Intersection Structures on Secants of Grassmannians

by

Dalton Bidleman

A dissertation submitted to the Graduate Faculty of
Auburn University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

Auburn, Alabama
August 05, 2023

Keywords: Grassmannians, Algebraic Geometry, Coding Theory

Approved by

Luke Oeding, Chair, Associate Professor of Mathematics and Statistics
Hal Schenck, Professor and Rosemary Kopel Brown Eminent Scholars Chair
Michael Brown, Assistant Professor of Mathematics and Statistics
Scott Martin, Assistant Professor of Mechanical Engineering

Abstract

Restricted secant varieties of Grassmannians are constructed from sums of points corresponding to k-planes with the restriction that their intersection has a prescribed dimension. This thesis calculates dimensions of restricted, cyclic, and path geometric secants of Grassmannians and relate them to the analogous question for secants of Grassmannians via an incidence variety construction. Next, it defines a notion of expected dimension and gives a formula, which holds if the BDdG conjecture [7, Conjecture 4.1] on non-defectivity of Grassmannians is true, for the dimension of all restricted secant varieties of Grassmannians. It also demonstrates example calculations in Macaulay 2 and points out ways to make these calculations more efficient. The thesis also shows a potential application to coding theory.

Acknowledgments

I would like to begin by thanking my mama, Billie Jo Monroe Bidleman. She inspired my love for reading and learning and for that I am grateful. Next, I would like to thank my dad, Floyd, for loving math and always asking me questions that helped clarify my understanding. To my brother Galen, thank you for always keeping me focused and helping me whenever I had any problems. My brother, Nolan, who lived with me the last year, was a sense of calm in a difficult process and someone that could speak the language of mathematics with me when most can not.

My advisor Dr. Luke Oeding took a chance on me and for that I am ever grateful. He pushed me and was always a model for what a mathematician should be. There are so many graduate students who I could thank, but I would specifically like to mention Matthew Speck. He and I share an advisor and many times he was the one person who gave me an idea or someone to talk to when I got stuck. Finally, Stacie Baumann and Josey Graves were my office mates and the two who would always talk through my problems and help me in any way they could.

Table of Contents

Chapter 1

Introduction

Secant varieties are fundamental objects in algebraic geometry. Given a projective variety $X \subset \mathbb{P}^n$, the k-secant variety is the closure of all points that have X-rank $\leq k$, i.e. those of the form $[v] = [x_1 + \cdots + x_k]$ with $[x_i] \in X$ for all i. Such decompositions have many applications since one can view an X-rank decomposition as recovering the information stored in the $[x_i]$ from $[v]$ [6, 23]. However, it is often not possible to choose the x_i independently and this could lead to defectivity of the secant variety.

For $X \subset \mathbb{P}V$ invariant under the action of a subgroup $G \subset \mathrm{GL}(V)$ secant varieties of X inherit this G-invariance. Hence secant varieties can be part of a classification of orbits [1, 5, 29, 36]. One seeks easy ways to compute invariants that permit the separation of orbits, perhaps the first of which is dimension.

Terracini's lemma [37] reduces the dimension of the secant variety of a variety X to a dimension count for a sum of linear spaces. This count is usually correct (as long as the spaces don't intersect), so when it fails for the k-secant variety one says that X is k-defective.

The problem of finding defectivity of varieties dates back to the 19th century with the proof of the defectivity of the Veronese surface and was studied by classical algebraic geometers; for example see the works of F. Palatini, A. Terracini and G.Scorza [30, 32–35]. While work in this area continued, the 1990s saw two major breakthroughs that reinvigorated the study of defectivity of varieties. First, Zak's 1993 work [37] studied tangential and secant varieties and classified Severi varieties via a connection to defectivity of $\sigma_2(X)$. From there, Alexander and Hirschowitz [3, 4] completely classified the dimensions of the Veronese variety, proving that apart from the quadratic Veronese varieties and four specific cases, all higher secant varieties of Veronese varieties have expected dimension.

Though the Veronese case is completely resolved, the non-symmetric (Segre) and skew-symmetric (Grassmann) analogues are still open. In this thesis we focus on the Grassmann case. In 2002, Geramita and Gimigiliano proved that $\sigma_3(\mathrm{Gr}(3,7))$ and $\sigma_4(\mathrm{Gr}(3,9))$ are defective [13]. Using a combination of theory and computation, Baur, Draisma and de Graaf in [7] conjectured that the list of known counterexamples was complete:

Conjecture 1.0.1. *Baur-Draisma-de Graaf* [7] *The only cases where the secant variety of the Grassmannian* $\sigma_s(\mathrm{Gr}(k,n))$ *is defective are the following:*

1. $\sigma_k(\mathrm{Gr}(2,n))$ *(skew symmetric matrices),*
2. $\sigma_3(\mathrm{Gr}(3,7))$,
3. $\sigma_4(\mathrm{Gr}(3,9))$,
4. $\sigma_3(\mathrm{Gr}(4,8))$,
5. $\sigma_4(\mathrm{Gr}(4,8))$.

In 2013, Boralevi improved the list of known cases and gave the following theorem which is the current state of known non-defective cases of secants of Grassmannians [11].

Theorem 1.0.2. *If* $k \geq 3, k \leq \frac{n}{2}$ *then* $\sigma_s(\mathrm{Gr}(k,n)$ *has the expected dimension:*

1. *for* $n \geq 15$, *all* k *and* s, *except* $(k,n;s) = (3,7;3),(4,8;3),(4,8;4),(3,9;4)$;
2. *for* $n > 15$, $k \geq 7$, $s \leq max\{111, \frac{n-k+3}{3}\}$;
3. *for* $n > 15$, $3 \leq k \leq 6$, *as follows:*
   (a) $k = 3$, $s \leq max\{12, \frac{n}{3}\}$
   (b) $k = 4$, $s \leq max\{30, \frac{n-1}{3}\}$
   (c) $k = 5$, $s \leq max\{59, \frac{n-2}{3}\}$
   (d) $k = 6$, $s \leq max\{90, \frac{n-3}{3}\}$.

A unifying feature of the defective cases and some of the larger cases that have not been classified is a complex intersection structure that affects the dimension. Fulton and Harris in their textbook on representation theory offer, as exercise [17, Ex. 15.44], an initial direction to follow in studying this more complicated intersection structure. A collection of k-planes that are all forced to share an r-dimensional overlap is called a restricted secant of Grassmannians. These restricted secants as well as secants of Grassmannians whose intersection structures resemble a path or cycle graphically, can be entirely described in terms of secants of Grassmannians. The techniques and constructions used within are a part of a larger program attempting to show that computing the dimension of secants of Grassmannians with more complex intersection structures can be reduced to computing dimensions of secants of Grassmannians.

In Chapter II basic definitions and concepts used to study Grassmannians, secants, and tensors are established. Notation is defined and the description of the necessary background information and proof of basic lemmas occurs. The focus is on inheritance, and this results in a proof of a version of the orbit-stabilizer theorem for subspaces to show how after a certain value of n, families of k-planes have the expected dimension and can be expressed in terms of the specific dimension at step n and a correction term.

The next section provides the algorithmic approach used to compute the dimension of a parameterized variety with a computer algebra software, in this case Macaulay2 [18]. It also highlights the drawbacks of the built in functions of M2 for the problem and provides the code to get around those. Next, is the classification of restricted secants of varieties. For a specific case of the restricted chordal variety, $\sigma_2^1(Gr(3,n))$, a version of Terracini's lemma is used to calculate the dimension which mimics the known approach. After this is a classification of the other restricted chordal varieties. Shifting to higher order restricted secants of Grassmannians, the main theorem of the work is provided. This is then used, along with the BDdG conjecture, to define the dimension of restricted secants of Grassmannians problem as a secant of Grassmannian problem. The section ends with an application to coding theory. Given the relevant definitions, creating a linear code using a restricted secant of Grassmannians is defined. For a specific code over a binary field, the code is shown to be identifiable, its orbits are classified, and it validates the construction utilized in the main theorem.

Chapter III is a study of two different intersection structures originating in graph theory. These Grassmannians have a Schouten diagram represented by a cycle or a path. Dimension of both cyclic and path secants of Grassmannians are classified again entirely in terms of secant of Grassmannians. The proof techniques in this chapter give a second description of the restricted chordal varieties.

Chapter VI is a collection of unsolved problems that will be the focus of future work. Within the appendix, there is a collection of Macaulay2 code used to generate examples.

Chapter 2

Restricted Secant Varieties of Grassmannians

Chapter 2 is taken from joint work completed with Dr. Luke Oeding [10].

2.1 Preliminaries and Notation

Let V, W denote complex finite-dimensional vector spaces. Given a projective variety $X \subset \mathbb{P}V$ let $\widehat{X}$ denote the cone in V. The Grassmann variety is the collection of k-dimensional subspaces of V denoted $\mathrm{Gr}(k, V)$, or $\mathrm{Gr}(k, n)$ if the ambient space V is n-dimensional. The Plücker embedding maps $\mathrm{Gr}(k, V)$ into $\mathbb{P}\bigwedge^k V$ as follows. Given a k-plane E, select a basis $e_1, \ldots, e_k$ of E and send it to the class of the wedge product $[e_1 \wedge \cdots \wedge e_k]$. We will write $\widehat{E} = e_1 \wedge \cdots \wedge e_k$ for a representative on that line. One checks that this map is well-defined independent of the choice of basis of E, and that it is an embedding.

Since we work in projective space and have a skew-symmetric product we often insist that $i_1 < i_2 < \cdots < i_k$. The general linear group acts transitively on the set of k-planes, hence the Grassmannian is also the $\mathrm{GL}(V)$-orbit

$$\mathrm{Gr}(k, V) = \mathrm{GL}(V).[e_1 \wedge \cdots \wedge e_k].$$

Any nonzero element $\delta \in \bigwedge^n V$ induces an isomorphism $\bigwedge^k V \to \bigwedge^{n-k} V$ given by contraction on simple elements and extending through linearity. For instance, if $\delta = e_1 \wedge \cdots \wedge e_n$, then $\delta(e_I) = \mathrm{sgn}(I, I^\star) e_{I^\star}$, where $\star$ denotes complement on multi-indices, and $\mathrm{sgn}(I, I^\star)$ denotes the sign of the corresponding permutation of $[n]$. This induces a duality on Grassmannians

$$\mathrm{Gr}(k,V) \cong \mathrm{Gr}(\dim(V)-k,V), \quad \text{or} \quad \mathrm{Gr}(k,n) \cong \mathrm{Gr}(n-k,n). \tag{2.1}$$

For parametrized varieties the differential-geometric view of the tangent space is useful:

Definition 2.1.1. *Let $x \in X$ be a smooth point on an algebraic variety $X \subset \mathbb{P}W$. The cone over the tangent space to X at x is*

$$\widehat{T_x}X = \{\gamma'(0) \mid \gamma\colon \mathbb{C}^1 \longrightarrow X, \gamma(0) = x\}.$$

It is a standard exercise in [24, Ch. 6] for instance to verify the following expression:

$$\widehat{T_E}\,\mathrm{Gr}(k,V) = E + E^* \otimes V/E,$$

where E^* is the dual vector space, and V/E is the quotient. One finds other useful characterizations of this tangent space in [14]. We prefer the following description. The tangent space to the Grassmannian at E is spanned by $e_1 \wedge \cdots \wedge e_k$ and all square-free monomials of the form $e_{I\setminus\{i\}\cup\{j\}}$, where $I = \{1,\ldots,k\}$, $i \in I$ and $j \in \{k+1,\ldots,n\}$. This description has an interpretation using simplices. Recall that the set of multi-indices of length k, denoted $\mathbb{S}_k = \{J \subset [n] \mid |J| = k\}$, parametrizes the space of k-simplices. There is a discrete distance function called the Hamming distance d_H on $\mathbb{S}_k$, which is defined as $d_H(I,J)$ the size of the symmetric difference of I and J.

The indices that occur in the monomials in $\widehat{T_E}\,\mathrm{Gr}(k,V)$ correspond to all simplices in a Hamming ball of radius 1 centered at the standard k-simplex, which one can show contains $k(n-k)+1$ simplices.

Given a variety $X \subset \mathbb{P}V$, the X-rank of a point $[p] \in \mathbb{P}V$ is the minimal number s of points $[x_1], \dots, [x_s]$ such that p lies in the span of the x_i. One notion of tensor rank (the CP rank) is defined when X is the variety of rank-1 tensors (indecomposable tensors). The s-secant variety of X, denoted $\sigma_s(X)$, is the Zariski closure of the points of $\mathbb{P}V$ with X-rank s. Points in $\sigma_s(X)$ are said to have X-border rank s. While X-rank is not semi-continuous, X-border rank is semi-continuous by construction.

Restricted secant varieties of Grassmannians generalize [17, Ex. 15.44] as follows:

Definition 2.1.2. *The r-restricted s-secant variety of* $\mathrm{Gr}(k, V)$ *in* $\mathbb{P}\bigwedge^k V$*, is*

$$\sigma_s^r(\mathrm{Gr}(k,V)) = \overline{\left\{[\lambda_1 \widehat{E}_1 + \cdots + \lambda_s \widehat{E}_s] \mid E_i \in \mathrm{Gr}(k,V), [\lambda] \in \mathbb{P}^{s-1}, \dim(\textstyle\bigcap_{i=1}^s E_i) \geq r\right\}}.$$

Note that it is necessary to define the dimension of the intersection as being $\geq r$ rather than $= r$ to ensure the variety is non-empty. When more than 2 k-planes are involved the intersection structure is more complicated and is not in general characterized by a single number. We find it already interesting to study this case.

2.1.1 Inheritance and orbit stability

Given a family $\mathcal{F}$ of algebraic varieties one can ask what properties a variety inherits from its subvarieties coming from the same family. For example, we define an orbit family $\mathcal{F} = \mathcal{F}(W_\bullet, G_\bullet, X_\bullet)$ by the data: a chain of vector subspaces $W_\bullet = W_0 \subset \cdots \subset W_i \subset \cdots \subset W_n$, a family of groups $G_\bullet$ with $G_i \subset \mathrm{GL}(W_i)$ and a family of varieties $X_\bullet$ with $X_i \subset \mathbb{P}W_i$ and we require the property that $G_j.X_i \subset X_j$ whenever $i \leq j$. We say that orbit stability occurs at step p if $G_j.X_i = X_j$ whenever $p \leq i \leq j$ [13]. When orbit stability occurs, we can use this structure to compute the dimensions of the X_i for $i \geq p$ precisely. Specifically, when the varieties X_i are defined by the closure of a single orbit, orbit stability at step p implies that all X_i for $i \geq p$ have the same normal form (representative of an orbit on a full-dimensional open set) $n \in X_i$. Hence, we can describe the tangent spaces to the X_i at n as follows for all $i \geq p$:

$$\widehat{T}_n X_i = \widehat{T}_n X_p + \text{a correction term}.$$

To determine this correction term, we recall the following version of an orbit-stabilizer theorem for subspaces (see [24, 6.9.4] for the case when V is a line, and $G.V = G/P$ is a homogeneous variety).

