Elementary Submodels of Function Spaces

by

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Abstract

In this thesis we develop the now-standard tool of elementary submodels and apply the technique to topological function spaces. We show that many cardinal relations regarding function spaces hold in suitable models. Furthermore, we give an example of a function space and submodel in which the tightness increases when passing to the model and prove that it is consistent that perfect normality is downwards preserved in submodels.

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Chapter 1

Introduction

Since the seminal paper of Alan Dow [Dow88], elementary submodels have found many applications in set-theoretic topology as a method of simplifying proofs which use a “closing off” argument. Elementary submodels allow us to easily define approximations of topological spaces we are interested in, and thus simplify our analysis. Many of these proofs work by finding a “small” approximation to a space and then showing that the approximation reflects enough information of the original space. Once we have done that, calculations done in the approximation can be used to prove results about the original space.

We make this idea rigorous using some important historical facts from set theory and logic. The standard set-theoretic universe, V is built recursively. However, by Gödel’s Incompleteness Theorems, ZFC cannot prove the existence of a model of ZFC. So, we must take some (sufficiently large) initial segment of this universe in which to do study our approximations. Once we have truncated the universe to some tractable level, we can use the Löwenheim-Skolem Theorem to get submodels of arbitrary size, containing as many parameters as we would like.

Within the submodel M, we can define a new topology on $X \cap M$, which allows M to reason about X. C_p-theory was developed notably by Alexander Arkhangel’skii, culminating in the 1992 book *Toplological Function Spaces*. It studies the relationships between X and the set of continuous maps from X to $\mathbb{R}$, denoted $C_p(X)$, with the topology of pointwise convergence. In [JT98], Lúcia Junqueira and Franklin Tall compiled many results about the relationships between X and X_M depending on the properties of X and M. In this thesis, we prove related results in the specific case of $C_p(X)$.

Chapter 2

Elementary Submodels

2.1 First Order Logic

In this section, we give an introduction to the necessary logic and model theory. More background on this material can be found in the classic text [Kun11], and this presentation very closely follows Chapter 4 of [HSW99] and [Gil21]. To start, we need to define the *language* of set theory. The language, denoted $\mathcal{L}$, consists of

(a) connectives: $\neg, \wedge, \vee, \implies, \iff$

(b) quantifiers: $\forall, \exists$

(c) infinitely many variables v_i

(d) the symbols $=$ and $\in$ which denote equality and membership. For our purposes, we will also have a well-ordering $<$ on our structures.

Formulas are built inductively. The atomic formulas are $(v_i = v_j)$, $(v_i \in v_j)$, and $(v_i < v_j)$. Then, if ϕ and φ are formulas, $\neg\phi$ and $\phi * \varphi$ are formulas where star is any of the other connectives. Additionally, $(\exists v_i \varphi)$ and $(\forall v_j \varphi)$ are formulas. Variables in a formula are of two types. If a variable appears in the scope of a quantifier, then it is bound, otherwise it is free. For example, in the formula $\exists v_0\ v_0 \in v_1$, the variable v_0 is bound, while v_1 is free. All of our formulas will be well-formed: variables which appear free will always appear free.

Formulas can be thought of as expressing properties of their free variables, and if a formula has no free variables, it should be either true or false. If a formula has no free variables, it is called a sentence. A collection of sentences is called a theory, and the elements of a theory T are its axioms.

Definition 2.1. *A structure $\mathcal{A} = (A, \in, <)$ consists of a non-empty set A called the universe, an inclusion relation $\in$, and a well-ordering $<$.*

Let $\mathcal{A}$ be a structure. If $\varphi(v_0, \ldots, v_{n-1})$ is a formula with free variables $v_0, \ldots, v_{n-1}$, and $a_0, \ldots, a_{n-1}$ is a collection of elements of A, we want to define the formula

$$\mathcal{A} \models \varphi(a_0, \ldots, a_n)$$

to mean that the property expressed by φ of the parameters is true. We define these formulas by induction on complexity. That is, we start with the atomic formula:

$$\mathcal{A} \models v_i \in v_j(a_0, \ldots, a_{n-1}) \iff a_i \in a_j$$
$$\mathcal{A} \models v_i = v_j(a_0, \ldots, a_{n-1}) \iff a_i = a_j$$
$$\mathcal{A} \models v_i < v_j(a_0, \ldots, a_{n-1}) \iff a_i < a_j$$

