leq t$. Define the graph $G_6 = (8x^2 - 4x)K_t$ with vertex set V by replacing each edge $\{u, v\}$ in G_6' colored j with $4x^2 - 2x$ edges joining u and v colored j in G_6.

(vii) Let G_7 be formed from $(8x^2)K_t$ on vertex set V by adding $8x^2-2x$ loops to each vertex. Partition the loops into edge-disjoint 2-factors (each loop contributes 2 to the degree of its incident vertex) and use Theorem 3.21 to partition $(8x^2)K_t$ into edge-disjoint 2-factors. Therefore, G_7 is decomposed into $8x^2-2x+\frac{8x^2(t-1)}{2}=4x^2-2x+4x^2t$ 2-factors. By appropriately combining the 2-factors, decompose G_7 into one $(8x^2-4x)$-factor, coloring its edges $(t+1)$, and t $(8x^2)$-factors, coloring their edges $t+2,\dots,2t+1$ in turn.

As indicated earlier in the proof, we now define $G'=\bigcup_{i=1}^{7}G_i$. Apply Theorem 3.20 to G' using the number function η defined earlier to form G. We now show that $G=K_v$ and that the resulting edge-coloring of G has the desired properties.

We first show the vertices in the set U, U', and W are incident in G' to $b(v)$ edges of each color with one exception, a color which appears on no edge, namely color i for each $u_i\in U\cup U'$ and color j for each $w_j\in W$. We also conclude what this means for G. For $1\le i\le t$, $1\le j\le 2t+1$, and $i\ne j$, using (i) and (ii), $d_{G'(j)}(u_i)=1+2x=b(v)$, so $d_{G(j)}(a)=d_{G'(j)}(u_i)/\eta(u_i)=b(v)/1=b(v)$ for each $a\in\phi^{-1}(u_i)$ by Theorem 3.20 (i). For $t+1\le i\le 2t-1$ and $t+1\le j\le 2t+1$, using (i) and (iii), $d_{G'(j)}(u_i)=1+2x=b(v)$, so $d_{G(j)}(a)=d_{G'(j)}(u_i)/\eta(u_i)=b(v)/1=b(v)$ for each $a\in\phi^{-1}(u_i)$ by Theorem 3.20 (i). For $t+1\le i\le 2t-1$ and $1\le j\le t$, using (i) and (iv), $d_{G'(j)}(u_i)=1+2x=b(v)$, so $d_{G(j)}(a)=d_{G'(j)}(u_i)/\eta(u_i)=b(v)/1=b(v)$ for each $a\in\phi^{-1}(u_i)$ by Theorem 3.20 (i). For $2t\le i\le 2t+1$, $t+1\le j\le 2t+1$, and $i\ne j$, using (i) and (iii), $d_{G'(j)}(w_i)=1+2x=b(v)$, so $d_{G(j)}(a)=d_{G'(j)}(w_i)/\eta(w_i)=b(v)/1=b(v)$ for each $a\in\phi^{-1}(w_i)$ by Theorem 3.20 (i). For $2t\le i\le 2t+1$ and $1\le j\le t$, using (i) and (v), $d_{G'(j)}(w_i)=1+2x=b(v)$, so $d_{G(j)}(a)=d_{G'(j)}(w_i)/\eta(w_i)=b(v)/1=b(v)$ for each $a\in\phi^{-1}(w_i)$ by Theorem 3.20 (i).

We will now check that each vertex detached from v_i with $1\le i\le t$ is incident in G to $b(v)$ edges of all colors with one exception, a color which appears on no edge, namely color i for each vertex detached from v_i. For $1\le i\le t$, $1\le j\le t$, and $i\ne j$, by (ii), (iv), (v), and (vi), $d_{G'(j)}(v_i)=2x+2x+2(2x)+2(4x^2-2x)=8x^2+4x$, so $d_{G(j)}(a)=d_{G'(j)}(v_i)/\eta(v_i)=(8x^2+4x)/(4x)=2x+1=b(v)$ for each $a\in\phi^{-1}(v_i)$ by Theorem 3.20 (i). For $1\le i\le t$ and $j=t+1$, by (ii), (iii), (v), and (vii), $d_{G'(j)}(v_i)=2x+2x+2(2x)+(8x^2-4x)=8x^2+4x$, so $d_{G(j)}(a)=d_{G'(j)}(v_i)/\eta(v_i)=(8x^2+4x)/(4x)=2x+1=b(v)$ for each $a\in\phi^{-1}(v_i)$ by

Theorem 3.20 (i). For $1 \leq i \leq t$ and $t+2 \leq j \leq 2t+1$, by (ii), (iii), and (vii), $d_{G'(j)}(v_i) = 2x+2x+8x^2 = 8x^2+4x$, so $d_{G(j)}(a) = d_{G'(j)}(v_i)/\eta(v_i) = (8x^2+4x)/(4x) = 2x+1 = b(v)$ for each $a \in \phi^{-1}(v_i)$ by Theorem 3.20 (i).