Proposition 2.1.3. *Let G be a connected compact complex semisimple Lie group contained in* $\mathrm{GL}(W)$*. Given a G-module $V \subset W$, set $H = \mathrm{Stab}_G(V)$. Then*

$$\dim(G.V) = \dim(G/H) + \dim(V). \tag{2.2}$$

Proof. The orbit $G.V$ can be seen as a parametrization:

$$\begin{array}{ccc} G \times V & \to & W \\ (g, v) & \mapsto & g.v \end{array}$$

The tangent space at $v = \mathrm{Id}.v$ can be computed via the Lie algebra action:

$$\widehat{T}_v G.V = v + [\mathfrak{g}, V], \tag{2.3}$$

see [22, Prop. 3.18]. The decomposition $W = V \oplus V/W$ induces a decomposition of the endomorphisms:

$$\mathrm{End}(W) = W^* \otimes W = (V^* \otimes V) \oplus (V^* \otimes W/V) \oplus (W/V^* \otimes V) \oplus (W/V^* \otimes W/V).$$

Define the corresponding subspaces $\mathfrak{g}_{ij}$ of $\mathfrak{g} \subset \mathrm{End}(W)$ via restriction. Seen as a matrix,

$$\mathfrak{g} = \begin{pmatrix} \mathfrak{g}_{00} & \mathfrak{g}_{01} \\ \mathfrak{g}_{10} & \mathfrak{g}_{11} \end{pmatrix}.$$

In addition, $\mathfrak{h} = \mathfrak{g}_{00} \oplus \mathfrak{g}_{01} \oplus \mathfrak{g}_{11}$ is the subalgebra of $\mathfrak{g}$ that stabilizes V. Hence

$$\begin{aligned}[]
[\mathfrak{g}, V] = [\mathfrak{g}_{00} \oplus \mathfrak{g}_{10} \oplus \mathfrak{g}_{01} \oplus \mathfrak{g}_{11}, V] &= [\mathfrak{g}_{00}, V] \oplus [\mathfrak{g}_{10}, V] \oplus [\mathfrak{g}_{01}, V] \oplus [\mathfrak{g}_{11}, V] \\
&= [\mathfrak{g}_{00}, V] \oplus [\mathfrak{g}_{10}, V]. \quad (2.4)
\end{aligned}$$

Since V is a G-module it is also a $\mathfrak{g}_{00}$-module and $[\mathfrak{g}_{00}, V] = V$. Moreover $v \in V$ so $v + [\mathfrak{g}, V] = v + [\mathfrak{g}_{00}, V] \oplus [\mathfrak{g}_{10}, V] = V \oplus [\mathfrak{g}_{10}, V]$. Finally, $\mathfrak{g}/\mathfrak{h} = \mathfrak{g}_{10}$, so $\dim(G/H) = \dim([\mathfrak{g}_{10}, V])$. □

We can then apply this to the family of varieties with symmetry.

Proposition 2.1.4. *Suppose an orbit family $\mathcal{F}$ achieves orbit stability at step p, and that G_i acts transitively on the set of $\dim(V_p)$-planes for each $i \geq p$. Define a fiber bundle $\Xi \to \mathrm{Gr}(\dim(V_p), V_i)$ with each fiber over $E \in \mathrm{Gr}(\dim(V_p), V_i)$ equal to a copy of $X_p \subset \mathbb{P}E$.*

Then for all $i \geq p$, X_i is birational to the total space of Ξ, and in particular

$$\dim(X_i) = \dim(X_p) + \dim(\mathrm{Gr}(\dim V_p, V_i)). \quad (2.5)$$

Proof. We will show that X_i is birational to the fiber bundle Ξ defined in the statement. Then the total space of Ξ has dimension equal to the dimension of the general fiber plus the dimension of the base, or $\dim(X_p) + \dim(\mathrm{Gr}(\dim V_p, V_i))$, so the "moreover" part follows.

Let $[x] \in X_i$ be a general point, so we can assume x is on the orbit $G_i.x_i$. Because of orbit stability at step p we can take x to be the normal form for X_p, $x = x_p \in X_p$. Consider the vector space $\widehat{T}_{x_p} X_p$ and take its orbit under the action of G_i. Since G_i acts transitively on $\dim(X_p)$-planes, this orbit is $\mathrm{Gr}(\dim(V_p), V_i)$, so we can send x_p to the pair $(\widehat{T}_p X_p, x_p)$, this is a rational mapping.

For the other direction, suppose we have a pair $(E, x) \in \Xi$ with $x \in \tilde{X}_p$, where $\tilde{X}_p$ denotes a copy of X_p in E. Then by the assumption that G_i acts transitively on $\dim(V_p)$-planes we can assume that the linear space E is a G_i translate of $\widehat{T}_{x_p} X_p$, hence $x \in g.X_p \subset g.\widehat{T}_n X_p$, with $g \in G_i$. What is left to show is that the composition of the two maps is the identity. Let $[x] \in X_i$ be a general point. Because of orbit stability, $x = x_p \in X_p$. Apply the first map. Take the orbit of x_p under the action of G_i. This produces a pair $(\widehat{T}_{x_p} X_p, x_p)$. Then, it is true that $x_p \in X_p$ and not just $\tilde{X}_p$, where $\tilde{X}_p$ was a copy of X_p in E. Further, since G_i acts transitively on $\dim(V_p)$-planes, applying the second map and acting on $\widehat{T}_{x_p} X_p$ by G_i means $x_p \in g.\widehat{T}_{x_p} X_p$. But we had $x_p = x$, therefore we arrive back at $[x]$. $\square$

We're interested in the case for fixed k, r, s with $W_i = \bigwedge^k V_i$, with $V_0 \subset \cdots \subset V_n$ and $V_i \cong \mathbb{C}^i$, $G_i = \mathrm{GL}(V_i)$ and $X_i = \sigma_s^r \,\mathrm{Gr}(k, V_i)$. We denote this family by $\mathcal{G}(r, s, k) = (\bigwedge^k V_\bullet, \mathrm{GL}(V_\bullet), \sigma_s^r \,\mathrm{Gr}(k, V_\bullet))$. These varieties are defined as orbit closures, and in this particular case orbit stability implies stability of normal forms.

Proposition 2.1.5. *The family $\mathcal{G}(r, s, k)$ obtains orbit stability (at least) when $p = r + s(k - r)$.*

Proof. Note the condition that $\mathcal{G}(r, s, k)$ obtains orbit stability at step p, where $p = \dim(V_p)$ and $\sigma_s^r(\mathrm{Gr}(k, V_p))$ can be guaranteed at the first instance where there are enough linearly independent basis vectors to define a general point $x \in \sigma_s^r(\mathrm{Gr}(k, V_p))$ with no additional intersection. Now count independent parameters. For a given, r, s, k there is an r-dimensional overlap which accounts for r elements $e_i \in V$ and additionally each of the s copies of the Grassmannian requires $k - r$ more e_i elements for a total of $r + s(k - r)$. These $e_i \in V$ can be chosen independently if $n \geq p = r + s(k - r)$. $\square$

Proposition 2.1.6. *Suppose $\mathcal{G}(r,s,k)$ attains orbit stability at step p. For all $n \geq p$ we have*

$$\dim(\sigma_s^r \operatorname{Gr}(k,V_n)) = \dim(\sigma_s^r \operatorname{Gr}(k,V_p)) + \dim(\operatorname{Gr}(p,n))$$
$$= r(p-r) + s((k-r)(p-k)) + s - 1 + p(n-p). \quad (2.6)$$

Proof. When orbit stability occurs there exists a birational morphism:

$$G_j \times X_j \dashrightarrow X_j$$

Using the orbit-stabilizer theorem 2.1.3 we obtain a dimension count: $r(p-r)+s((k-r)(p-k))+s-1+p(n-p)$. □

Example 2.1.7. Apply this argument above to the 1-restricted case for $\sigma_s^1(\operatorname{Gr}(3,V))$. We have the following chain of inclusions

$$\operatorname{Gr}(k,a) \subset \mathbb{P}\bigwedge^k \mathbb{C}^a \subset \mathbb{P}\bigwedge^k \mathbb{C}^n.$$

Then, $\sigma_s^1(\operatorname{Gr}(k,V))$ can be found by taking the appropriate orbits of $\sigma_s^1(\operatorname{Gr}(k,a))$:

$$\operatorname{GL}(n).\sigma_s^1(\operatorname{Gr}(k,a)) \subset \sigma_s^1(\operatorname{Gr}(k,n)).$$

Consider $\sigma_3^1(\operatorname{Gr}(3,n))$ and take $a = 7$ in this case. So, $\sigma_3^1(\operatorname{Gr}(3,V))$ is birational to $\operatorname{Gr}(7,V)\times \sigma_3^1(\operatorname{Gr}(3,7))$. Therefore,

$$\dim(\sigma_3^1(\operatorname{Gr}(3,n))) = \dim(\operatorname{Gr}(7,n)) + \dim(\sigma_3^1(\operatorname{Gr}(3,7))).$$

So, since $\sigma_3^1(Gr(3,7)) = 20$ we have $\dim(\sigma_3^1(\operatorname{Gr}(3,n))) = 7\cdot(n-7)+20 = 7n-29$ for $n \geq 7$.

This leads to other explicit formulas for 1-restricted chordal varieties such as:

$$\dim(\sigma_2^1(\mathrm{Gr}(4,n))) = 7n - 24, \quad \text{for} \quad n \geq 7,$$

and

$$\dim(\sigma_2^1(\mathrm{Gr}(5,n))) = 9n - 40, \quad \text{for} \quad n \geq 9.$$

The following result of Boralevi has a similar flavor to the work in this section:

Theorem 2.1.8 ([11]Lem. 3.2, Thm. 3.3)**.** *If $\sigma_s(\mathrm{Gr}(k,n))$ has the expected dimension and does not fill its ambient space then $\sigma_s(\mathrm{Gr}(k,n+t))$ and $\sigma_s(\mathrm{Gr}(k+t,n+t))$ both have the expected dimension for every $t \geq 0$.*

2.2 Computing Dimensions Using Macaulay2

A standard method to compute the dimension of a parametrized projective variety is via differentials. Recall a parametrization is a rational mapping

$$\varphi\colon \mathbb{P}^M \to \mathbb{P}^N,$$

defined on a non-trivial open subset $U \in \mathbb{P}^M$. That the map φ is rational means that $[\varphi(x)] = [\varphi_0(x) : \ldots : \varphi_N(x)]$ with $\varphi_i(x)$ a rational function for each coordinate i. Recall that the image X of a rational mapping φ is the Zariski closure $\overline{\varphi(U)}$ and note that the definition doesn't depend on which non-trivial open subset we choose as long as φ is defined on that set. Work on the cone over U and take the total differential (the Jacobian):

$$d\varphi\colon \widehat{U} \to \mathbb{C}^{N+1},$$

noting that $\widehat{T}_pU = \mathbb{C}^{M+1}$, and $\widehat{T}_{\varphi(p)}\mathbb{C}^{N+1} = \mathbb{C}^{N+1}$. At a point $[p] \in U$, the linear mapping

$$d\varphi_p \colon \mathbb{C}^{M+1} \to \mathbb{C}^{N+1},$$

may be represented by a matrix with (i, j) entry $\frac{\partial \varphi_i}{\partial x_j}(p)$, with $0 \leq i \leq N$ and $0 \leq j \leq M$. The image of $d\varphi_p$ for general $[p] \in U$ is the tangent space $\widehat{T}_{\varphi(p)}X$, and hence the rank of $d\varphi_p$ computes the dimension of the cone $\widehat{X}$.

In summary, to compute the dimension of a parametrized variety we may

1. Generate sufficiently many random points $[p]$ of the source.

2. Compute the partial derivatives $\frac{\partial \varphi_i}{\partial x_j}(p)$ and populate the matrix $d\varphi_p$.

3. Compute the rank of the matrix $d\varphi_p$.

2.2.1 Computing dimensions of $\sigma_s^r \operatorname{Gr}(k, n)$

We verified the calculation of dimension for several families of restricted chordal varieties with Macaulay2 [18]. An example computation can be found in the ancillary files associated with the arXiv version of this article.

Since any k-dimensional subspace of an n-dimensional space can be represented as the row space of a $k \times n$ matrix, a parametrization for $\sigma_s(\operatorname{Gr}(k, n))$ is given by

$$\varphi \colon \mathbb{P}(\mathbb{C}^{k \times n})^{\times s} \to \mathbb{P}{\textstyle\bigwedge}^k \mathbb{C}^n,$$

which takes an s-tuple of $k \times n$ matrices (up to scale) to the sum of their vectors of k-minors. The open set we work on is the one where all of the matrices in question have full rank.

The Jacobian at a point p is a linear mapping

$$d\varphi \colon (\mathbb{C}^{k \times n})^{\times s} \to {\textstyle\bigwedge}^k \mathbb{C}^n,$$

whose coordinates are evaluations of derivatives of sums of minors. Its size is $\binom{n}{k} \times (kns)$. We write $d\varphi(A)$ (and similar) to indicate a symbolic Jacobian, and $d\varphi_C$ to indicate the evaluation at a point parametrized by an s-tuple of matrices C.

Similarly, a parametrization φ^r for $\sigma_s^r(\mathrm{Gr}(k,n))$ is given by restricting the source of φ to a set where the s-tuple of matrices mutually share r row vectors. This restricted source is

$$(\mathbb{C}^{r\times n}) \times (\mathbb{C}^{(k-r)\times n})^{\times s},$$

where the first factor is the shared rows. So the Jacobian $d\varphi^r$ has size $\binom{n}{k} \times (rn + (k-r)ns)$.

Focus on the case $s = 2$ for the moment, the case of general s is similar. Given two symbolic matrices A and B in $\mathbb{C}^{k\times n}$ with the first r rows of B the same as those of A (to reflect the overlap in their row spaces) we can represent the structure of the sum of the Jacobians of their Plücker images. So $d\varphi(A + B) = d\varphi(A) + d\varphi(B)$. Let $A = (a_{ij})$ with $0 \le i \le k-1$, $0 \le j \le n-1$ and $B = (b_{ij})$, with $0 \le i \le k-r-1$, $0 \le j \le n-1$. Let $d = \binom{n}{k}$ and A_I represent the maximal minor of A with columns I and order the multi-indices I lexicographically and re-name them $m_1, \ldots, m_d$. The Jacobians of the Plücker maps of A and B are the following.

$$d\varphi(A) = \begin{pmatrix} \frac{\partial A_{m_1}}{\partial a_{00}} & \cdots & \frac{\partial A_{m_d}}{\partial a_{00}} \\ \vdots & \ddots & \vdots \\ \frac{\partial A_{m_1}}{\partial a_{(k-1)(n-1)}} & \cdots & \frac{\partial A_{m_d}}{\partial a_{(k-1)(n-1)}} \end{pmatrix}$$

$$d\varphi(B) = \begin{pmatrix} \frac{\partial B_{m_1}}{\partial a_{00}} & \cdots & \frac{\partial B_{m_1}}{\partial a_{(r-1)(n-1)}} & \frac{\partial B_{m_1}}{\partial b_{00}} & \cdots & \frac{\partial B_{m_1}}{\partial b_{(k-r-1)(n-1)}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial B_{m_d}}{\partial a_{00}} & \cdots & \frac{\partial B_{m_d}}{\partial a_{(r-1)(n-1)}} & \frac{\partial B_{m_d}}{\partial b_{00}} & \cdots & \frac{\partial B_{m_d}}{\partial b_{(k-r-1)(n-1)}} \end{pmatrix}^{\top}$$

Therefore, $d\varphi(A) + d\varphi(B) =$

$$\begin{pmatrix} \frac{\partial A_{m_1}}{\partial a_{00}} + \frac{\partial B_{m_1}}{\partial a_{00}} & \cdots & \frac{\partial A_{m_1}}{\partial a_{(r-1)(n-1)}} + \frac{\partial B_{m_1}}{\partial a_{(r-1)(n-1)}} & \frac{\partial A_{m_1}}{\partial a_{(r-1)(n-1)+1}} & \cdots & \frac{\partial A_{m_1}}{\partial a_{(k-1)(n-1)}} & \frac{\partial B_{m_1}}{\partial b_{00}} & \cdots & \frac{\partial B_{m_1}}{\partial b_{(k-r-1)(n-1)}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial A_{m_d}}{\partial a_{00}} + \frac{\partial B_{m_d}}{\partial a_{00}} & \cdots & \frac{\partial A_{m_d}}{\partial a_{(r-1)(n-1)}} + \frac{\partial B_{m_d}}{\partial a_{(r-1)(n-1)}} & \frac{\partial A_{m_d}}{\partial a_{(r-1)(n-1)+1}} & \cdots & \frac{\partial A_{m_d}}{\partial a_{(k-1)(n-1)}} & \frac{\partial B_{m_d}}{\partial b_{00}} & \cdots & \frac{\partial B_{m_d}}{\partial b_{(k-r-1)(n-1)}} \end{pmatrix}^{\top}. \quad (2.7)$$

We use this block structure to make our computations more efficient.

We generate a collection $C = (C_1, \ldots, C_s)$ of random matrices $C_i \in \mathbb{C}^{k\times n}$ with the appropriate overlap of their row spaces. Via Terracini's Lemma the Jacobian $d\varphi_C$ is the sum of the differentials of the Plücker maps, $d\varphi_C(A_i)$. The rank of the resulting matrix is equal to the dimension of that restricted chordal variety (as long as the initial choice of C was sufficiently general, which it will be with probability 1).