The definitions extend naturally to the quantifiers and connectives. As an example, we say

$$\mathcal{A} \models \exists v_n \psi(a_0, \ldots, a_{n-1}) \iff \exists a \in A \; \mathcal{A} \models \psi(a_0, \ldots, a_{n-1}, a)$$

Now, if the universal (with regards to A!) closure $\forall \vec{v} \varphi$ of a formula φ is provable in $\mathcal{A}$, then we say $\mathcal{A}$ models φ, denoted $\mathcal{A} \models \varphi$. Similarly, if $\mathcal{A}$ models every sentence of a theory T, we write $\mathcal{A} \models T$.

Definition 2.2. *If $\mathcal{A} = (A, \in, <_A)$ and $\mathcal{B} = (B, \in, <_B)$ are structures, we say that $\mathcal{A}$ is a substructure of $\mathcal{B}$ if $A \subseteq B$ and the well-ordering on A is the induced well-order from $<_B$. $\mathcal{A}$ is an elementary substructure of $\mathcal{B}$ if for every formula φ and for every $a_0, \ldots, a_{n-1} \in A$ we have*

$$\mathcal{A} \models \varphi(a_0, \ldots, a_{n-1}) \iff \mathcal{B} \models \varphi(a_0, \ldots, a_{n-1})$$

In other words, for $\mathcal{A}$ to be an elementary substructure of $\mathcal{B}$ means that $\mathcal{A}$ and $\mathcal{B}$ agree about the properties of elements of A. The following is a convenient characterization of elementary substructures which we will use to prove the next theorem.

Lemma 2.1 (Tarski-Vaught Criterion). *Assume that $\mathcal{A} \subseteq \mathcal{B}$. Then $\mathcal{A} \preceq \mathcal{B}$ if and only if for any formula ψ and for any $a_0, \ldots, a_{n-1} \in A$*

$$\mathcal{B} \models \exists x_n \psi(a_0, \ldots, a_{n-1}) \implies (\exists a \in A)\ \mathcal{B} \models \psi(a_0, \ldots, a_{n-1}, a)$$

Theorem 2.1 (Downward Löwenheim-Skolem). *Let $\mathcal{A}$ and $\mathcal{B}$ be infinite structures such that $\mathcal{A} \subseteq \mathcal{B}$ and $Y \subseteq B$. Then there exists and elementary substructure $\mathcal{M} \preceq \mathcal{B}$ such that $\mathcal{A} \subseteq \mathcal{M}$, $Y \subseteq M$ and $|M| = \max\{|A|, |Y|\}$*

Proof. Let $M_0 = Y \cup A$, and let $\kappa = |M_0|$. By the previous lemma, we only need to make sure that the structure we construct contains witnesses to all existential statements that $\mathcal{B}$ models. So, for every formula φ and set $a_0, \ldots, a_{n-1} \in M_0$ such that $\mathcal{B} \models \exists x\, \varphi(a_0, \ldots, a_{n-1})$, we choose a witness a. Let M_0^* be the set of all of these witnesses. Clearly $M_0 \subseteq M_0^*$, and we just need to check that we didn't increase the cardinality. However, $|[M_0]^{<\omega}| = |M_0|$, and there are only countably many formulas, so everything works. Let $M_1 = M_0^*$, and analogously define M_{n+1} from M_n. Set $M = \cup_{n\in\omega} M_n$ as the universe of $\mathcal{M}$ with the inherited well-order. Then by construction and the Tarski-Vaught criterion, $\mathcal{M} \preceq \mathcal{B}$. $\square$

This theorem will allow us to put all of the relevant information into a submodel of desired cardinality.

2.2 $H(\theta)$

We would like to say that we are taking submodels of the set theoretic universe V which satisfy ZFC. Unfortunately, by Gödel's famous theorem, ZFC cannot prove the existence of such a model. However, we can get sufficiently good approximations of the universive satisfying nearly all of ZFC.

Definition 2.3. *A set x is transitive if $y \in x$ implies $y \subseteq x$. We define $tc(x)$ to be the transitive closure of x, the intersection of all transitive sets containing x. For any cardinal κ, let $H(\kappa)$ be the collection of all sets x such that $|tc(x)| < \kappa$*

When κ is regular and uncountable, $H(\kappa)$ is a model of ZFC minus the power set axiom.