We will now check there is exactly one edge between each pair of vertices in G. By (i), $m_{G'}(b,b') = 1$ for $b,b' \in (U \cup U' \cup W)$, so $m_G(a,a') = m_{G'}(b,b')/(\eta(b)\eta(b')) = 1/((1)(1)) = 1$ for each $a \in \phi^{-1}(b)$ and $a' \in \phi^{-1}(b')$ by Theorem 3.20 (iii). For each $u \in U$ and $v \in V$, by (ii), $m_{G'}(u,v) = 4x$, so $m_G(a,a') = m_{G'}(u,v)/(\eta(u)\eta(v)) = 4x/((1)(4x)) = 1$ for each $a \in \phi^{-1}(u)$ and $a' \in \phi^{-1}(v)$ by Theorem 3.20 (iii). For $u' \in U'$ and $v \in V$, by (iii) and (iv), $m_{G'}(u',v) = 2x+2x = 4x$, so $m_G(a,a') = m_{G'}(u',v)/(\eta(u')\eta(v)) = 4x/((1)(4x)) = 1$ for each $a \in \phi^{-1}(u')$ and $a' \in \phi^{-1}(v)$ by Theorem 3.20 (iii). For $w \in W$ and $v \in V$, by (iii) and (v), $m_{G'}(w,v) = 2x+2x = 4x$, so $m_G(a,a') = m_{G'}(w,v)/(\eta(w)\eta(v)) = 4x/((1)(4x)) = 1$ for each $a \in \phi^{-1}(w)$ and $a' \in \phi^{-1}(v)$ by Theorem 3.20 (iii). For $v,v' \in V$, by (v), (vi), and (vii), $m_{G'}(v,v') = 4x + (8x^2-4x) + 8x^2 = 16x^2$, so $m_G(a,a') = m_{G'}(v,v')/(\eta(v)\eta(v')) = 16x^2/((4x)(4x)) = 1$ for each $a \in \phi^{-1}(v)$ and $a' \in \phi^{-1}(v')$ by Theorem 3.20 (iii). Finally, for $1 \leq i \leq t$ by (vii), v_i has $8x^2-2x$ loops, so $m_G(a,a') = \ell_{G'}(v_i)/\binom{\eta(v_i)}{2} = (8x^2-2x)/(\frac{(4x)(4x-1)}{2}) = 1$ for each distinct $a,a' \in \phi^{-1}(v_i)$ by Theorem 3.20 (ii). Therefore, every vertex in G is attached to each other vertex with exactly one edge. Thus, $G = K_v$. $\square$

Proposition 3.35. *Let $v = 4tx+2t+1$ for some integer x. In any equitable (s,p)-edge-coloring of K_v with $c_t = \psi'_t(K_2,K_v)$, $c_1 = c_2 = \cdots = c_t = \psi'_t(K_2,K_v)$ and $c_{t+1} = c_{t+2} = \cdots = c_s = v-1$.*

Proof. Since $\psi'_t(K_2,K_v) = 4tx+2t-4x$, by definition of color vector and Lemma 3.18,

$$\sum_{i=1}^{s} c_i \leq t(4tx+2t-4x) + (s-t)(v-1)$$

with equality if and only if

$$c_i = \begin{cases} 4tx+2t-4x & \text{for } 1 \leq i \leq t \text{, and} \\ v-1 & \text{for } t+1 \leq i \leq s. \end{cases}$$

Because $p = 2t$ and $s = 2t+1$, we know each vertex must be missing exactly one color; so $s-1$ colors appear at each vertex. Counting in two ways, we have $v(s-1) = \sum_{i=1}^{s} c_i(E)$. The result follows since,

$$\begin{aligned}
\textstyle\sum_{i=1}^{s} c_i &= (s-1)v \\
&= 2t(4tx + 2t + 1) \\
&= t(4tx + 2t - 4x) + t(4tx + 2t) + 4xt + 2t \\
&= t(4tx + 2t - 4x) + (t+1)(4tx + 2t) \\
&= t(4tx + 2t - 4x) + (s-t)(v-1).
\end{aligned}$$

□

The following corollary of our main theorems obtains additional values $\psi_i'(K_2, K_v)$.