2.2.2 Computing the dimension of secants in M2

A naive implementation to compute the dimension of a secant variety of the Grassmannian in M2 is given below:

```
testnk = (n,k) -> (
  R = QQ[a_(0,0)..a_(k-1,n-1),b_(0,0)..b_(k-1,n-1)];
  A = transpose genericMatrix(R, a_(0,0), n,k);
  B = transpose genericMatrix(R, b_(0,0), n,k);
  fun = matrix{apply(subsets(n,k), s-> det A_s + det B_s )};
  jac = diff(transpose basis(1, R), fun);
  val = map(QQ,R, random(QQ^1,QQ^(dim R)));
  rank val jac
)
```

The first 3 lines define the source variables (ring) and matrices. It then defines the mapping, `fun`, and differentiates with respect to the column vector of variables to calculate the Jacobian as a matrix. Note `M2` also has a command for Jacobian.

This gives us a set of polynomials which we can evaluate and then find the rank of the corresponding numerical matrix. Note that in this case with negligible computational time we see that the rank is 26, which is indeed the dimension of the cone over the secant of the Grassmannian $\sigma_2(\mathrm{Gr}(3,7))$.

One can modify the procedure as follows to handle the restricted secant case:

```
n = 7; k=3; r=1;
R = QQ[a_(0,0)..a_(k-1,n-1),b_(0,0)..b_(k-r-1,n-1)];
A = transpose genericMatrix(R, a_(0,0), n,k);
B = A^{0..r-1}||transpose genericMatrix(R, b_(0,0), n,k-r);
```

Note there are fewer variables needed because of the overlap, and we force the matrices to share an r dimensional overlap (the first r rows). The exact same functions as before compute the Jacobian and its rank. For example, in the case of $k = 3$, $n = 7$, $r = 1$ we find the dimension of the cone over $\sigma_2^1(\mathrm{Gr}(3,7))$ is 20. We tested this straightforward calculation for $r = 1, 2$ and $k, n = 2, \ldots, 10$, as well as for $s = 3$, $r = 1$, $k, n = 2, \ldots, 10$.

2.2.3 Computational efficiency

The above naive implementation for calculating the dimension of the restricted chordal variety is not efficient enough to handle larger computations. There is a trade-off of easy-to-implement formulas that ignore redundancy versus more careful implementation that is aware of these redundancies. In addition, we should pay attention to the order of operations for evaluation, in order to limit the size of intermediate computations.

In the naive implementation we take s symbolic matrices with the required r-dimensional overlap, and for each of those matrices determine the symbolic Jacobian, and then evaluate at a random point. However, the corresponding computation of differentials of minors is very inefficient for even very small cases. For example, a case as small as $s = 2$,$r = 1$,$k = 8$,$n = 10$ has Jacobian consisting of more than $100,000$ total terms and takes at least 20 minutes on a local system to evaluate. This inefficiency can be avoided noting redundancies from the fact that the differential of a minor is a linear combination of smaller minors and representing these entries as *unevaluated* determinants, or subfunctions, (rather than sums of monomials).

This point is illustrated by the following. Suppose A is a matrix of variables, and φ is the determinant function. Compute the Jacobian of φ in this case. We don't need to expand a determinant and then take derivatives in order to find an expression for the derivative $\frac{\partial \det A}{\partial a_{ij}}$. Instead use Laplace expansion on the i-th row,

$$\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij} \quad \Longrightarrow \quad \frac{\partial \det(A)}{\partial a_{ij}} = C_{ij}$$

where C_{ij} is the cofactor corresponding to the entry, which does not use the variable a_{ij}. Now we can treat C_{ij} as an unevaluated subfunction. For sums of determinants this same general principle can be applied. Every entry of equation 2.7 has this format.

Another efficiency consideration is order of operations, particularly evaluation and minor determinants. Generally, it is better to compute the determinant of a numerical matrix instead of evaluating it determinant at a point.

Return to our example. The coordinate functions are (sums of) minors, and hence their partial derivatives are also (sums of) minors. Realizing this allows us to define the Jacobian with subfunctions that evaluate these minors rather than compute the minors as derivatives. Computing a vector of minors at a point allows one to make use of reductions like Gaussian elimination which speed the computation of determinants greatly (on the order of n^3 operations rather than $n!$).

To implement this idea, we wrote functions (essentially linear combinations of determinants) to populate the entries of the Jacobian, instead of relying on functions from `M2` like `diff` or `jacobian`. To replace our use of `diff` we explicitly populated the Jacobian matrix utilizing appropriate subfunctions (cofactors) depending on the row and column labels. The Jacobian has column labels representing differentiation with respect to variables and row labels representing maximal minors.

Specifically, we populate this matrix utilizing these rules: in row m_i and column x_{ij} we put a 0 if maximal minor A_{m_i} does not contain the variable x_{ij}, or we put the (numerical) determinant of the A_{m_i} cofactor. This procedure, for each of the s matrices, defines the Jacobian with respect to the collection of variables defined only by that individual matrix and not the variables defined by every one of the s matrices. This directly produces the block structure (seen at (2.7)) of the Jacobian. Then we add the numerical Jacobians of each of the matrices together and calculate the rank.

Here is an implementation of this strategy as a function that eats a matrix M and spits out the column of the Jacobian of the Plücker map at M corresponding to the differential with respect to variable (i, j) for M.

```
par = (j, s) -> (
c=0;
for i to (length(s) -1) do(
    if j ==s_i then return i
    else continue;);
    return c);
```

The function `par` determines the sign of the cofactor. We loop over the subsets representing the maximal minors. The if-then statement determines whether or not the given minor contains the variable at (i, j) and sends it to 0 if it doesn't otherwise it evaluates the necessary numerical cofactor. We compute the full numerical Jacobian by using the function `dM` to populate the relevant non-zero columns.

```
t = set(0..k-1);
dM = (i,j,M)->apply(subsets(n,k),s->
   if not member(j,s) then 0
   else (-1)^(par(j,s)+i)*
     det(submatrix(M,toList(t-set{i}),toList(s-set{j}))));
```

Remark 2.2.1. With these changes we notice the following differences in speed (on a laptop) for computing the dimension of $\sigma_2^1(\mathrm{Gr}(8,10))$: after 20 minutes we force the code for the naive implementation to end with no answer, while the new code calculated the dimension in .09 seconds.

2.3 Dimensions of 1-Restricted Chordal Varieties

The main tool used to calculate the dimension of the secant variety is:

Lemma 2.3.1 (Terracini[33]). *Suppose $X \subset \mathbb{P}W$ is an algebraic variety and suppose $[x_1], \ldots, [x_s]$ are smooth general points of X such that $[x_1 + \cdots + x_s]$ is a smooth general point of $\sigma_s(X)$. Then*

$$\widehat{T}_{x_1+\cdots+x_s}\sigma_s(X) = \left\langle \widehat{T}_{x_1}X, \ldots \widehat{T}_{x_s}X \right\rangle.$$

There is a one-to-one correspondence between nonzero k-vectors $e_I := e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}$ and square-free monomials $e_{i_1} \ldots e_{i_k}$, so we often omit the $\wedge$ symbols. For shorthand, we write $\widehat{T}_{i_1,\ldots,i_k} := T_{e_{i_1},\ldots,e_{i_k}} \widehat{\mathrm{Gr}}(k,V)$.

2.3.1 The case of 2-planes

Asking for too much overlap causes collapsing, such as the following.

Proposition 2.3.2. *Suppose* $r \geq k-1$ *and* $\dim(V) = n \geq k$. *Then,*

$$\sigma_s^r(\mathrm{Gr}(k,n)) = \mathrm{Gr}(k,n). \tag{2.8}$$

Proof. When $r \geq k$ then the proof is trivial. Now consider the case $r = k-1$. An open subset of points in the cone $\sigma_s^{k-1}\widehat{(\mathrm{Gr}(k,V))}$ can be written as

$$v_1 \cdots v_k + v_1 \cdots v_{k-1} v_{k+1} + \cdots + v_1 \cdots v_{k-1} v_{k+s-1},$$

for $v_i \in V$. This expression factors as

$$v_1 \cdots v_{k-1} v_{k+1} (v_k + \cdots v_{k+s-1}),$$

which is clearly an element of $\widehat{\mathrm{Gr}(k,V)}$. So, the result follows. □

2.3.2 The case of 3-planes

The first non-trivial case of restricted secant varieties is that of the 1-restricted chordal variety of $\mathrm{Gr}(3,V)$, with $\dim(V) \geq 5$.

Proposition 2.3.3. *Consider* $X = \mathrm{Gr}(3,V)$ *with* $\dim(V) = n \geq 5$. *Then the following hold:*

1. $\widehat{T}_{e_1e_2e_3+e_1e_4e_5}\sigma_2^1(X) = V \cdot \{e_2e_3 + e_4e_5, e_1e_2, e_1e_3, e_1e_4, e_1e_5\}$,
2. $\widehat{T}_{e_1e_2e_3+e_4e_5e_6}\sigma_2(X) = \widehat{T}_{e_1e_2e_3+e_1e_4e_5}\sigma_2^1(X) \oplus \{e_ie_2e_3 - e_ie_4e_5 \mid i \geq 6\} \backslash (\widehat{T}_{123} \cap \widehat{T}_{145})$,
3. *and* $\dim(\sigma_2^1(X)) = 5n - 16$ *and* $\sigma_2^1(X)$ *has codimension* $n-1$ *in* $\sigma_2(X)$.

Remark 2.3.4. Note that $\widehat{T}_{123} \cap \widehat{T}_{145} = e_1 \cdot \{e_3e_4, e_2e_4, e_3e_5, e_2e_5\} = \mathbb{C}^4$ and that $\dim(\sigma_2^1(X)) = 5n - 16 = \dim(\sigma_2(X)) - 4 - (n-5)$, where the "4" in the right-hand side signifies an extra intersection. It is also notable that the monomials in (1) correspond to the triangles in a square triangulated by adding a central point and all edges to that point.

Proof. We mimic how one would prove Terracini's lemma. First we recall how to compute the cone over the tangent space to the Grassmannian. Use Def. 2.1.1 and construct a curve $\gamma(t) = e_1(t)e_2(t)e_3(t)$ such that $e_i(0) = e_i$ and let e_i' denote $e_i'(0)$, for $1 \leq i \leq 3$. Then

$$\gamma(t)'_{|t=0} = e_1'e_2e_3 + e_1e_2'e_3 + e_1e_2e_3'.$$

Since the vectors e_i' are arbitrary in V,

$$\widehat{T}_{123} = V \cdot \{e_1e_2, e_1e_3, e_2e_3\} \cong \{e_1e_2e_3\} \oplus (V/\{e_1, e_2, e_3\}) \cdot \{e_1e_2, e_1e_3, e_2e_3\} \cong \mathbb{C}^{(n-3)3+1},$$

which agrees with the description given in [14, p 638]. The spaces $\widehat{T}_{145}$ and $\widehat{T}_{456}$ are similarly defined. Now let $\gamma(t) = e_1(t)e_2(t)e_3(t) + e_1(t)e_4(t)e_5(t) = e_1(t)(e_2(t)e_3(t) + e_4(t)e_5(t))$. Then

$$\gamma(t)'_{|t=0} = e_1'(e_2e_3 + e_4e_5) + e_1e_2'e_3 + e_1e_2e_3' + e_1e_4'e_5 + e_1e_4e_5',$$

with $e_i(0) = e_i$ and $e_i'(0) = e_i'$ for $1 \leq i \leq 5$. Since e_i' are arbitrary in V, we arrive at (1).

For (2), note that by a similar calculation we have

$$\widehat{T}_{e_1e_2e_3+e_4e_5e_6}\sigma_2(X) = V \cdot \{e_1e_2, e_1e_3, e_2e_3, e_4e_5, e_4e_6, e_5e_6\}.$$

Compare the tangent spaces $\widehat{T}_{e_1e_2e_3+e_1e_4e_5}\sigma_2^1(X)$ and $\widehat{T}_{e_1e_2e_3+e_4e_5e_6}\sigma_2(X)$. The required overlap on $\sigma_2^1(X)$ forces elements of the form $\{e_ie_2e_3 - e_ie_4e_5 \mid i \geq 6\}$ to be excluded. Therefore,

$$\widehat{T}_{e_1e_2e_3+e_4e_5e_6}\sigma_2(X) = \widehat{T}_{e_1e_2e_3+e_1e_4e_5}\sigma_2^1(X) \oplus \{e_ie_2e_3 - e_ie_4e_5 \mid i \geq 6\}.$$

Now we prove (3). In the case $n = 5$, one shows that a general point on $\sigma_2(\mathrm{Gr}(3,5))$ can be written (after a possible change of basis) as $[e_1e_2e_3 + e_1e_4e_5]$, which is in $\sigma_2^1(\mathrm{Gr}(3,5))$, hence $\sigma_2(\mathrm{Gr}(3,5)) = \sigma_2^1(\mathrm{Gr}(3,5))$.

For $n \geq 6$, a general point of $\sigma_2(X)$ is (up to a change of basis) $e_1e_2e_3+e_4e_5e_6$. Therefore, we obtain orbit stability, and by Proposition 2.1.6 we get the dimension count $6k-17$.

So, by Terracini's lemma and Grassmann's formula

$$\widehat{T}_{e_1e_2e_3+e_4e_5e_6}\sigma_2(X) = \widehat{T}_{123} + \widehat{T}_{456} = \{\widehat{T}_{123} \cup \widehat{T}_{456}\} - \{\widehat{T}_{123} \cap \widehat{T}_{456}\} = \{\widehat{T}_{123} \cup \widehat{T}_{456}\}.$$

Now compare the two tangent spaces, and notice that

$$\begin{aligned} \widehat{T}_{e_1e_2e_3+e_4e_5e_6}\sigma_2(X) &= \widehat{T}_{e_1e_2e_3+e_1e_4e_5}\sigma_2^1(X) \oplus \{e_ie_2e_3 - e_ie_4e_5 \mid i \geq 6\} \\ &= \widehat{T}_{123} \oplus \widehat{T}_{456} = (\widehat{T}_{123} + \widehat{T}_{145}) \oplus \{e_ie_2e_3 - e_ie_4e_5 \mid i \geq 6\} \\ &= \widehat{T}_{123} \oplus \widehat{T}_{456} = ((\widehat{T}_{123} \cup \widehat{T}_{145}) - (\widehat{T}_{123} \cap (\widehat{T}_{145}))) \oplus \{e_ie_2e_3 - e_ie_4e_5 \mid i \geq 6\}. \end{aligned} \tag{2.9}$$

So the formula for (3) follows by noting that the "-4" comes from (1) and the "$-(n-5)$" comes from the complement on the right hand side of (2.9). □

Examples like these and computations done in `M2` led to the generalizations in Section 2.4.

2.4 Dimensions for r-restricted secant varieties

Recall for varieties $X, Y \subset \mathbb{P}V$ the abstract join variety is

$$J(X,Y) = \overline{\{([x],[y],[p]) \mid p \in \operatorname{span}\{x,y\}\}} \subset \mathbb{P}V \times \mathbb{P}V \times \mathbb{P}V,$$

where the overline denotes Zariski closure. The abstract s-secant variety of X is denoted $\Sigma_s(X) \subset (X)^{\times s} \times \mathbb{P}V$ and can be constructed inductively as the s-fold join of X with itself:

$$\Sigma_s(X) = \overline{\{([x_1],[x_2],\ldots,[x_s],[p]) \mid p \in \operatorname{span}\{x_1,\ldots,x_s\}\}} \subset \mathbb{P}V^{\times s} \times \mathbb{P}V.$$

The **embedded s-secant variety** is the projection to the last factor, denoted $\sigma_s(X) \subset \mathbb{P}V$. The **virtual dimension** of the s-secant variety is the dimension of the abstract s-secant variety:

$$\mathrm{v.dim}(\sigma_s(X)) = \dim(\Sigma_s(X)) = s \cdot \dim(X) + s - 1.$$

The **expected dimension** of $\sigma_s(X)$ is

$$\exp.\dim(\sigma_s(X)) = \min\{\dim(\mathbb{P}V), \dim(\Sigma_s(X))\} = \min\{\dim(\mathbb{P}V), s\dim(X) + s - 1\}.$$

Similarly, the abstract r-restricted s-secant variety is the incidence variety

$$\mathcal{I} \subset \mathrm{Gr}(r, V) \times \mathrm{Gr}(k-r, V)^{\times s} \times \mathbb{P}{\textstyle\bigwedge^k} V,$$

defined by

$$\mathcal{I} := \overline{\{(E, F_1, \ldots, F_s, [z]) \mid z \in \mathrm{span}\{\widehat{E} \wedge \widehat{F}_1, \ldots, \widehat{E} \wedge \widehat{F}_s\}\}}.$$

This incidence variety is natural as it mimics the way one might choose a point in $\sigma_s^r(\mathrm{Gr}(k, n))$. That is, select an r-plane for the overlap, then select the s $(k-r)$-planes in the complement. Finally, select the $(s-1)$ points needed to define the secant variety. However, what we say is "expected" should change based on how many k-planes we are trying to fit into a vector space V with an r-dimensional overlap, and we handle this in several cases.