Suppose that M is transitive, and φ is a formula. We define the relativization of φ to M, denoted φ^M by induction on the complexity of the formula. For atomic formula, nothing changes, i.e. $(x \in y)^M$ is just $x \in y$. For formula built using connectives, we have

$$(\neg\varphi)^M \text{ is } \neg(\varphi^M) \text{ and } (\varphi \vee \phi)^M \text{ is } \varphi^M \vee \psi^M.$$

Quantifiers just become restricted to M. That is, $(\exists x\varphi)^M$ becomes $(\exists x \in M)\varphi$.

Definition 2.4. *A formula $\varphi(v_0, \ldots, v_{n-1})$ is called absolute for M if*

$$\forall\, x_0, \ldots, x_{n-1} \in M\, (\varphi \iff \varphi^M)$$

is provable in ZF.

So, if formulas are absolute, then what the model (either $H(\theta)$ or some $\mathcal{M} \preceq H(\theta)$ for our purposes) witnesses about elements of the model is actually true in the set theoretic universe. However, for any proof, we only need finitely many statements.

Theorem 2.2 (Reflection Principle). *Let φ_i for $i = 0, \ldots, n$ be finitely many formulas. Then there exists θ uncountable such that for every each φ_i is absolute for $H(\theta)$.*

In applications of submodels to topology, we know beforehand the cardinalities of all of the objects we would like to consider. Thus, we can pick θ big enough so that $H(\theta)$ satisfies "enough" power set. At this point, $H(\theta)$ is for our purposes a true model of the universe, and so what $H(\theta)$ "thinks" about objects is really true.

One of the most important properties of elementary submodels of $H(\theta)$ is *closure under definability*.

Definition 2.5. *Suppose that $\mathcal{M} \preceq H(\theta)$, that $\varphi(x, a_0, \ldots, a_{n-1})$ is a formula, and that $a_0, \ldots, a_{n-1} \in M$. Then the set*

$$b \coloneqq \{x \in H(\theta) : H(\theta) \models \varphi(x, a_0, \ldots, a_{n-1})\}$$

is said to be definable by parameters from M.

Lemma 2.2. *If $\mathcal{M} \preceq H(\theta)$, and $b \in H(\theta)$ is definable by parameters from M, then $b \in M$.*

Proof. We have

$$H(\theta) \models \exists\, z\, \forall\, x\, (x \in z \iff \varphi(x, a_0, \ldots, a_{n-1})).$$

By elementarity, $\mathcal{M}$ models the same thing. So, there exists $c \in M$ such that

$$c = \{x \in H(\theta) : H(\theta) \models \varphi(x, a_0, \ldots, a_{n-1})\}.$$

By Extensionality, $b = c$. □

This lemma is extremely useful, and immediately gives us the following:

Lemma 2.3. *Suppose that $\mathcal{M} \preceq H(\theta)$. Then the following are members of M if they are members of $H(\theta)$:*

(a) $\omega, \mathbb{R}, \mathbb{Q}$. *Additionally,* $\omega \subseteq M$

(b) κ^+ *for any cardinal* $\kappa \in M$

(c) $|b|$ *for any* $b \in M$

(d) $dom(f), ran(f)$, *and* $f|b$ *for any* $b \in M$ *and function* $f \in M$

(e) $f(b)$ *for any function* $f \in M$ *and* $b \in M \cap dom(f)$

In general, it is not always true that $x \in M$ implies $x \subseteq M$ or vice-versa. Many times though, small elements are subsets.

Lemma 2.4. *Suppose that $\mathcal{M} \preceq H(\theta)$, and that $\kappa \subseteq M$ and $\kappa \in M$. If $x \in M$ is such that $|x| = \kappa$, then $x \subseteq M$*

Proof. Since $|x| = \kappa$, we have

$$H(\theta) \models \exists\, f : \kappa \to x\, (f \text{ is bijective}).$$

The two parameters of this, κ and x, are both in M, so we have

$$\mathcal{M} \models \exists\, f : \kappa \to x\, (f \text{ is bijective}).$$

And now we can repeatedly apply the Lemma. $\square$

However, sometimes it is beneficial to have small subsets be elements as well. We can construct such models and not increase the size of the model by too much.