Corollary 3.36. *If $i < j$ and*

$$\max\{b(v)+1, \lceil \frac{v(s-1)-(s-i)(v-1)}{i} \rceil_e\} = \psi_j'(K_2, K_v),$$

then $\psi_i'(K_2, K_v) = \psi_j'(K_2, K_v)$.

Proof. By Lemma 3.19, $\psi_i'(K_2, K_v) \geq \max\{b(v)+1, \lceil \frac{v(s-1)-(s-i)(v-1)}{i} \rceil_e\}$. Using assumption, $\psi_i'(K_2, K_v) \geq \psi_j'(K_2, K_v)$. By definition of color vector, $\psi_i'(K_2, K_v) \leq \psi_j'(K_2, K_v)$. Thus, $\psi_i'(K_2, K_v) = \psi_j'(K_2, K_v)$. □

The significance of the previous corollary is demonstrated in the following example. It is extremely useful when $j > t$ which is seen in Example 3.37 (ii).

Example 3.37.

(i) Let $x = 1$ and $t = 5$. Thus, $v = 4tx + 2t + 1 = 31$. By Theorem 3.34, $\psi_5'(K_2, K_v) = 4tx + 2t - 4x = 26$. Also, for $i = 4$,

$$\begin{aligned}
&\max\{b(v)+1, \lceil \tfrac{v(s-1)-(s-i)(v-1)}{i} \rceil_e\} \\
&= \max\{(2(1)+1)+1, \lceil \tfrac{31(11-1)-(11-4)(31-1)}{4} \rceil_e \\
&= 26.
\end{aligned}$$

By Corollary 3.36, $\psi_4'(K_2, K_v) = \psi_5'(K_2, K_v) = 26$.

(ii) Let $x = 2$ and $t = 4$. Thus, $v = 4tx + 2t + 1 = 41$. By Theorem 3.24, $\psi_8'(K_2, K_v) = 4tx + 2t - 2x = 36$. Also, for $i = 6$ and $i = 7$, $\max\{b(v) + 1, \lceil \frac{v(s-1)-(s-i)(v-1)}{i} \rceil_e\} = 36$. By Corollary 3.36, $\psi_6'(K_2, K_v) = \psi_7'(K_2, K_v) = \psi_8'(K_2, K_v) = 36$.

3.3.3 Extension to C_4-Decompositions

In order to be consistent with notation in previous literature, we define $v' = 2v \equiv 4t + 2 \pmod{8t}$. Now we will combine the main theorems from the last section with the techniques used in [22, 21, 25] to find $\psi_{2t}'(C_4, K_{v'} - F)$, $\psi_2'(C_4, K_{v'} - F)$, and $\psi_t'(C_4, K_{v'} - F)$. Although they could be called corollaries, we call them theorems here because they may be of most interest to design theorists.

Recall, Lemma 3.1 which is stated in more detail below.

Lemma 3.1. *If there exists an equitable (s, p)-edge-coloring E of G, then there exists an equitable (s, p)-C_4-coloring E' of $G \times 2$. Futhermore, $2c_i(E) = c_i(E')$ for $1 \leq i \leq s$.*

Lemma 3.38, proved in [25], will be used to establish lower bounds for $\psi_i'(C_4, K_{v'} - F)$ in Lemma 3.39.

Lemma 3.38 ([25])**.** *Let $v' \equiv 4t + 2 \pmod{8t}$. In any equitable $(2t + 1, 2t)$-C_4-coloring E of $K_{v'} - F$, for $1 \leq i \leq 2t + 1$,*

(i) 4 *divides* $c_i(E)$,

(ii) $c_i(E) \geq 2(b'(v') + 1) = 2(\frac{v-1}{p} + 1)$, *and*

(iii) $c_i(E) \leq v' - 2$.

Lemma 3.39 generalizes results in [21] and [25] that considered the $i = 2t + 1$ and $i = 1$ cases respectively. The statement and proof of Lemma 3.39 is similar to Lemma 3.19. Let $\lceil a \rceil_{d:4}$ be the smallest integer greater than or equal to a and divisible by 4.