As the restricted secant variety depends on the intersection of s linear spaces, Grassmann's formula calculates the size of the intersection of exactly two vector spaces. We apply this to the case of the restricted chordal variety below, where let $E \in \mathrm{Gr}(r, V)$ and V/E to respectively denote the r-dimensional space and its quotient.

Remark 2.4.1. Recall that the set of skew-symmetric matrices of rank $\leq r$ corresponds to the secant variety $\sigma_r(\mathrm{Gr}(2, V))$, which is always defective.

Proposition 2.4.2. *Let $n = k + 2$ and $r = \max(r, 2k - n)$. Then,*

$$\exp.\dim(\sigma_2^r(\mathrm{Gr}(k,n))) = \min\left\{\binom{n}{k} - 1, r(n-r) + 2((k-r)(n-k)) + 1\right\},$$

$$\mathrm{v.}\dim(\sigma_2^r(\mathrm{Gr}(k,n))) = \min\left\{\binom{n}{k} - 1, r(n-r) + 2((k-r)(n-k)) - 3\right\}.$$

Further $\sigma_2^{r+1}(\mathrm{Gr}(k,n))) = \mathrm{Gr}(k,n)$.

Proof. Let $n = k + 2$. This case handles spaces that have greater than a one-dimensional overlap $(2k - (k+2) = k - 2)$. The corresponding incidence variety is composed of a secant of Grassmannian of lines $\sigma_2(Gr(2, V))$, which are known to be defective. Redefine, if necessary, $r := \max(r, 2k - n)$. An isomorphic incidence variety to the one given in the proposition above has the form

$$\mathcal{I} \subset \mathrm{Gr}(r, V) \times \mathrm{Gr}(k - r, V/E)^{\times 2} \times \mathbb{P}\bigwedge^k V.$$

Let $N = \binom{n}{k} - 1$. The expected dimension is

$$\begin{aligned} \exp.\dim(\sigma_2^r(\mathrm{Gr}(k,V))) &:= \min\{\dim(\mathcal{I}), N\} \\ &= \min\{r(n-r) + (k-r)(n-(k-r)) + 1, N\}. \end{aligned} \tag{2.10}$$

Consider another representation of the same restricted chordal variety in the form

$$\mathcal{I} \subset \mathrm{Gr}(r, V) \times \sigma_2(\mathrm{Gr}(k-r, n-(k-r))).$$

All secant varieties of lines are defective [14]. Therefore, as these restricted chordal varieties are composed of a Grassmannian and a point in $\mathbb{P}\bigwedge^k V$ which have full dimension and one piece that is defective, namely the secant variety of lines, the restricted chordal variety is defective. Then, by direct calculation the actual dimension is the expected dimension minus one copy of $\mathrm{Gr}(k-r, n-(k-r))$. $\square$

Example 2.4.3. Consider $\sigma_2^1(\mathrm{Gr}(6,8))$. By Grassmann's formula any pair of 6-dimensional subspaces of an 8-dimensional space has at least a 4-dimensional intersection. Therefore $\sigma_2^i(\mathrm{Gr}(6,8))$ with $1 \leq i \leq 4$ are all equal. Those varieties have dimension 21 but from the incidence description the expected dimension would be $16+(4+4+1)=25$. This is exactly the known defect for $\sigma_s(Gr(2,V)$ which is $2s(s-1)$ or 4 when $s=2$ [14].

Proposition 2.4.4. *Let $k=r+2$. Then, $\sigma_2^r(\mathrm{Gr}(k,n))$ is defective, with defect $2s(s-1)$.*

Proof. Let $k=r+2$. Then, $\sigma_2^r(\mathrm{Gr}(k,n))$ is defective. Here the removal of the r-dimensional overlap leaves $\sigma_2(\mathrm{Gr}(k-r,V/E))$ and $k-r=2$ meaning it is also a secant variety of a Grassmannian of lines which is known to be defective. Construct the incidence variety as follows:

$$\mathcal{I} \subset \mathrm{Gr}(r,V) \times \sigma_2(\mathrm{Gr}(k-r,V/E)).$$

The incidence variety for this restricted chordal variety is also composed of a secant variety of lines which we know to be defective. The expected and virtual dimension counts are then exactly the same as 2.4.2, however $\sigma_2^{r+1}(\mathrm{Gr}(k,n))) \neq \mathrm{Gr}(k,n)$. □

Proposition 2.4.5. *Suppose $2k-1 \leq n \leq k+2$ and $r = \max(r, 2k-n)$. Then the virtual and expected dimensions for $\sigma_2^r(\mathrm{Gr}(k,n))$ are:*

$$\mathrm{v.dim}(\sigma_2^r(\mathrm{Gr}(k,V))) := \dim(\mathcal{I}) = r(n-r)+2(k-r)(n-k)+1,$$

and

$$\mathrm{exp.dim}(\sigma_2^r(\mathrm{Gr}(k,V))) = \min\left\{\mathrm{v.dim}(\sigma_2^r(\mathrm{Gr}(k,V))), \binom{n}{k}-1\right\}.$$

Proof. When $n \geq 2k-1$, count parameters in the following manner. First, choose $E \in \mathrm{Gr}(r,V)$, then choose $F_1, F_2 \in \mathrm{Gr}(k-r,V/E)$, and finally z on the line $\{\widehat{E} \wedge \widehat{F}_1, \widehat{E} \wedge \widehat{F}_2\}$. This gives the dimension counts listed in the proposition. □

The r-restricted chordal variety may also be defined as the following orbit closure

$$\sigma_2^r(\mathrm{Gr}(k,V)) := \overline{\mathrm{GL}(V).[e_1e_2\dots e_r(e_{r+1}\dots e_k + e_{k+1}\dots e_{2k-r})]},$$

which is equivalent to Def. 2.1.2. The dimension of $\sigma_2^r(\mathrm{Gr}(k,V))$ is the dimension of the tangent space at a general point (i.e., on the orbit).

Remark 2.4.6. We also have a nice description of the tangent space of the restricted chordal variety using $\mathcal{I}$, that is it is the image of the tangent space to $\mathcal{I}$ under the projection:

$$\begin{array}{c} \widehat{T}_E\,\mathrm{Gr}(r,V) \times (\widehat{T}_A\,\mathrm{Gr}(k-r,V) + \widehat{T}_B\,\mathrm{Gr}(k-r,V)) \;\subset \bigwedge^r V \times \bigwedge^{k-r} V \;\subset \bigwedge^r V \otimes \bigwedge^{k-r} V \\ \xrightarrow{\ \pi\ } \bigwedge^k V \end{array}$$

It turns out that restricted secant varieties are birational to a fiber bundle, which can be used to understand their dimension. It may be possible to further exploit this connection like what was done in [25], which applied Weyman's Geometric Technique to a similar partial desingularization to obtain generators of the ideal.

Theorem 2.4.7. *Let* $\dim(V) = n$ *and* $r, s, \geq 0$ *and* $0 \leq k \leq n$*. Then the restricted secant variety* $\sigma_s^r(\mathrm{Gr}(k,V))$ *is birationally isomorphic to the fiber bundle, denoted* Ξ*, with base* $\mathrm{Gr}(r,V)$ *and whose fiber over a point* $E \in \mathrm{Gr}(r,V)$ *is* $\sigma_s(\mathrm{Gr}(k-r,V/E))$*.*

Proof. Let Ξ denote the fiber bundle in the statement of the theorem. Recall the tautological sequence of bundles over the Grassmannian $\mathrm{Gr}(r,V)$:

$$0 \longrightarrow \mathcal{S} \longrightarrow \underline{V} \longrightarrow \mathcal{Q} \longrightarrow 0$$

where over a point $E \in \mathrm{Gr}(r,V)$ the fiber of the subspace bundle $\mathcal{S}$ is E, the fiber of the trivial bundle $\underline{V}$ is V and the fiber of $\mathcal{Q}$ is V/E. Applying the Schur functor $\bigwedge^{k-r}$ we obtain a vector bundle:

$$\begin{array}{c} \bigwedge^{k-r} Q \\ \downarrow \\ \mathrm{Gr}(r, V) \end{array}$$

whose fiber over E is $\bigwedge^{k-r}(V/E)$. In each fiber we have (a copy of) $\sigma_s(\mathrm{Gr}(k-r, V/E))$. We depict this in the following diagram.

$$\begin{array}{ccccc} \sigma_s(\mathrm{Gr}(k-r, V/E)) & \hookrightarrow & \mathbb{P}\bigwedge^{k-r} V/E & \hookrightarrow & \mathbb{P}\bigwedge^{k-r} Q \\ & \searrow & & & \downarrow \\ & & E \in & & \mathrm{Gr}(r, V) \end{array}$$

The total space of the fiber bundle Ξ consists of pairs $(E, [t])$ with $[t] \in \sigma_s(\mathrm{Gr}(k-r, V/E))$, and on an open subset we can assume that t has rank at most k (not just border rank k). Select such a pair $(E, [t])$. For $E \in \mathrm{Gr}(r, V) \subset \mathbb{P}\bigwedge^r V$ we write $E = [e_1 \wedge \cdots \wedge e_r]$ for independent elements $e_i \in V$. Elements in an open subset of $\sigma_s(\mathrm{Gr}(k-r, V/E))$ are of the form $[t] = [t^{(1)} + \cdots + t^{(k)}]$, with $[t^{(i)}] = [a_1^{(i)} \wedge \cdots \wedge a_{k-r}^{(i)}] \in \mathrm{Gr}(k-r, V/E)$ for each i.

Define a rational map $\Phi\colon \Xi \dashrightarrow \sigma_s^r \mathrm{Gr}(k, V)$ via

$$\Phi(E, [t]) = [e_1 \wedge \cdots \wedge e_r \wedge t]$$

on the open subset of points $(E, [t])$ in Ξ such that $e_1 \wedge \cdots \wedge e_r \wedge t$ is non-zero and rank $t \leq k-r$.

The image is indeed in $\sigma_s^r(\mathrm{Gr}(k, V))$ since the collection $(\widehat{E} \wedge t^{(1)}, \ldots, \widehat{E} \wedge t^{(s)})$ is a set of forms representing k-planes with (at least) an r-dimensional intersection. This mapping is dominant because an open subset of points of $\sigma_s^r \mathrm{Gr}(k, V)$ have a representation as $[\widehat{E} \wedge t]$.

Now we describe a rational map $\Psi\colon \sigma_s^r(\mathrm{Gr}(k, V)) \dashrightarrow \Xi$. Choose a basis $\{v_1, \ldots, v_n\}$ of V and volume form $\Omega_V := v_1 \wedge \cdots \wedge v_n \in \bigwedge^n V$. This induces isomorphisms $\bigwedge^j V \to \bigwedge^{n-j} V^*$ via contraction (Hodge star) with Ω_V. This mapping is graded in the following sense.

Lemma 2.4.8. *Suppose A, B are respectively vector spaces of dimensions a, b, and let $A \oplus B$ denote their external direct sum. Let $\alpha \in \bigwedge^i A$ and $\beta \in \bigwedge^j B$. Then $\alpha \wedge \beta \in \bigwedge^{i+j}(A \oplus B)$. Moreover,*

$$\Omega_{A\oplus B}(\alpha \wedge \beta) = (-1)^{i+j}\Omega_A(\alpha) \wedge \Omega_B(\beta),$$

in $\bigwedge^{a-i} A^* \otimes \bigwedge^{b-j} B^* \subset \bigwedge^{a+b-(i+j)}(A \oplus B)^*$.

Proof. Since the mappings $\Omega_{A\oplus B}, \Omega_A, \Omega_B$ are all linear, it suffices to prove the statement on rank-one elements, $\alpha = a_1 \wedge \cdots \wedge a_i$ and $\beta = b_1 \wedge \cdots \wedge b_j$. We may choose an adapted basis $\{a_1, \ldots, a_a, b_1, \ldots, b_b\}$ of $A \oplus B$ so that the first a vectors come from A and the next b vectors come from B. Moreover, we can select the first i vectors from the terms of α, and extend to a basis of A to obtain the next $a - i$ vectors. Similarly, for the last b be choose a basis of B starting from the terms of β. We also choose a dual basis $\{a^1 \cdots a^a, b^1 \cdots b^b\}$ of $(A \oplus B)^*$. Now apply the contraction operator to $\alpha \wedge \beta = a_1 \wedge \cdots \wedge a_i \wedge b_1 \wedge \cdots \wedge b_j$:

$$\Omega_{A\oplus B}(\alpha \wedge \beta) = (-1)^{i+j} \times a^1 \wedge \cdots \wedge a^{a-i} \wedge b^1 \wedge \cdots \wedge b^{b-j}.$$

where $(-1)^{i+j}$ defines the sign of the permutation that passes the a_i's through the b_j's to get it in the form $a^1 \wedge \cdots \wedge a^{a-i} \wedge b^1 \wedge \cdots \wedge b^{b-j}$. Then, as $\Omega_A(\alpha) = a^1 \wedge \cdots \wedge a^{a-i}$ and $\Omega_B(\beta) = b^1 \wedge \cdots \wedge b^{b-j}$, substituting into the right-hand side yields:

$$\Omega_{A\oplus B}(\alpha \wedge \beta) = (-1)^{i+j}\Omega_A(\alpha) \wedge \Omega_B(\beta).$$

One checks that the result is independent of the choice of bases of A and B. □

Now let $[w] \in \sigma_s^r(\mathrm{Gr}(k, n))$ be a general point, so that

$$w = \sum_{i=1}^{s} e_1^{(i)} \wedge \cdots \wedge e_k^{(i)},$$

with $E_i = [e_1^{(i)} \wedge \cdots \wedge e_k^{(i)}] \in \mathrm{Gr}(k,n)$ for each i, and with $\cap_i E_i = E$ an r-dimensional subspace of V. More explicitly, let π denote the projection from the abstract secant variety. General points are selected from the complement of the following closed subset:

$$\{\pi(E_1, \ldots, E_s, [w]) \mid \mathrm{rank}(E_i) < k \text{ for some } i \text{ or } \dim(\cap_i E_i) < r\}.$$

We wish to find an expression (after a possible change of basis) like

$$w = e_1 \wedge \cdots \wedge e_r \wedge (a_1^{(1)} \wedge \cdots \wedge a_{k-r}^{(1)}) + \cdots + e_1 \wedge \cdots \wedge e_r \wedge (a_1^{(s)} \wedge \cdots \wedge a_{k-r}^{(s)}),$$

which factors as

$$w = e_1 \wedge \cdots \wedge e_r \wedge \left(a_1^{(1)} \wedge \cdots \wedge a_{k-r}^{(1)} + \cdots + a_1^{(s)} \wedge \cdots \wedge a_{k-r}^{(s)}\right),$$

and hence can be readily seen to be an element in $\bigwedge^r E \otimes \bigwedge^{k-r} V/E$. If we can do this, then the mapping from such a point to Ξ will be clear.