Lemma 2.5. *Suppose $\theta > \mathfrak{c}$, and $X \subseteq H(\theta)$ is such that $|X| \leq \mathfrak{c}$. Then there exists $\mathcal{M} \preceq H(\theta)$ such that $X \subseteq M$, $|M| = \mathfrak{c}$, and $[M]^\omega \subseteq M$.*

There are similar lemmas for other cardinals, but this one will be sufficient for our purposes. In general, if a model has the property that $[M]^\kappa \subseteq M$, then we say that M is κ-closed. In particular, we can use this lemma to say that M can reason about countable covers, networks, etc. Going forward, we will slightly abuse notation and drop the distinction between $\mathcal{M}$ and M and only write M.

Chapter 3

Function Spaces

3.1 Basic Facts about $C_p(X)$

Given a $T_{3\frac{1}{2}}$ topological space $\langle X, \tau \rangle$, we will be concerned with the following function space:

Definition 3.1. *$C_p(X)$ is the set of continuous functions from X to $\mathbb{R}$ with the topology of pointwise convergence. A basic open set is of the form*

$$B(f, x_1, \ldots, x_n, \epsilon) \coloneqq \{g \in C_p(X) : \forall i \leq n \ |g(x_i) - f(x_i)| < \epsilon\},$$

and we denote this standard basis $\mathcal{B}_p = \{B(f, x_1, \ldots, x_n, \epsilon) : f \in C_p(X) \wedge \epsilon \in \mathbb{R}\}$.

$C_p(X)$ is a dense subspace of $\mathbb{R}^X$ with the product topology, almost immediately by the definition and the fact that X is completely regular. In fact, if X is discrete, then the two spaces are equal. An important property of these function spaces is that we can embed X into $C_p(C_p(X))$ using the evaluation map:

$$j : X \to C_p(C_p(X))$$

given by

$$x \mapsto j(x) : C_p(X) \to \mathbb{R}$$

where $j(x)$ is "evaluate at x": $j(x)(f) = f(x)$.

In addition to topological structure, the space $C_p(X)$ has algebraic structure, making it into a *topological ring* under function addition and multiplication, meaning that addition and multiplication by elements are continuous maps. The algebraic properties of $C_p(X)$ will be important for the analysis in the next section, as seen by the following fact:

Lemma 3.1. *Topological groups are homogeneous.*

Proof. Let G be a topological group, and $x, y \in G$ arbitrary. Then, since the group operation is continuous, $\varphi_{xy} : G \to G$ defined by $a \mapsto ax^{-1}y$ is a homeomorphism sending x to y. $\square$

This means that when considering local cardinal functions such as (pseudo)character, it suffices to look at, for example, the function which is everywhere 0. We now recall the definitions of the cardinal functions we will look at. All of the definitions are standard and can be found in [Ark92] or [HOD84]

Definition 3.2. *Let X be an arbitrary space.*

(a) The weight of X, denoted $w(X)$, is the minimum cardinality of a basis.

(b) The character of X, denoted $\chi(X)$, is the supremum over all points $x \in X$ of the minimum cardinality of a local base at x.

(c) The pseudocharacter of X, denoted $\psi(X)$ is the supremum over all points $x \in X$ of the minimum cardinality of a family of open sets γ such that $\cap \gamma = \{x\}$.

(d) The tightness of X, denoted $t(X)$, is the supremum over all points $x \in X$ of the minimum cardinality κ such that $x \in \overline{A}$ implies that there exists $B \subseteq A$ with $|B| \leq \kappa$ and $x \in \overline{B}$.

(e) A network for X is a collection of sets $\mathcal{N}$ such that for every $x \in X$ and U open with $x \in U$, there exists $N \in \mathcal{N}$ with $x \in N \subseteq U$. The netweight, denoted $nw(X)$ is the minimum cardinality of a network.

(f) The i-weight, denoted $iw(X)$, is the minimum weight of a space Y such that there exists a bijective continuous $f : X \to Y$.

(g) The density, denoted $d(X)$, is the minimum cardinality of a dense subspace of X.

(h) The Lindelöf number of X, denoted $L(X)$, is the smallest cardinal κ such that every open cover $\mathcal{U}$ of X has a subcover of cardinality κ.