Lemma 3.39. *Let $v' = 8tx+4t+2$ for some integer x. Then, $\psi_i'(C_4, K_{v'}-F) \geq \max\{2(b'(v')+1), \lceil \frac{v'(s-1)-(s-i)(v'-2)}{i} \rceil_{d:4}\}$ for $1 \leq i \leq s$.*

Proof. Let E be any equitable $(2t+1, 2t)$-coloring of a C_4 decompostion of $K_{v'} - F$. Because $p = 2t$ and $s = 2t+1$, we know each vertex must be missing exactly one color; so $s-1$ colors appear at each vertex. Counting in two ways, we have $v'(s-1) = \sum_{i=1}^{s} c_i(E)$. By the definition of the color vector, $c_1(E) \leq c_2(E) \leq \cdots \leq c_i(E)$. By Lemma 3.38 (iii), $c_j(E) \leq v' - 2$ in particular for the $s-i$ values $(i+1) \leq j \leq s$. Thus, $\sum_{i=1}^{s} c_i(E) \leq ic_i(E) + (s-i)(v'-2)$. So, $v'(s-1) \leq ic_i(E) + (s-i)(v'-2)$. Therefore, $\lceil \frac{v'(s-1)-(s-i)(v'-2)}{i} \rceil \leq c_i(E)$. By Lemma 3.38 (i), $c_i(E)$ must be divisible by 4. Thus, $\lceil \frac{v'(s-1)-(s-i)(v'-2)}{i} \rceil_{d:4} \leq c_i(E)$. By Lemma 3.38 (ii) $c_i(E) \geq 2(b'(v')+1)$. Therefore, $\psi'_i(K_2, K_v) \geq \max\{2(b'(v')+1), \lceil \frac{v'(s-1)-(s-i)(v'-2)}{i} \rceil_{d:4}\}$ for $1 \leq i \leq s$. □

Next, we use Lemma 3.1, Lemma 3.39, and the main theorems from Subsection 3.3.2 to find $\psi'_{2t}(C_4, K_{v'} - F)$, $\psi'_2(C_4, K_{v'} - F)$, and $\psi'_t(C_4, K_{v'} - F)$.

Theorem 3.40. *Let* $v' = 8tx+4t+2$ *for some integer* x*. Then,* $\psi'_{2t}(C_4, K_{v'}-F) = 8tx+4t-4x$.

Proof. By Lemma 3.1 and Theorem 3.24, $\psi'_{2t}(C_4, K_{v'} - F) \leq 2(4tx + 2t - 2x)$. Also, using the postulated values of v', s, and i,

$$\begin{aligned}
&\lceil \tfrac{v'(s-1)-(s-i)(v'-2)}{i} \rceil_{d:4} \\
&= \lceil \tfrac{(8tx+4t+2)((2t+1)-1)-((2t+1)-2t)((8tx+4t+2)-2)}{2t} \rceil_{d:4} \\
&= \lceil \tfrac{16t^2x+8t^2-8tx}{2t} \rceil_{d:4} \\
&= 8tx + 4t - 4x
\end{aligned}$$

By Lemma 3.39, $\psi'_{2t}(C_4, K_{v'}-F) \geq 8tx+4t-4x$. Thus, $\psi'_{2t}(C_4, K_{v'}-F) = 8tx+4t-4x$. □

Theorem 3.41. *Let* $v' = 8tx + 4t + 2$ *for some integer* x*. Then,* $\psi'_2(C_4, K_{v'} - F) = 4tx + 4t$.

Proof. By Lemma 3.1 and Theorem 3.27, $\psi'_2(C_4, K_{v'} - F) \leq 2(2tx + 2t)$. Also, using the postulated values of v', s, and i,

$$\begin{aligned}
&\lceil \tfrac{v'(s-1)-(s-i)(v'-2)}{i} \rceil_{d:4} \\
&= \lceil \tfrac{(8tx+4t+2)((2t+1)-1)-((2t+1)-2)((8tx+4t+2)-2)}{2} \rceil_{d:4} \\
&= \lceil \tfrac{8tx+8t}{2} \rceil_{d:4} \\
&= 4tx + 4t.
\end{aligned}$$

By Lemma 3.39, $\psi_2'(C_4, K_{v'} - F) \geq 4tx + 4t$. Thus, $\psi_2'(C_4, K_{v'} - F) = 4tx + 4t$. $\square$

Next, we will find $\psi_t'(C_4, K_{v'} - F)$. We first note that the case where $t = 1$ has been settled since $\psi_1'(C_4, K_{v'} - F)$ was found in [25], as stated below.