Apply Ω_V to this expression for w to obtain (via Lemma 2.4.8)

$$\Omega_V(w) = \Omega_E(e_1 \wedge \cdots \wedge e_r) \cdot \Omega_{V/E}(\left(a_1^{(1)} \wedge \cdots \wedge a_{k-r}^{(1)} + \cdots + a_1^{(s)} \wedge \cdots \wedge a_{k-r}^{(s)}\right).$$

We can take the scalar factor $\Omega_E(e_1 \wedge \cdots \wedge e_r)$ to be equal to 1 so that

$$\Omega_V(w) = \Omega_{V/E}\left(a_1^{(1)} \wedge \cdots \wedge a_{k-r}^{(1)} + \cdots + a_1^{(s)} \wedge \cdots \wedge a_{k-r}^{(s)}\right),$$

and by construction the summands in $\Omega_V(w)$ live in $\bigwedge^{n-r} V/E$. Moreover,

$$[\Omega_V(w)] \in \sigma_s(\mathrm{Gr}(n-r, V/E)).$$

Note that $\Omega_V(w) \in \bigwedge^{n-r} V/E$ in particular. Consequently, one can find E from $\Omega_V(w)$ as the annihilator in the dual of the kernel of the 1-flattening defined for $T \in \bigwedge^{n-r} V^*$ as

$$F_T \colon V \to \bigwedge^{n-r-1} V^*$$

applied to $T = \Omega_V(w)$. Once $E = \ker F_{\Omega_V(w)}$ is found, one can find an expression for $[t] \in \sigma_k(\mathrm{Gr}(n-k, V/E))$ by applying the projection operator $\Omega_{V/E}$ to $\Omega_V(w)$.

This process gives a method for producing from $[w] \in \sigma_s^r \mathrm{Gr}(k, V)$ a pair $(E, [t]) \in \Xi$. In particular $\Psi([w]) \mapsto (E, [\Omega_{V/E}(\Omega_V(w))])$, with $E = \ker F_{\Omega_V(w)}$. By construction the composition of these two mappings is the identity on the open sets where they are defined. □

The description of the restricted secant varieties suggests the following regarding the minimal defining equations of the ideals of secants of restricted secant varieties, which was studied in the case of usual secants by one of us [15].

Conjecture 2.4.9. *Consider $X = \sigma_s^r(\mathrm{Gr}(k, n))$ with parameters s, r, k, n so that X is non-trivial. Then the ideal of X is generated by two types of polynomials:*

1. *polynomials inherited from the ideal of $\sigma_s(\mathrm{Gr}(k-r, n-r))$, i.e. the polynomials coming from the condition that $\Omega(w) \in \sigma_s(\mathrm{Gr}(k-r, n-r))$ for $w \in \sigma_s^r(\mathrm{Gr}(k, n))$;*
2. *polynomials coming from the $(r+1)\times(r+1)$ minors of the 1-flattening $F_T \colon V \to \bigwedge^{n-r-1}$ for $T = \Omega(w)$.*

Some consequences of this birational isomorphism are the following.

Corollary 2.4.1. *The restricted secant $\sigma_s^r(\mathrm{Gr}(k, n))$ has the expected dimension if and only if $\sigma_s(\mathrm{Gr}(k-r, n-r))$ is not s-defective.*

The BDdG conjecture [7, Conjecture 4.1] has the following implication.

secant variety	secant defect	defective restricted secant	reference
$\sigma_s(\mathrm{Gr}(2,n))$	$2s(s-1)$ for $2s \leq n$	$\sigma_2^r(\mathrm{Gr}(k,2k-2))$ $\sigma_s^r(\mathrm{Gr}(2+r,n))$	Prop. 2.4.2 Prop. 2.4.4
$\sigma_3(\mathrm{Gr}(3,7))$	1	$\sigma_3^r(\mathrm{Gr}(3+r,7+r))$	Thm. 2.4.7
$\sigma_3(\mathrm{Gr}(4,8))$	1	$\sigma_3^r(\mathrm{Gr}(4+r,8+r))$	Thm. 2.4.7
$\sigma_4(\mathrm{Gr}(4,8))$	4	$\sigma_4^r(\mathrm{Gr}(4+r,8+r))$	Thm. 2.4.7
$\sigma_4(\mathrm{Gr}(3,9))$	2	$\sigma_4^r(\mathrm{Gr}(3+r,9+r))$	Thm. 2.4.7

Table 2.1: The conjecturally complete list of defectivity for secants of Grassmannians [7, 14] and the consequences for restricted secants assuming $r, s \geq 0$ and $0 \leq k \leq n$.

Corollary 2.4.2. *If* [7, Conjecture 4.1] *is true, then* $\sigma_s^r(\mathrm{Gr}(k,V))$ *has no additional defect other than the defect coming from (usual) secant varieties of Grassmannians, see Table 2.1.*

Non-defectivity is known to hold in several cases listed below. Different authors use different conventions on projective and affine dimensions, so we have translated the results in the references to the conventions we use in this article.

Corollary 2.4.3. *In each of the cases listed below and outside the defective cases listed in Table 2.1 it is known that* $\sigma_s(\mathrm{Gr}(k,n))$ *is not defective and hence for those values of* k, n *we also have* $\sigma_s^r(\mathrm{Gr}(k+r,n+r))$ *is not defective for all* r.

- [11] *If* $n \leq 15$ *and* $s \leq 14$.
- [11] *If* $n > 15$, $k \geq 7$, *and* $s \leq max\{111, \frac{n-k+3}{3}\}$, *and in this case* $\sigma_s^r(\mathrm{Gr}(k+r,n+r))$ *is not defective except for* $\sigma_s^{k-2}(\mathrm{Gr}(k,n))$.
- [11] *If* $n > 15$, $3 \leq k \leq 6$, *and any of the following*

  1. $k = 3, s \leq max\{12, \frac{n}{3}\}$,
  2. $k = 4, s \leq max\{30, \frac{n-1}{3}\}$,
  3. $k = 5, s \leq max\{59, \frac{n-2}{3}\}$,
  4. $k = 6, s \leq max\{90, \frac{n-3}{3}\}$.

- *Asymptotically:* $s \leq \binom{n-k}{3} + 1$ [1], *and* $s \leq \binom{n}{r}^{\lfloor \log_2(k-1) \rfloor}$, *(better for* $k \geq 5$*)* [26].

A conjecturally complete list (from [7]) of known defective secant varieties of Grassmannians can be found at Table 2.1. We can combine the considerations above with the BDdG-Conjecture [2,7,11,26,27] to say that the defectivity of r-restricted higher order secant varieties only depends on the usual notion of k-defectivity of secant of Grassmannians.

Corollary 2.4.2. *If* [7, Conjecture 4.1] *is true, then* $\sigma_s^r(\mathrm{Gr}(k,V))$ *has no additional defect other than the defect coming from (usual) secant varieties of Grassmannians, see Table 2.1.*

Proof. Let $\sigma_s^r(\mathrm{Gr}(k,V))$ be the r-restricted s-secant variety and define the corresponding incidence variety $\mathcal{I} \subset \mathrm{Gr}(r,V) \times \sigma_s(\mathrm{Gr}(k-r,V/E))$. We showed in Theorem 2.4.7 that the restricted secant is birational to this incidence variety, and its dimension is completely determined by the dimension of the usual secant variety. Therefore, any defect must come from $\sigma_s(\mathrm{Gr}(k-r,V/E))$. The current list of known defective cases are exactly those in the BDdG conjecture. □

The following is the special case of Corollary 2.4.2 for r-restricted chordal variety.

Proposition 2.4.10. *The projection from the incidence variety*

$$\mathcal{I} \subset \mathrm{Gr}(r,V) \times \sigma_2(\mathrm{Gr}(k-r,V/E)) \to \mathbb{P}(\textstyle\bigwedge^k V),$$

whose image is $\sigma_2^r(\mathrm{Gr}(k,V))$*, has finite fibers. Hence given the BDdG conjecture* $\sigma_2^r(\mathrm{Gr}(k,V))$ *has no additional defect other than the defect coming from (usual) secant varieties of Grassmannians. The only defective restricted chordal varieties of Grassmannians are when* $n = k+2$ *or when* $k-r=2$*.*

We confirmed this statement for those r-restricted chordal varieties composed of $\mathrm{Gr}(2,n)$ for several examples in Macaulay2. We also calculated the dimension for several other known cases. For example, $\sigma_3^1(\mathrm{Gr}(4,8))$ which is composed of $\sigma_3^1(\mathrm{Gr}(3,7))$ has dimension 40, however its expected dimension is 45 indicating it is in fact defective. We also performed similar checks of other r-restricted chordal varieties composed of a defective secant variety.

2.5 Coding Theory

Let us recall several relevant coding theory definitions from [20]. Let F denote an alphabet, which is a set of digits. A sequence of digits from F is called a codeword. The length of a codeword is the number of digits in the codeword. The collection of codewords, denoted C, is called a dictionary. A code of length n is a collection of codewords. A code is called a binary code if $F = \{0, 1\} =: \mathbb{F}_2$. A code is transmitted by sending the digits of its codewords in sequence across a channel. The Hamming distance between two codewords of equal length $u, v \in C$, denoted $d(u, v)$, is the number of places that u and v differ. For a codeword u, the weight of u is defined as, $w(u) = d(u, 0)$ where 0 corresponds to the 0 digit in the given alphabet. Abo-Ottaviani-Peterson gave the following connection to geometry.

Theorem 2.5.1. [1, Theorem 4.1] *Let $A(n, 6, w)$ be the cardinality of the largest binary code of length n, constant weight w, and Hamming distance between any two codewords at least 6. If $s \leq A(n + 1, 6, k + 1)$ then $\sigma_s(\mathrm{Gr}(k, n))$ has the expected dimension.*

A Grassmann code is a special case of a linear code. Let $\mathbb{F}_q$ denote the field with q elements. Then, it is well-known that $\mathrm{Gr}_{\mathbb{F}_q}(k, n)$ contains P points where

$$P = \frac{(q^n - 1)(q^{n-1} - 1) \dots (q^{n-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1) \dots (q - 1)}. \tag{2.11}$$

To define the Grassmann code as a linear code first pick a Plücker representative of each of the P points as a column vector in $(\mathbb{F}_q)^I$ for $I = \binom{n}{k}$ and form an $I \times P$ matrix M (the generator matrix) with these P vectors as columns. Grassmann codes (in the identifiable case) correspond to sums of k-fold wedge products [8, 28, 31]. Vectors in the Plücker embedding of $\mathrm{Gr}(k, n)$ are the codewords in a Grassmann code. So a general $x \in \sigma_s(\mathrm{Gr}(k, n))$ can be thought of as an unordered collection of s codewords [16]. The codewords are uniquely recoverable as long as $\mathrm{Gr}(k, n)$ is identifiable in rank s, which we expect is true for small s [9, 12].

The Grassmannian distance for $A, B \in \mathrm{Gr}(k, n)$ is $d_G(A, B) = k - dim(A \cap B)$. Note, points of the restricted chordal variety $\sigma_2^r(\mathrm{Gr}(k, n))$ are of the form $[\hat{A} + \hat{B}]$, with $d_G(A, B) = k - r$.

A code corresponding to a point of $\sigma_s^r(\mathrm{Gr}(k, n))$ (again assuming identifiability), consists of a collection of s codewords with the restriction that (pairwise) codewords must have distance $k - r$ between them, and that the intersection is the same for all pairs. This leads to a trade-off between redundancy and the capacity of the coding scheme. The restriction limits the number of possible codewords available, corresponding to an increase in the amount of information necessary to ensure accurate decoding. The max number of codewords in a signal for a given coding scheme can be considered the capacity of the channel, which is, in turn, found by determining the dimension of the variety (i.e. $\dim(\sigma_s^r(\mathrm{Gr}(k, n)))$ and $\dim(\sigma_s(\mathrm{Gr}(k, n))))$ corresponding to the coding scheme.

Section 2.4 provides a method involving the contraction operator to determine whether a given point lies on a restricted chordal variety. The contraction determines the common intersection and the remaining information could be computed separately by tensor decomposition. Therefore, with an appropriate choice of collections of codewords on restricted secants one could build extra information for decoding as redundancies in each codeword. This redundancy could permit an error-correcting mechanism.

Theorem 2.4.7 says the following in terms of the coding theory. Codes for restricted secants of Grassmannians can be thought of as Grassmann codes except that the codewords are padded with an additional overlap. Therefore, [1, Theorem 4.1] says: Let $A(n, 6, w)$ be the cardinality of the largest binary code of length n, constant weight w, and distance 6. If $s \leq A(n + 1, 6, k + 1)$ then $\sigma_s^r(\mathrm{Gr}(k + r, n + r))$ has the expected dimension.

We end this section with an extended example.

Example 2.5.2. Consider binary codes in the case of $\mathrm{Gr}(3,\mathbb{F}_2^6) \subset \mathbb{P}\bigwedge^3\mathbb{F}_2^6$. By (2.11) there are $1,395$ points in $\mathrm{Gr}(3,\mathbb{F}_2^6)$. The corresponding linear code has a $20 \times 1,395$ generator matrix, M, whose columns are the Plücker coordinates of each of the $1,395$ points. Then, one encodes a message b as the product Mb.

Special subsets of possible messages come from points of a given orbit (like the secant or restricted secant, or tangent to the Grassmannian). For a variety X "the orbit" is the set, denoted X°, of points that are equivalent to the normal form on the respective variety up to change of coordinates by $\mathrm{SL}_6(\mathbb{F}_2)$. We are interested in the numbers of points in each orbit.

For a pair of codewords $x, y \in \mathrm{Gr}(3,\mathbb{F}_2^6)$, construct the message b consisting of two non-zero entries. This represents a code in $\sigma_2(Gr(3,\mathbb{F}_2^6))$. Changing the codewords $x, y \in \mathrm{Gr}(3,\mathbb{F}_2^6)$ so that they share an r-dimensional overlap results in a message in $\sigma_2^r(\mathrm{Gr}(3,\mathbb{F}_2^6))$.

Here we can completely describe the $\mathrm{SL}_6(\mathbb{F}_2)$-orbits in $\bigwedge^3\mathbb{F}_2^6$. To count the number of points in an orbit of a finite matrix group we repeatedly apply random non-singular matrices to the set of known points in the orbit until the number of unique elements in the set stabilizes. This indicates that it is likely that all the points in that orbit have been obtained. To see the code for this and an explanation see Appendix A.0.4 If the list of orbits obtained this way fills out the entire ambient space we are ensured that no points were missed. On the other hand, if there are missing points one can take the orbit of a point not already on a known orbit, and compute its orbit. The results are listed in Table 2.2.

X°	0	$\mathrm{Gr}(3,\mathbb{F}_2^6)^\circ$	$\sigma_2^1(\mathrm{Gr}(3,\mathbb{F}_2^6))^\circ$	$\tau(\mathrm{Gr}(3,\mathbb{F}_2^6))^\circ$	$\sigma_2(\mathrm{Gr}(3,\mathbb{F}_2^6))^\circ$	Z°
$\#X^\circ$	1	1,395	54,684	468,720	357,120	166,656

Table 2.2: The orbits of $\bigwedge^3\mathbb{F}_2^6$ under the $\mathrm{SL}_6(\mathbb{F}_2)$-action.

The classical orbit closures are linearly ordered: $\mathrm{Gr}(3,\mathbb{F}_2^6) \subset \sigma_2^1 \mathrm{Gr}(3,\mathbb{F}_2^6) \subset \tau(\mathrm{Gr}(3,\mathbb{F}_2^6)) \subset \sigma_2(\mathrm{Gr}(3,\mathbb{F}_2^6)) = \mathbb{P}\bigwedge^3\mathbb{F}_2^6$. We found precisely one new orbit, with normal form:

$$\xi = e_1e_2e_4 + e_0e_3e_4 + e_0e_2e_5 + e_0e_3e_5 + e_1e_3e_5 = (e_1e_2 + e_0e_3)e_4 + (e_0e_2 + (e_0 + e_1)e_3)e_5.$$

Taking a limit that sends $e_5 \to 0$ one sees that the closure of Z contains $\sigma_2^1 \mathrm{Gr}(3,6)$. Experiments suggest that that Z is not contained in τ. Indeed, the Grassmann discriminant [21, Ex. 6.1], the defining polynomial for the hypersurface $\tau(\mathrm{Gr}(3,\mathbb{F}_2^6))$, evaluates at ξ to $15 \not\equiv 0 \mod 2$, hence implying non-membership: τ, i.e. $Z \not\subset \tau(\mathrm{Gr}(3,\mathbb{F}_2^6))$.