Some of these cardinal functions, such as weight, netweight, and i-weight, are *monotone*, meaning that if $Y \subseteq X$, then $\phi(Y) \leq \phi(X)$ where ϕ is one of those cardinal functions. For cardinal functions ϕ which are not monotone, it makes sense to define a new cardinal function $h\phi = \sup\{\phi(Y) : Y \subseteq X\}$, the hereditary version. It is easily seen that the weight is an upper bound on all of the other cardinal functions listed, and this fits with the fact that a basis carries all of the necessary information about a topological space. We also have some elementary facts.

Lemma 3.2. $d(X) \leq nw(X)$

Lemma 3.3. $nw(X) = nw(C_p(X))$

Proof. Let $\mathcal{N}$ be a network for X, and let $\mathcal{B}$ be a countable basis for $\mathbb{R}$. We will define a network for $C_p(X)$ as follows. For every $S_1, \ldots, S_k \in \mathcal{N}$ and $U_1, \ldots, U_k \in \mathcal{B}$ define

$$W(S_1, \ldots, S_k, U_1, \ldots, U_k) \coloneqq \{f \in C_p(X) : \forall i \leq k \; [f(S_i) \subseteq U_i]\}.$$

Let $\mathcal{N}_{C_p}$ be the collection of all such sets as the choices range over $\mathcal{N}$ and $\mathcal{B}$. Pick $f \in C_p(X)$ and let $W(f, x_1, \ldots, x_n, \epsilon)$ be an arbitrary open set. Pick $U_i \in \mathcal{B}$ such that $f(x_i) \in U_i \subseteq B_\epsilon(f(x_i))$. By continuity, $V_i = f^{-1}(U_i)$ is open, so we can pick $S_i \in \mathcal{N}$ such that $x_i \in S_i \subseteq V_i$. Then

$$W(S_1, \ldots, S_k, U_1, \ldots, U_k) \subseteq W(f, x_1, \ldots, x_n, \epsilon).$$

This proves that $\mathcal{N}_{C_p}$ is a network. Since $|\mathcal{N}_{C_p}| = |\mathcal{N}|$, we get that $nw(C_p(X)) \leq nw(X)$. From the above discussion and the fact that net-weight is monotone, we get that

$$nw(X) \leq nw(C_p(C_p(X))) \leq nw(C_p(X)).$$

□

Lemma 3.4. $iw(X) \geq \psi(X)$.

Proof. Pick $x \in X$ arbitrary, and let $f : X \to Y$ be a condensation. Fix a basis $\mathcal{B}$ for Y such that $|\mathcal{B}| = w(Y)$, and let $x \in X$ be arbitrary. Define

$$\mathcal{B}_x := \{U \in \mathcal{B} : f(x) \in U\}.$$

I claim that

$$\bigcap_{U \in \mathcal{B}_x} f^{-1}(U) = \{x\}.$$

Clearly, x is in the intersection. If $y \neq x$, then $f(y) \neq f(x)$, so there exists $U \in \mathcal{B}_x$ such that $f(y) \notin U$. Then $y \notin f^{-1}(U)$. □

3.2 X_M and $C_p(X)_M$

Given a topological space $\langle X, \tau \rangle$ and a submodel $M \preceq H(\theta)$, there are two interesting topologies to put on $X \cap M$. The first is the standard subspace topology, i.e. $\{U \cap M : U \in \tau\}$. However, it is often advantageous to consider the following *submodel topology*:

$$\{U \cap M : U \in \tau \cap M\}.$$

Now M can reason about open sets since they are elements of M, and we denote $X \cap M$ with this topology by X_M. Note that this is in general a coarser topology. Useful results can be derived by comparing $X \cap M$ and X_M. Here we prove some important facts about $C_p(X)_M$.

Lemma 3.5. *(1)* $\mathcal{B}_p \cap M = \{B(f, F, n) : f \in M,\ F \subseteq M\}$.

(2) $C_p(X)_M$ *densely embeds into* $C_p(X_M)$ *(hence in* $\mathbb{R}^{X \cap M}$*).*

(3) The canonical embedding $j : X \to C_pC_p(X)$ *is an element of* M *and* $j|_M$ *is an embedding of* X_M *into* $(C_pC_p(X))_M$.

Proof. (1) If $F \subseteq M$, then $F \in M$, and together with $f \in M$ they give $B(f, F, n) \in M$. On the other hand, if B is a basic open set in M then there are f, F, n such that $B = B(f, F, n)$ and by elementarity, they can be picked in M.