Theorem 3.42 ([25])**.** *Let $v' = 8tx + 4t + 2$ for some integer x. Then, $\psi_1'(C_4, K_{v'} - F) = \max\{2(b'(v') + 1), 4t\}$.*

Theorem 3.43. *Let $v' = 8tx+4t+2$ for some integer x. Assume $t \geq 2$. Then, $\psi_t'(C_4, K_{v'}-F) = 8tx + 4t - 8x$.*

Proof. By Lemma 3.1 and Theorem 3.34, $\psi_t'(C_4, K_{v'} - F) \leq 2(4tx + 2t - 4x)$. Also, using the postulated values of v', s, and i,

$$\begin{aligned}
&\lceil \tfrac{v'(s-1)-(s-i)(v'-2)}{i} \rceil_{d:4} \\
&= \lceil \tfrac{(8tx+4t+2)((2t+1)-1)-((2t+1)-t)((8tx+4t+2)-2)}{t} \rceil_{d:4} \\
&= \lceil \tfrac{8t^2x+4t^2-8tx}{t} \rceil_{d:4} \\
&= 8tx + 4t - 8x.
\end{aligned}$$

By Lemma 3.39, $\psi_{2t}'(C_4, K_{v'}-F) \geq 8tx+4t-8x$. Thus, $\psi_t'(C_4, K_{v'}-F) = 8tx+4t-8x$. $\square$

Proposition 3.44. *Let $v' = 8tx+4t+2$ for some integer x. In any equitable (s,p)-C_4-coloring of $K_{v'}$ with $c_i = \psi_i'(K_2, K_v)$ for $i \in \{2, t, 2t\}$, $c_1 = c_2 = \cdots = c_i = \psi_i'(K_2, K_v)$ and $c_{i+1} = c_{i+2} = \cdots = c_s = v - 1$.*

The proof of the previous proposition is similar to the proofs of Proposition 3.25, Proposition 3.29, and Proposition 3.35.

Corollary 3.45. *If $i < j$ and $\max\{2(b'(v') + 1), \lceil \frac{v'(s-1)-(s-i)(v'-2)}{i} \rceil_{d:4}\} = \psi_j'(C_4, K_{v'})$, then $\psi_i'(C_4, K_{v'}) = \psi_j'(C_4, K_{v'})$.*

The proof of the previous corollary is similar to the proof of Corollary 3.36.

3.4 Future Directions

This section summarizes possible future directions. The first question is a natural generalization of our main theorems from Section 3.3 and would find the remaining minimum values of the color vector.

Question 3.46. *Let $v = 4tx + 2t + 1$ for some integer x. What is $\psi_i'(K_v)$ for $3 \leq i \leq t - 1$ and $t + 1 \leq i \leq 2t - 1$?*

We conjecture the correct answer is equal to the lower bound established in Lemma 3.19. With the previous techniques this seems like a difficult problem, so new techniques may need to be explored.

The next question was described in detail in Subsection 3.2.2.

Question 3.47. *What is $\chi_p'(K_2, \lambda K_v)$?*

Partial results for this question were stated in Subsection 3.2.2. If Conjecture 3.16 (stated below again for convenience) is true and we modify Theorem 3.3 appropriately, we will have a complete solution to this question.

Conjecture 3.16. *Suppose v is odd, $p \mid \lambda(v-1)$, and $\lambda(v-1)/p$ is odd. Then there exists an equitable $(p + \lceil \frac{p}{v-1} \rceil, p)$-edge-coloring of λK_v.*

We have made progress toward this conjecture using the amalgamation technique.

Finally, a common question to ask in design theory is whether one design can be embedded into a larger design. This leads to the next question.

Question 3.48. *Let $v_1 = 4t_1x + 2t_1 + 1$ and $v_2 = 4t_2x + 2t_2 + 1$ where $x \geq 0$ and $t_1, t_2 \geq 1$ are integers and $t_1 < t_2$. Can an equitable $(2t_1 + 1, 2t_1)$-edge-coloring of K_{v_1} be embedded in an equitable $(2t_2 + 1, 2t_2)$-edge-coloring of K_{v_2}?*

This problem also seems approachable with the amalgamation technique.

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