We note a bijection between $\sigma_2^1(\mathrm{Gr}(3,\mathbb{F}_2^6))^\circ$ and $\mathrm{Gr}(1,\mathbb{F}_2^6)^\circ \times \sigma_2(\mathrm{Gr}(2,\mathbb{F}_2^5))^\circ = (\mathbb{F}_2^6 \setminus 0) \times \mathbb{P}\bigwedge^2\mathbb{F}_2^5 \setminus \mathrm{Gr}(2,\mathbb{F}_2^5))$, which is the fiber bundle from Theorem 2.4.7. The number of points of the latter is, using (2.11), $(2^6 - 1) \cdot (2^{\binom{5}{2}} - \frac{(2^5-1)(2^4-1)}{2^2-1}) = 54,684$, which agrees with the exhaustive count. Further, we have an identifiability over $\mathbb{F}_2$ for $\sigma_2^1(\mathrm{Gr}(3,\mathbb{F}_2^6))^\circ$, whose points correspond uniquely to pairs of a non-zero vector in $\mathbb{F}_2^6$ and a full rank skew-symmetric 5×5 matrix over $\mathbb{F}_2$.

Chapter 3

Graph Theoretic Intersection Structures and Related Problems

3.1 Tensors and Related Problems

Tensors are an important object of study not only in mathematics but in physics and chemistry as well because representing information as a multi-dimensional array is necessary. There are two common ways to define a tensor that will be listed below as sometimes it is more convenient to use one definition in comparison to the other. For each definition, the tensor product will be defined on 3 arbitrary vector spaces, and it will generalize. As the tensor product is an associative operator, the tensor product without coordinates is the vector space $A \otimes B \otimes C$. This is the $\mathcal{F}$-linear span of $a \otimes b \otimes c$ where $a \in A, b \in B, c \in C$. With coordinates, let $\{c_1, \ldots, c_n\}$ be a basis of C and A and B be vector spaces. The tensor product can be thought of in terms of a basis of $A \otimes B \otimes C$ defined as:

$$\{[a_i \otimes b_j \otimes c_k \mid 1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq p\} \subset \mathbb{P}\bigwedge^k V.$$

A first order tensor is a vector and a tensor of order 2 is a matrix. For an arbitrary tensor, T , its' rank is the minimum number r such that T can be written as the sum of r rank one tensors. These rank one tensors are the building blocks and decomposing a tensor is a hard and unsolved question in most cases.

One reason to define all of the relevant information about tensors and tensor rank is that tensor rank corresponds to generic rank, which for a given projective variety X, corresponds to the first r for which $\sigma_r(X)$ fills the ambient space over $\mathbb{C}$. Therefore, the question of tensor decomposition is synonymous to studying secant varieties. Given a tensor space like

$\mathcal{C}^{n_1 \times \cdots n_k}$, one might want to find the number of rank 1 tensors that fits in the sum of an element in that space. This question can be asked in the most obvious sense where all terms are required to be completely independent. But it also can be asked where the only requirement is that all the terms must be identifiable. The restricted secant case falls under this question. As shown in the restricted secant section, one has complete identifiability, at least in the initial case of $\mathrm{Gr}(3, \mathbb{F}_2^6)$ and it might be expected in many others as well. This happens because the contraction operator can separate the r-plane the restricted secants share from the independent parts that are left.

3.2 Description of Defective Secants of Grassmannians

In an attempt to describe the entire list of defective secants of Grassmannians, the first family of defective cases is the family of skew symmetric matrices that have the form $\sigma_2(\mathrm{Gr}(k, n))$. Up to $s = 15$ there are only 4 other defective cases. Connecting these cases to one another has been elusive, however a connection from $\sigma_3(\mathrm{Gr}(3, 7))$ to skew symmetric matrices has been found utilizing the Schouten diagrams. Each Schouten diagram can be thought of as graph G where the vertices are basis vectors of V, and the edges represent a wedge product of two basis vectors they connect. Therefore, a side of 3 vertices and two edges can be thought of as $e_i \wedge e_j \wedge e_k$.

3.2.1 $\sigma_3(\mathrm{Gr}(3, 7))$

Let V be a vector space such that the $\dim(V) = 7$ and a corresponding basis be $e_1 \ldots e_7$. There is a one-to-one correspondence between nonzero k-vectors $e_I := e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}$ and square-free monomials $e_{i_1} \ldots e_{i_k}$, so we often omit the $\wedge$ symbols. The expected dimension of $\sigma_3(\mathrm{Gr}(3, 7))$ is 34 with 33 being the actual dimension.

From the Schouten diagram of $\sigma_3(\mathrm{Gr}(3, 7))$, (see Figure 3.1) a general point in the variety can be expressed as: $e_1e_2e_3 + e_3e_4e_5 + e_5e_6e_1 + e_2e_6e_7$. However, this can be constructed in the

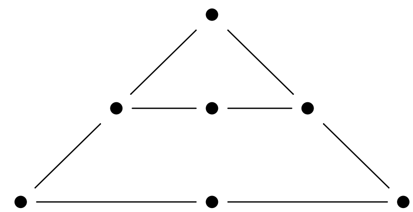

Figure 3.1: A Schouten diagram corresponding to a point
$e_1 \wedge e_2 \wedge e_3 + e_3 \wedge e_4 \wedge e_5 + e_5 \wedge e_6 \wedge e_1 + e_2 \wedge e_6 \wedge e_7$

following manner. Let $X = \sigma_2^1(\mathrm{Gr}(3,7))$, $Y = \mathrm{Gr}(1,7)$, and $Z = \mathrm{Gr}(1,7)$, where $X \in \mathbb{P}\bigwedge^3\mathbb{C}^7$ and $Y, Z \in \mathbb{P}\bigwedge^1\mathbb{C}^7$. Within the Schouten diagram, this corresponds to the outer "v" shape, then the two single points left.

Take the incidence variety constructed as follows: $\mathcal{I} \subset (X \times (Y \times Z \times \mathbb{P}^1) \times \mathbb{P}^1) \times \mathbb{P}\bigwedge^3\mathbb{C}^7$, defined by

$$\mathcal{I} := \overline{\{([x],([y],[z],[p_1]),[p_2]),[p_3] \mid p_1 \in \mathrm{span}\{y,z\}, p_2 \in \mathrm{span}\{x,p_1\}, p_3 \in \mathrm{span}\{p_1,p_2\}\}}$$

Then, by taking the projection of $\pi : \mathcal{I} \to \mathbb{P}\bigwedge^3\mathbb{C}^7$, the calculation of the dimension of the fiber gives an upper bound on the dimension of the corresponding secant variety. Since the expected dimensions of the varieties are $\dim(X) = 19$, $\dim(Y) = 6$, and $\dim(Z) = 6$, the dimension count is:

$$\dim(X + (Y + Z + 1) + 1 = 19 + (6 + 6 + 1) + 1 = 33$$

However, the expected dimension of $\sigma_3(\mathrm{Gr}(3,7))$ was 34 and the upper bound found here is 33, thus explaining the defectivity.

3.2.2 Other defective cases

There are 3 other defective cases not included in the family of skew symmetric matrices. However, since $\sigma_3(\mathrm{Gr}(4,8))$ is defective and $\sigma_4(\mathrm{Gr}(4,8))$ is not the entire ambient space it must also be defective so there are really two cases that need to be studied in the same way as $\sigma_3(\mathrm{Gr}(3,7))$. At this point, utilizing M2 code and the list of normal forms found in [22, 36], normal forms for $\sigma_3(\mathrm{Gr}(4,8))$ and $\sigma_4(\mathrm{Gr}(3,9))$ have been found. The normal forms are respectively: $e_{2345}+e_{1347}+e_{1567}+e_{1268}$ and $e_{123}+e_{456}+e_{789}+e_{149}+e_{157}+e_{168}+e_{247}+e_{348}$. The next steps are to factor these normal forms and define maps that contain pieces which are skew-symmetric matrices.

3.3 Schouten Diagrams

Schouten graphs are used as a way to describe the orbits of GL_6 acting on trivectors of a 6-dimensional vector space. Call the generalization of these graphs Schouten diagrams [1, 19]. More specifically, for $\bigwedge^k \mathbb{C}^n$, a Schouten diagram is a discrete geometry on points with labels from $\{1, \ldots, n\}$, and lines each consisting of k points. One associates an expression of a point of $\bigwedge^k \mathbb{C}^n$ with a Schouten diagram D via a sum-product formula:

$$D \dashrightarrow \sum_{\ell \text{ a line in } D} \prod_{e \in \ell} e,$$

where the product is the wedge product and the direction on the line is given by the following rule: for $e_i, e_j \in \ell$, e_i leads to e_j if $i < j$. It is clear that such a discrete set of objects, Schouten diagrams, cannot be sufficient to enumerate all orbits of GL_n acting on $\bigwedge^k \mathbb{C}^n$ for $k \geq 8$. However, certain families of secants of Grassmannians are still interesting to study.

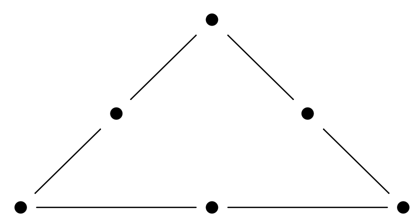

Figure 3.2: A Schouten diagram corresponding to a point $e_1 \wedge e_2 \wedge e_3 + e_3 \wedge e_4 \wedge e_5 + e_5 \wedge e_6 \wedge e_1$

3.3.1 Cyclic Schouten Diagrams

In this note the focus is on the family of cyclic Schouten diagrams, and the algebraic varieties associated to the orbits of their associated forms in $\bigwedge^k \mathbb{C}^n$. A cycle is a diagram that has only one path and begins and ends at the same vertex. From this, define a cyclic secant of Grassmannians in the following way:

Definition 3.3.1. *Let* $\dim(V) \geq sk - js$. *The cyclic s-secant variety of* $\mathrm{Gr}(k, V)$ *is* $\sigma_s^{c_j}(\mathrm{Gr}(k,V)) =$

$$cl\Big\{[E_1 + \cdots + E_s] \mid [E_i] \in \mathrm{Gr}(k,V), \dim(\textstyle\bigcap_{i=1}^{s-1} E_i \cap E_{i+1}) = j \text{ and } \dim(E_s \cap E_1) = j)\Big\} \subset \mathbb{P}\textstyle\bigwedge^k V.$$

Theorem 3.3.2. *Given* $s \geq 4, k \geq 4, n \geq 12$ *the cyclic Secant of Grassmannian* $\sigma_s^{c_j}(\mathrm{Gr}(k,n))$ *is birationally isomorphic to* $\sigma_s(\mathrm{Gr}(k-j,n))$.

Proof. Suppose V is a vector space with basis $\{e_1, \ldots, e_n\}$. Let $E_1 + E_2 + \cdots + E_s$ be an general element in (an open subset of) $\sigma_s^c(\mathrm{Gr}(k,n))$. Then, up to change of coordinates assume that the E_i are spanned by basis vectors such that $\forall i, i+1 \in \{1 \cdots s\}, \dim(E_i \cap E_{i+1}) = j$ as each k-plane intersects the adjacent one at exactly j distinct points by construction. Now, define the rational map $\gamma\colon \sigma_s^{c_j}(\mathrm{Gr}(k,n)) \dashrightarrow \sigma_s(\mathrm{Gr}(k-j,n))$ by

$$\gamma(E_1 + E_2 + \cdots + E_s) = \frac{E_1}{E_1 \cap E_2} + \frac{E_2}{E_2 \cap E_3} + \frac{E_{s-1}}{E_{s-1} \cap E_s} + \cdots \frac{E_s}{E_s \cap E_1},$$

where the quotient $\frac{E_i}{E_i \cap E_{i+1}}$ is identified with the subspace $(E_i \cap E_{i+1}) \star E_i \subset E_i \subset V$ where $\star$ denotes contraction.

Then, mapping the basis vectors, $\sigma(e_1 \wedge \cdots \wedge e_k + e_{k-j+1} \wedge \cdots \wedge e_{2k-j} + \cdots + e_{(s-1)(k-j)+1} \cdots \wedge e_1 \wedge \cdots e_j) \mapsto (e_1 \wedge \cdots \wedge e_{k-j} + e_{k-j+1} \wedge \cdots \wedge e_{2k-2j} + \cdots + e_{(s-1)(k-j)+1} \wedge \cdots \wedge e_{sk-sj})$. But this is exactly an element in $\sigma_s(Gr(k-j,n))$.

For the other direction, up to a change of coordinates a general point in $\sigma_s(\mathrm{Gr}(k-j,n))$ is of the form $\widetilde{E}_1 + \widetilde{E}_2 + \cdots + \widetilde{E}_s = e_1 \wedge \cdots \wedge e_{k-j} + e_{k-j} \wedge \cdots \wedge e_{2s-2j} + \cdots + e_{(s-1)(k-j)+1} \cdots \wedge e_{sk-sj}$. Then, for each $\widetilde{E}_i$ select an l- plane, l_i, corresponding to a $Gr(j,k-j) \in \widetilde{E}_I$. Define a rational map $\Phi \colon \sigma_s(\mathrm{Gr}(k-j,n)) \dashrightarrow \sigma_s^{c_j}(\mathrm{Gr}(k,n))$ via

$$\Phi(\widetilde{E}_1 + \widetilde{E}_2 + \cdots \widetilde{E}_s, (l_1, l_2, \cdots l_s)) = (\widetilde{E}_1 \wedge l_2 + \widetilde{E}_2 \wedge l_3 + \cdots \widetilde{E}_s \wedge l_1).$$

Note that each $\widetilde{E}_I \wedge l_J$ is now a k-plane that, by choice of the l_j, is forced to intersect the adjacent k-plane in a j-dimensional space. This new element must be in $\sigma_s^{c_j}(\mathrm{Gr}(k,n))$ and the result follows. □

Theorem 3.3.2 requires $k > 3$. If the theorem above were to hold then the expected map would send $\sigma_3^c(\mathrm{Gr}(3,n)) \dashrightarrow \sigma_3(Gr(2,n))$. Note that this is a defective secant of Grassmannians with defect 12. Calculating an expected dimension, including the defect, the formula would be $6n - 22$. This is incorrect because the formula is $6n - 18$ matching [1] when $n = 7$ for the dimension of the tangential variety $\tau(\mathrm{Gr}(3,7))$. To explain this note that the normal form of $\sigma_3^c(\mathrm{Gr}(3,n))$ is (up to scale) $e_1e_2e_3 + e_3e_4e_5 + e_5e_6e_1$. Following the mappings set up in Theorem 3.3.2, rewrite this normal form as $e_1(e_2 + e_6) + e_3e_4e_5$. However, $e_1(e_2 + e_6) \in \sigma_2^1(\mathrm{Gr}(2,n))$. But $\sigma_2^1(\mathrm{Gr}(2,n)) = \mathrm{Gr}(2,n)$, so the two points collapse into a single 2-plane. Therefore, the supposed normal form in $\sigma_3^p(\mathrm{Gr}(3,n))$ really gives only 2 2-planes after the collapse. But the rational map in the theorem sends s k-planes to s $k-1$ planes. So, the mapping does not apply. Taking this one step further, there is a conjectured known list of defective restricted secants of Grassmannians. When looking at the normal form of a $\sigma_s(\mathrm{Gr}(k,n))$ that has a more complicated intersection structure, if there exists some subset of the k-planes that form a defective variety it gives a criterion for defectivity of the larger variety.

This description and proof of the cyclic secant variety combined with the discussion on defectivity of normal forms gives a second way to define restricted chordal varieties strictly as a join of two different sized Grassmannians.

Theorem 3.3.3. *Let $\sigma_2^1(Gr(k,n))$ be a restricted chordal variety not given in Table 2.1. Then, for $(X,Y) \in (\bigwedge^k \mathbb{C}^n, \bigwedge^{k-1}\mathbb{C}^n.)$, take the Cartesian product $(X \times Y)$ in $\bigwedge^k \mathbb{C}^n \times \bigwedge^{k-1}\mathbb{C}^n$. By, Theorem 3.3.2 there exists a birational map from $\sigma_2^1(Gr(k,n)) \dashrightarrow X \times Y$.*

Proof. Let $e_1 \cdots e_k + e_1 e_{k+1} \cdots e_{2k-1}$, up to relabeling, be a general point in $\sigma_2^1(Gr(k,n))$. By, Theorem 3.3.2 there exists a map that sends the point to $e_1 \cdots e_k + e_{k+1} \cdots e_{2k-1}$. This is in $X \times Y$ where $e_1 \cdots e_k \in \mathrm{Gr}(k,n) = X$ and $e_{k+1} \cdots e_{2k-1} \in \mathrm{Gr}(k-1,n) = Y$. For the other direction, take $e_1 \cdots e_k \in \mathrm{Gr}(k,n)$ and $e_{k+1} \cdots e_{2k-1} \in \mathrm{Gr}(k-1,n)$. Next, take the Cartesian product to get $e_1 \cdots e_k + e_{k+1} \cdots e_{2k-1}$. By, Theorem 3.3.2 there exists a map to $\sigma_2^1(Gr(k,n))$ sending the point to $e_1 \cdots e_k + e_1 e_{k+1} \cdots e_{2k-1}$. This completes the proof. □

This corollary allows for the derivation of the formula for $\dim(\sigma_3^c(\mathrm{Gr}(3,n)))$. Suppose the two varieties are $X = \sigma_2^1(Gr(3,n))$ and $Y = Gr(1,n-2)$. Then, $\dim(X) = 5n-16$ and $\dim(Y) = n-3$. Calculating the dimension of the join, $\dim(J(X,Y)) = 5n-16+n-3+1 = 6n-18$.

3.3.2 Schouten Diagrams that Form Paths

Within the language of graph theory, define a path to be a collection of lines, each having the same number of points in the geometry, that are connected in sequence and no lines are used more than once. As a result of that definition, every cycle has a path but

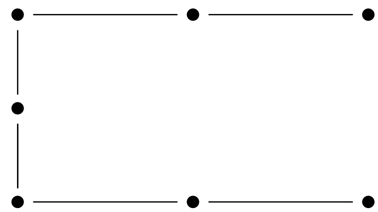

Figure 3.3: A Schouten diagram corresponding to a point
$e_1 \wedge e_2 \wedge e_3 + e_3 \wedge e_4 \wedge e_5 + e_5 \wedge e_6 \wedge e_7$

every path does not have a cycle. To turn a cycle into a path, simply remove the last edge. Therefore, considering a collection of k-planes whose intersection structure forms a path should be studied independently. To define this, remove one condition from the definition of cyclic secants of Grassmannians.

Definition 3.3.4. *Let* $\dim(V) \geq sk - js$. *The path s-secant variety of* $\mathrm{Gr}(k, V)$ *is* $\sigma_s^{p_j}(\mathrm{Gr}(k, V)) =$

$$cl\Big\{[E_1 + \cdots + E_s] \mid [E_i] \in \mathrm{Gr}(k, V), \dim(\textstyle\bigcap_{i=1}^{s-1} E_i \cap E_{i+1}) = j)\Big\} \subset \mathbb{P}\bigwedge^k V.$$

For motivation, consider the singular locus of the 3rd secant, $\mathrm{Sing}(\sigma_3(\mathrm{Gr}(3, 7))$, whose Schouten diagram can be seen in [1], and below, and has normal form $e_1e_2e_3 + e_3e_4e_5 + e_5e_6e_7$.

This, when viewed as a graph is a 1-path. A combination of the previous techniques used and theorems proved allow us to say the following.

Theorem 3.3.5. *Given* $s \geq 2, k \geq 3, n \geq 3$ *the path geometric secant of Grassmannians* $\sigma_s^{p_j}(\mathrm{Gr}(k, n))$ *is birationally isomorphic to* $J(\sigma_{s-1}(\mathrm{Gr}(k - j, n)), Gr(k - j + 1, n))$.

Proof. Since the given Grassmannians are of different sizes, given $(U, W) \in (\sigma_{s-1}(\mathrm{Gr}(k - j, n), \mathrm{Gr}(k, n))$ use the Plucker embedding to map it to $\bigwedge^{k-j} V \bigoplus \bigwedge^k V$. Let $e_1 \wedge \cdots \wedge e_k + e_{k-j} \wedge e_{2k-j} + \cdots + e_{(s-1)(k-j)+1} \cdots \wedge e_{sk-sj}$ represent a point on an s secant variety whose Schouten diagram is represented by a path, up to a change of coordinates. Then, taking the first $(s - 1)$ $k - j$-planes, by the birational map in Theorem 3.3.2, this is birationally isomorphic to $\sigma_{s-1}(\mathrm{Gr}(k - j, n))$. This leaves a $k - j + 1$-plane. Next, take a point in the span of the spaces found above. The result follows. □

Consider the following example from [1] as an example of counting the dimension in another way. The dimension for $\mathrm{Sing}(\sigma_3(\mathrm{Gr}(3,7)))$ is 30. This is also, $\sigma_3^{p_1}(\mathrm{Gr}(3,7))$. Calculating the dimension

$$\dim(\sigma_2(\mathrm{Gr}(2,7)) + \dim(\mathrm{Gr}(3,7)) + 1 = 17 + 12 + 1 = 30.$$

which is expected.

Chapter 4

Future Works

Below are planned areas of future research.

A restricted Grassmann code is an extension of the Grassmann code. An obvious next step is to fully flesh out the encoding and decoding scheme as well as understanding the capacity of such a code. With the use of the contraction operator, one can find the overlapping r-dimensional space shared by the s k-dimensional linear spaces. Therefore, one potential idea is to utilize the overlap as the extra information to send for error correction and then with the contraction operator decode and remove it after it has been sent. A point of interest still not fully understood is the ideal situation of when to use such a code, as the tradeoff is not fully understand. A restricted Grassmann code by definition has reliably less information it is able to send. Yet, by optimizing the code words chosen and the overlap longer messages still have the possibility of being sent. Determining for what values of s, r, k, n $\sigma_s^r(\mathrm{Gr}(k, n))$ is a more effective space to work in versus $\sigma_s(\mathrm{Gr}(k, n))$ is an open question.

The research done up to this point shows a direct connection between restricted secant varieties and secant varieties through the incidence variety. Every variety in turn can be defined by a series of equations which forms an ideal. Determining the ideal for this space and the syzygies associated with it is another area yet to be explored.

Terracini's Lemma is a tool for computing dimensions of varieties constructed as joins or secants by viewing their tangent spaces as sums of linear spaces. Linear spaces are parameterized by the Grassmannian and can be evaluated with tools from linear algebra. Therefore, this can lead to the development of an algorithm for testing defectivity as follows. First, parameterize a variety as a set of linear spaces. Next, determine which Grassmannian or restricted chordal Grassmannian it corresponds to. If it is defective determine if that is enough to say the initial variety is defective. This was done with the first known defective Veronese varieties, and it worked. Another motivation from this same line of thinking is attempting to prove the BDdG conjecture. Reminder, this says the expected list of defective secants of Grassmannians should be exactly those they provided [1]. Yet, one attempt toward a solution could be as follows. Start with the known defective cases. Construct the secant variety of Grassmannians as an incidence variety composed of the restricted chordal variety as these cases all have actual dimension less than the expected dimension. Then, from what is known about the restricted secant varieties, if that restricted secant variety is defective or it can further be decomposed into a defective secant variety then an algorithm exists. Upon the existence of a concrete connection, this could reverse engineer another defectivity or show that the list is complete.

The method of calculating the dimension of spaces with certain types of intersections can be extended to other more complex intersections. A simple extension of the work currently in progress involves r-restricted higher order secant varieties of Grassmannian's. Results have been generated for $r = 1$, $s = 3$, and $k, n \leq 10$, and. Attempts to find the formula for the virtual dimension in this case and others are in progress.

Our initial results focus on the collection of spaces that all had the same r-dimensional subspace in common. However, suppose the slight change was made, so that for a collection of three or more spaces pairwise they share an r-dimensional subspace. What could be said about the dimension of those spaces? This is one specific example of a generalized problem. Also, one approach that is currently being studied involves the known defective cases that are not the skew symmetric matrices. The goal is to find a unifying feature that they all share.

The underlying techniques used to develop this theorem are transmitted into any type of study on defective varieties and dimensions. Given a variety, parameterize it and calculate a collection of test cases. Attempt, by incidence correspondence, to map the unknown variety to one where the dimension is known. A simple first result is to find defective cases. Next, if there are none, attempt to write a proof for the equality of the expected dimension and actual dimension. Otherwise, work towards a proof of the entire list of defective cases or at least conjecture one to start. One place to apply this process is $\mathrm{Seg}((\mathrm{Gr}(k_1, n_1) \times \mathrm{Gr}(k_2, n_2)$. This is the case when there is a product of Grassmannians.

Chapter 5

Appendix

5.1 Naive Implementation for the cone over secant of Grassmannians

```
testnk = (n,k) -> (

R = QQ[a_(0,0)..a_(k-1,n-1),b_(0,0)..b_(k-1,n-1)];

A = transpose genericMatrix(R, a_(0,0), n,k);

B = transpose genericMatrix(R, b_(0,0), n,k);

fun = matrix{apply(subsets(n,k), s-> det A_s + det B_s )};

jac = diff(transpose basis(1, R), fun);

val = map(QQ,R, random(QQ^1,QQ^(dim R)));

rank val jac
```

5.1.1 Improved Computation for cone over restricted secant of Grassmannians

```
R=QQ[x]

M=random(QQ^8,QQ^10);

N= M^{0}||random(QQ^7,QQ^10);

ss = subsets(10,8);

t={0,1,2,3,4,5,6,7}

par = (j, s) -> (c=0; for i to (length (s) -1) do

( if j ==s_i then return i else continue;); return c);
```

```
A=apply(ss,s->flatten apply(8,i->
apply(10,j->if not member(j,s) then 0
else (-1)^(par(j,s)+i)*det(submatrix(M,toList(t-set{i}),toList(s-set{j}))))));
B=apply(ss,s->flatten apply(8,i->
apply(10,j->if not member(j,s) then 0
else (-1)^(par(j,s)+i)*det(submatrix(N,toList(t-set{i}),toList(s-set{j}))))));
F=transpose matrix(A);
G= transpose matrix(B);
C=map(R^70,R^45,0);
D=F||C;
E=submatrix(G,{0..9},)||C||submatrix(G,{10..79},);
L=matrix(entries(D+E));
rank(L)
```

5.1.2 The dimension of the cone over cyclical secant of Grassmannians

```
R=QQ[x]
M=random(QQ^4,QQ^9);
N= M^{3}||random(QQ^3,QQ^9);
O= M^{0}||N^{3}||random(QQ^2,QQ^9);
ss = subsets(9,4);
t=deepSplice{0..3};
par = (j, s) -> (c=0; for i to (length (s) -1) do
( if j ==s_i then return i else continue;); return c);
A=apply(ss,s->flatten apply(4,i->
apply(9,j->if not member(j,s) then 0 else
(-1)^(par(j,s)+i)*det(submatrix(M,toList(t-set{i}),toList(s-set{j}))))));
```

```
B=apply(ss,s->flatten apply(4,i->
apply(9,j->if not member(j,s) then 0 else
(-1)^(par(j,s)+i)*det(submatrix(N,toList(t-set{i}),toList(s-set{j}))))));
C=apply(ss,s->flatten apply(4,i->
apply(9,j->if not member(j,s) then 0 else
(-1)^(par(j,s)+i)*det(submatrix(O,toList(t-set{i}),toList(s-set{j}))))));
D=transpose matrix(A);
E= transpose matrix(B);
F=transpose matrix(C);
G=map(R^27,R^126,0);
H=map(R^18,R^126,0);
I=D||G||H;
J=G||E||H;
K=submatrix(F,{0..8},)||G||H||submatrix(F,{9..35,);
L=(I+J+K);
rank(T)
```

5.1.3 Classifying the orbits of $\bigwedge^3\mathbb{F}_2^6$ under the $\mathrm{SL}_6(\mathbb{F}_2)$-action

The code below gives the user one way to classify the orbits of $\bigwedge^3\mathbb{F}_2^6$ under the $\mathrm{SL}_6(\mathbb{F}_2)$-action. It is very much a brute force approach to find everything in the orbits that are already known and then from there determine what is left and see if it matches the number of points in $\bigwedge^3\mathbb{F}_2^6$. The first section defines functions to calculate the Plucker coordinates, and also maps for the tangential, restricted secant, and secant variety. The next block of code generates the list of the 1395 matrices (points) in $\mathrm{Gr}(3,\mathbb{F}_6^2)$. It then takes pairs of those points, finds the differential, adds them together and uses the rank of the resulting matrix to store its' Plucker coordinates in the list representing the orbit it lies in. After sorting these lists and combining them into one unique bigList the code has to find a representative for the orbit, or potentially orbits, that are unaccounted for. To do this, we need to take a random matrix of the right size, calculate its' Plucker coordinates, and compare them to the bigList. If it is already in the list we continue until we find something new. Once the new element was found, calculating the number of elements in that orbit gave us all points that were missing and completed the entire classification of the orbits in this specific case.

```
R = ZZ/2[e_0..e_5,SkewCommutative => true]
plucker = M -> sum(subsets(6,3), ss-> (det M_ss^{0,1,2})*product(ss, s-> e_s));
tanMap = M -> sum(subsets(6,3), ss-> (det M_ss^{0,1,2})*product(ss, s-> e_s)) +
               sum(subsets(6,3), ss-> (det M_ss^{0,3,4})*product(ss, s-> e_s)) +
      sum(subsets(6,3), ss-> (det M_ss^{1,3,5})*product(ss, s-> e_s));
rsMap = M -> sum(subsets(6,3), ss-> (det M_ss^{0,1,2})*product(ss, s-> e_s)) +
              sum(subsets(6,3), ss-> (det M_ss^{0,3,4})*product(ss, s-> e_s));
sMap = M -> sum(subsets(6,3), ss-> (det M_ss^{0,1,2})*product(ss, s-> e_s)) +
              sum(subsets(6,3), ss-> (det M_ss^{3,4,5})*product(ss, s-> e_s));


L = toList  {0,0,0,0,0,0}..{1,1,1,1,1,1};
rowVecs = drop(L,1);


time rk3 = for xx in subsets(rowVecs,3) list
           if rank(matrix xx) ==3 then sub(matrix xx,R) else continue;


grassMats = {};
grassPlucker = {};
time for xx in rk3 do ( tmpPl = plucker xx; if not member(tmpPl,grassPlucker) then(
           grassMats = append(grassMats,xx);
```

```
           grassPlucker = append(grassPlucker, tmpPl);
   ));


#grassMats
grassPairs = for i to 1395  list
             if grassPlucker#i !=0 then {sub(grassMats#i,ZZ/2),grassPlucker#i}
             else continue;
#grassPairs
grassPairs#0#1
secant = {};
restrictedSecant = {};
tangential = {};
 for AB in subsets(grassPairs, 2) do(
    tmpA = AB#0#0;
    tmpApl = AB#0#1;
    tmpB = AB#1#0;
    tmpBpl = AB#1#1;
    tmpM = tmpA||tmpB;
    tmpMpl = tmpApl + tmpBpl;
    if rank tmpM == 5 then if not member(tmpMpl, restrictedSecant) then (
restrictedSecant = append(restrictedSecant, tmpMpl);
);
    if rank tmpM == 6 then(
        if not member(tmpMpl, secant) then (
secant = append(secant, tmpMpl);
);
```

```
        if not member(tanMap tmpM, tangential) then (
tangential = append(tangential, tanMap tmpM);
);
        )
    );

#restrictedSecant
#tangential
#secant
#grassPlucker
member(0, restrictedSecant)
member(0, tangential)
member(0, secant)
#restrictedSecant + #tangential + #secant + #grassPlucker
2^20
perms = permutations(6);
p2 = map(R, R, (basis(1, R))_(perms_2))
shuffle = f -> sub(f, apply(flatten entries basis(3,R), ee-> ee=> p2 ee))
g2 = p2 \ grassPlucker;
sort g2 == sort grassPlucker

sort p2 secant == sort secant
sort p2 restrictedSecant == sort restrictedSecant

perms = permutations(6);
time for perm in perms do (
```

```
print(#tangential);
p2 = map(R, R, (basis(1, R))_(perm));
tan2 = p2 \ tangential;
tangential = unique ( tan2| tangential);
)

time for perm in perms do (
print(#secant);
p2 = map(R, R, (basis(1, R))_(perm));
sec2 = p2 \ secant;
secant = unique ( sec2| secant);
)

time for perm in perms do (
print(#restrictedSecant);
p2 = map(R, R, (basis(1, R))_(perm));
rsec2 = p2 \ restrictedSecant;
restarictedSecant = unique ( rsec2| restrictedSecant);
)

time bigList = sort(grassPlucker|restrictedSecant|tangential|secant);
#bigList

newList = {};
for i to 100 do (
M = random(R^9,R^6);
```

```
pt = (plucker M^{0,1,2}) + (plucker M^{3,4,5}) + (plucker M^{6,7,8});
if not member(bigList,pt) then newList = unique append(newList, pt);
)
newList
```

5.1.4 Finding the Normal Forms of Defective Secants of Grassmannians

Calculating a normal form for each of the known defective secants of Grassmannians allows the Schouten Diagram and the contraction operator to be used as tools to study those cases for commonalities. One brute force approach that worked to find these normal forms was to use the tables provided in [36] and [21] to check each normal form listed with the appropriate dimension restrictions. Below, the Macaulay2 code checking the dimension of the specific normal form for each defective case is provided.

```
FINAL CODE for Normal Form \sigma_3(Gr(3,7))
restart
R=QQ[x]
M=random(QQ^3,QQ^7);
N= M^{2}||random(QQ^2,QQ^7);
O= N^{2}||random(QQ^1,QQ^7)||M^{0};
P= M^{1}||O^{1}||random(QQ^1,QQ^7);
ss = subsets(7,3);
t=deepSplice{0..2};
par = (j, s) -> (c=0; for i to (length (s) -1) do
( if j ==s_i then return i else continue;); return c);
A=apply(ss,s->flatten apply(3,i->apply(7,j->
if not member(j,s) then 0
else (-1)^(par(j,s)+i)*det(submatrix(M,toList(t-set{i}),toList(s-set{j}))))));
```

```
AA=apply(ss,s->flatten apply(3,i->apply(7,j->
if not member(j,s) then 0
else (-1)^(par(j,s)+i)*det(submatrix(N,toList(t-set{i}),toList(s-set{j}))))));
B=apply(ss,s->flatten apply(3,i->apply(7,j->
if not member(j,s) then 0
else (-1)^(par(j,s)+i)*det(submatrix(O,toList(t-set{i}),toList(s-set{j}))))));
BB=apply(ss,s->flatten apply(3,i->apply(7,j->
if not member(j,s) then 0
else (-1)^(par(j,s)+i)*det(submatrix(P,toList(t-set{i}),toList(s-set{j}))))));

D=transpose matrix(A);
DE=transpose matrix(AA);
E= transpose matrix(B);
EF=transpose matrix(BB);
G=map(R^14,R^35,0);
GG=map(R^21,R^35,0);
GGG=map(R^7,R^35,0);
GGGG=map(R^28,R^35,0);

H=D||GGGG;
I=G||DE||G;
II=submatrix(E,{14..20},)||GG||submatrix(E,{0..13},)||GGG;
J=GGG||submatrix(EF,{0..6},)||GG||submatrix(EF,{7..20},);

K=matrix(entries(H+I+II+J));
rank(K)
```

FINAL CODE for Normal Form \sigma_4(Gr(3,9))

```
restart
R=QQ[x]
M=random(QQ^3,QQ^9);
N= M^{0}||random(QQ^2,QQ^9);
O= M^{0}||random(QQ^2,QQ^9);
P= random(QQ^1,QQ^9) ||M^{1}||N^{2};
PP=random(QQ^1,QQ^9)||M^{1}||O^{2};
MMM=M^{0}||P^{0}||random(QQ^1,QQ^9);
NNN=M^{1}||N^{1}||O^{1};
OOO=O^{2}||N^{2}||M^{2};
ss = subsets(9,3);
--t is list of row indices
t=deepSplice{0..2};
par = (j, s) -> (c=0; for i to (length (s) -1) do
( if j ==s_i then return i else continue;); return c);
A=apply(ss,s->flatten apply(3,i->apply(9,j->
if not member(j,s) then 0
else (-1)^(par(j,s)+i)*det(submatrix(M,toList(t-set{i}),toList(s-set{j}))))));
AA=apply(ss,s->flatten apply(3,i->apply(9,j->
if not member(j,s) then 0
else (-1)^(par(j,s)+i)*det(submatrix(N,toList(t-set{i}),toList(s-set{j}))))));
B=apply(ss,s->flatten apply(3,i->apply(9,j->
if not member(j,s) then 0
else (-1)^(par(j,s)+i)*det(submatrix(O,toList(t-set{i}),toList(s-set{j}))))));
```

```
BB=apply(ss,s->flatten apply(3,i->apply(9,j->
if not member(j,s) then 0
else (-1)^(par(j,s)+i)*det(submatrix(P,toList(t-set{i}),toList(s-set{j}))))));
C=apply(ss,s->flatten apply(3,i->apply(9,j->
if not member(j,s) then 0
else (-1)^(par(j,s)+i)*det(submatrix(PP,toList(t-set{i}),toList(s-set{j}))))));
CCC=apply(ss,s->flatten apply(3,i->apply(9,j->
if not member(j,s) then 0
else (-1)^(par(j,s)+i)*det(submatrix(MMM,toList(t-set{i}),toList(s-set{j}))))));
CCCC=apply(ss,s->flatten apply(3,i->apply(9,j->
if not member(j,s) then 0
else (-1)^(par(j,s)+i)*det(submatrix(NNN,toList(t-set{i}),toList(s-set{j}))))));
CCCCC=apply(ss,s->flatten apply(3,i->apply(9,j->
if not member(j,s) then 0
else (-1)^(par(j,s)+i)*det(submatrix(OOO,toList(t-set{i}),toList(s-set{j}))))));

D=transpose matrix(A);
DE=transpose matrix(AA);
E= transpose matrix(B);
EF=transpose matrix(BB);
F=transpose matrix(C);
FG=transpose matrix(CCC);
FFG=transpose matrix(CCCC);
FFGG=transpose matrix(CCCCC);
G=map(R^18,R^84,0);
GG=map(R^27,R^84,0);
```

```
GGG=map(R^9,R^84,0);
GGGG=map(R^36,R^84,0);


H=submatrix(D,{0..8},)||G||submatrix(D,{9..17},)||GGGG||submatrix(D,{18..26},);
I=submatrix(DE,{0..8},)||GG||submatrix(DE,{9..17},)||GGG||submatrix(DE,{18..26},)||G;
II=submatrix(E,{0..8},)||GGGG||submatrix(E,{9..17},)||GGG||submatrix(E,{18..26},)||GGG;
J=GGG||submatrix(EF,{0..8},)||GGG||submatrix(EF,{9..17},)||G||submatrix(EF,{18..26},)||G
JJ=G||submatrix(F,{0..17},)||GG||submatrix(F,{18..26},)||GGG;
KK=FG||GG||GG;
LL=GG||FFG||GG;
MM=GG||GG||FFGG;
K=matrix(entries(H+I+II+J+JJ+KK+LL+MM));
rank(K)


--FINAL CODE for Normal Form \sigma_3(Gr(4,8))
restart
R=QQ[x]
M=random(QQ^4,QQ^8);
N= random(QQ^1,QQ^8)||M^{1}||M^{2}||random(QQ^1,QQ^8);
O= N^{0}||M^{3}||random(QQ^1,QQ^8)||N^{3};
P= N^{0}||M^{0}||O^{2}||random(QQ^1,QQ^8);
ss = subsets(8,4);
t=deepSplice{0..3};
par = (j, s) -> (c=0; for i to (length (s) -1) do
```

```
( if j ==s_i then return i else continue;); return c);
A=apply(ss,s->flatten apply(4,i->apply(8,j->
if not member(j,s) then 0
else (-1)^(par(j,s)+i)*det(submatrix(M,toList(t-set{i}),toList(s-set{j}))))));
AA=apply(ss,s->flatten apply(4,i->apply(8,j->
if not member(j,s) then 0
else (-1)^(par(j,s)+i)*det(submatrix(N,toList(t-set{i}),toList(s-set{j}))))));
B=apply(ss,s->flatten apply(4,i->apply(8,j->
if not member(j,s) then 0 else
(-1)^(par(j,s)+i)*det(submatrix(O,toList(t-set{i}),toList(s-set{j}))))));
BB=apply(ss,s->flatten apply(4,i->apply(8,j->
if not member(j,s) then 0
else(-1)^(par(j,s)+i)*det(submatrix(P,toList(t-set{i}),toList(s-set{j}))))));


D=transpose matrix(A);
DE=transpose matrix(AA);
E= transpose matrix(B);
EF=transpose matrix(BB);
G=map(R^16,R^70,0);
GG=map(R^24,R^70,0);
GGG=map(R^8,R^70,0);
GGGG=map(R^32,R^70,0);

H=GGG||D||GG;
I=submatrix(DE,{0..7},)||GGG||submatrix(DE,{8..23},)||G||submatrix(DE,{24..31},)||GGG;
```

```
II=submatrix(E,{0..7},)||GG||submatrix(E,{8..31},)||GGG;
J=submatrix(EF,{0..15},)||GG||submatrix(EF,{16..23},)||GGG||submatrix(EF,{24..31},);

K=matrix(entries(H+I+II+J));
rank(K)
```

Bibliography

[1] H. Abo, G. Ottaviani, and C. Peterson, *Non-defectivity of Grassmannians of planes*, J. Algebraic Geom. **21** (2012), no. 1, 1–20.

[2] H. Abo, *On three conjectures about the secant defectivity of classically studied varieties*, Proceedings of the algebraic geometry symposium, Waseda University, Tokyo, Japan, 2010.

[3] J. Alexander and A. Hirschowitz, *La méthode d'Horace éclatée: application à l'interpolation en degré quatre*, Invent. Math. **107** (1992), no. 3, 585–602.

[4] ———, *Polynomial interpolation in several variables*, J. Algebraic Geom. **4** (1995), no. 2, 201–222.

[5] L. V. Antonyan, *Classification of four-vectors of an eight-dimensional space*, Trudy Sem. Vektor. Tenzor. Anal. **20** (1981), 144–161.

[6] E. Arrondo, A. Bernardi, P. M. Marques, and B. Mourrain, *Skew-symmetric tensor decomposition*, Commun. Contemp. Math. **23** (2021), no. 02, 1950061.

[7] K. Baur, J. Draisma, and W. A. de Graaf, *Secant dimensions of minimal orbits: computations and conjectures*, Experiment. Math. **16** (2007), no. 2, 239–250.

[8] P. Beelen, S. R Ghorpade, and T. Hoholdt, *Affine Grassmann codes*, IEEE Trans. Inform. Theory **56** (2010), no. 7, 3166–3176.

[9] A. Bernardi and D. Vanzo, *A new class of non-identifiable skew-symmetric tensors*, Ann. Mat. Pura Appl. (4) **197** (2018), no. 5, 1499–1510.

[10] D. Bidleman and L. Oeding, *Restricted secant varieties of grassmannians*, Collectanea Mathematica (2023), 1–21.

[11] A. Boralevi, *A note on secants of Grassmannians*, Rend. Istit. Mat. Univ. Trieste **45** (2013), 67–72.

[12] A. Casarotti and M. Mella, *Tangential weak defectiveness and generic identifiability*, Int. Math. Res. Not. **2022** (2022), no. 19, 15075–15091.

[13] M. V. Catalisano, A. V. Geramita, and A. Gimigliano, *On the rank of tensors, via secant varieties and fat points. Zero-dimensional schemes and applications*, Proceedings of the Workshop, Naples, Italy, Queen's Papers Pure Appl. Math. **123** (2002), 133–147.

[14] ———, *Secant varieties of Grassmann varieties*, Proc. Amer. Math. Soc. **133** (2005), no. 3, 633–642.

[15] N. S Daleo, J. D Hauenstein, and L. Oeding, *Computations and equations for Segre-Grassmann hypersurfaces*, Port. Math. **73** (2016), no. 1, 71–90.

[16] D. Di Tullio and M. Gyawali, *A post-quantum signature scheme from the secant variety of the Grassmannian*, Cryptology ePrint Archive (2020). `https://eprint.iacr.org/2020/1141`.

[17] W. Fulton and J. Harris, *Representation theory, a first course*, Graduate Texts in Mathematics, vol. 129, New York: Springer-Verlag, 1991.

[18] D. R. Grayson and M. E. Stillman, *Macaulay2, a software system for research in algebraic geometry.* Available at `http://www.math.uiuc.edu/Macaulay2/`.

[19] G. B. Gurevich, *Foundations of the theory of algebraic invariants*, Translated by J. R. M. Radok and A. J.M. Spencer, P. Noordhoff Ltd., Groningen, 1964.

[20] D. Hankerson, G. Hoffman, D. A Leonard, C. C Lindner, K. T Phelps, C. A Rodger, and J. R Wall, *Coding theory and cryptography: the essentials*, CRC Press, 2000.

[21] F. Holweck and L. Oeding, *Hyperdeterminants from the E8 discriminant*, J. Algebra **593** (2022), 622–650.

[22] F. Holweck and L. Oeding, *Jordan decompositions of tensors* (2022). `arXiv:2206.13662`.

[23] T. Kolda and B. Bader, *Tensor decompositions and applications*, SIAM Rev. **51** (2009), no. 3, 455–500.

[24] J. M. Landsberg, *Tensors: geometry and applications*, Graduate Studies in Mathematics, vol. 128, American Mathematical Society, Providence, RI, 2012.

[25] J. M. Landsberg and J. Weyman, *On the ideals and singularities of secant varieties of Segre varieties*, Bull. Lond. Math. Soc. **39** (2007), no. 4, 685–697.

[26] A. Massarenti and R. Rischter, *Non-secant defectivity via osculating projections*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **19** (2019), no. 1, 1–34.

[27] B. McGillivray, *A probabilistic algorithm for the secant defect of Grassmann varieties*, Linear Algebra Appl. **418** (2006), no. 2-3, 708–718.

[28] D Y. Nogin, *Codes associated to Grassmannians*, Arithmetic, Geometry, and Coding Theory: Proceedings of the International Conference held at Centre International de Rencontres de Mathématiques (CIRM), Luminy, France, June 28 - July 2, 1993, 2011, pp. 145–154.

[29] L. Oeding, *A translation of "Classification of four-vectors of an 8-dimensional space," by Antonyan, L. V. , with an appendix by the translator*, Trudy Mosk. Mat. Obs. **83** (2022).

[30] F. Palatini, *Sulle varieta algebriche per le quali sono di dimensione minore dell'ordinario senza riempire lo spazio ambiente una o alcune delle varieta formate da spazi seganti*, 1909.

[31] C. Ryan, *An application of Grassmannian varieties to coding theory*, Congr. Numer **57** (1987), 257–271.

[32] G. Scorza, *Sopra la teoria delle figure polari delle curve piane del 4. o ordine*, Annali di Matematica Pura ed Applicata (1898-1922) **2** (1899), 155–202.

[33] A. Terracini, *Sulle V_k per cui la variet'a degli S_h $(h+1)$-secanti ha dimensione minore dell'ordinario*, Rend. Circ. Mat. Palermo **Selecta vol. I** (1911), 392–396.

[34] ———, *Sulla rappresentazione delle forme quaternarie mediante somme di potenze di forme lineari*, 1916.

[35] A. Terracini, *On the vk for which the variety à of the sh (h+ 1) segments has dimension smaller than the ordinary*, Reports of the Mathematical Circle of Palermo (1884-1940) **31** (1911), 392–396.

[36] È. B. Vinberg and A. G. Èlašvili, *A classification of the three-vectors of nine-dimensional space*, Trudy Sem. Vektor. Tenzor. Anal. **18** (1978), 197–233.

[37] F. L. Zak, *Tangents and secants of algebraic varieties*, Transl. Math. Monographs, vol. 127, Providence: Amer. Math. Soc., 1993.