(2) Define $\phi : C_p(X)_M \to C_p(X_M)$ to be the restriction map: $\phi(f) = f|_M$ for each $f \in C_p(X)_M$. We need to check that for each $f \in C_p(X)_M$, we have that $\phi(f) : X_M \to \mathbb{R}$ is continuous. Both the standard basis $\mathcal{B}$ of $\mathbb{R}$ and f are in M, and for all $f \in M$, and $U \in \mathcal{B}$, $f^{-1}(U)$ is definable, thus in M. Therefore,

$$H(\theta) \models \ \forall U \in \mathcal{B}\ (f^{-1}(U) \in \tau)$$

and by elementarity,

$$M \models \forall U \in \mathcal{B}\,(f^{-1}(U) \in \tau).$$

This shows that ϕ truly maps into $C_p(X_M)$. Further ϕ is injective. Suppose that $f \neq g$. Then

$$H(\theta) \models \exists\, x \in X\ (f(x) \neq g(x))$$

and by elementarity

$$M \models \exists\, x \in X\ (f(x) \neq g(x))$$

which is equivalent to

$$H(\theta) \models \exists\, x \in X \cap M\,(f(x) \neq g(x)).$$

Thus, $\phi(f) \neq \phi(g)$. Part (1) immediately implies that ϕ is an embedding. It remains to show that the image is dense. Fix a finite $F \in X \cap M$, and $\epsilon > 0$. Then for each $x \in F$, there exists $q_x \in \mathbb{Q} \cap B_\epsilon(x)$ (remember that $\mathbb{Q} \subseteq M$), and thus we can find $f \in C_p(X)$ such that $f(x) = q_x$ for each $x \in F$. By elementarity, there must exist such an f in M.

(3) The space $C_pC_p(X)$ and its standard base are in M and j is definable from $X, C_p(X)$, and $C_pC_p(X)$ and thus is an element of M. Since $j : X \to C_pC_p(X)$ is an embedding and bases of X and $C_pC_p(X)$ are elements of M, elementarity implies that $j|_M : X_M \to (C_pC_p(X))_M$ is also an embedding. $\square$

Part 2 of this lemma is very useful. In particular, it implies that if ϕ is a monotone cardinal function, then $\phi(C_p(X)_M) \leq \phi(C_p(X_M))$. Using information about X_M, we can use known results on function spaces to calculate cardinal functions of $C_p(X_M)$ since it is just the usual function space of X_M. Thus, we get many results about $C_p(X)_M$.

Lemma 3.6. *$C_p(X)_M$ is a topological ring.*

The homogeneity of $C_p(X)_M$ will be important in the next section.

3.3 Downwards preservation in $C_p(X)_M$

In this section, we try to answer analogous questions from [JT98] for $C_p(X)$. Namely, under what circumstances does $C_p(X)$ having property $\mathcal{P}$ imply that $C_p(X)_M$ have property $\mathcal{P}$. First, we investigate cardinal functions.

Proposition 3.1. *$|X_M| = \chi(C_p(X)_M) = w(C_p(X)_M)$.*

Proof. We adjust the standard proof.

By Lemma 3.5 part (2), we have

$$\chi(C_p(X)_M) \leq w(C_p(X)_M) \leq w(\mathbb{R}^{X\cap M}) \leq |X \cap M| = |X_M|$$

Suppose $\chi(C_p(X)_M) < |X \cap M|$. Let f_0 be the everywhere zero function on X. Since $C_p(X)_M$ is a topological ring and $f_0 \in M$, f_0 witnesses our assumption: there is a local base $\gamma \subseteq \mathcal{B}_p \cap M$ at 0_X consisting of sets of the form $B(f_0, F, n)$, such that $|\gamma| < |X \cap M|$. Let $W = \bigcup\{F : B(f_0, F, n) \in \gamma\}$. Then $|W| < |X \cap M|$. By Lemma 3.5 part (1), $W \subseteq X \cap M$. Pick $x \in (X \cap M)\backslash W$ and let $B = B(f_0, \{x\}, 1)$. Then $B \in M \cap \mathcal{B}_p$. Fix any finite $F \subseteq W \subseteq M$. Then there is $g \in C_p(X)$ that maps all of F to zero and x to 1. Since all parameters are in M, we can assume $g \in M$. So $g \in C_p(X)_M \cap V$ for all $V \in \gamma$ but $g \notin B$. This contradicts γ being a local base. $\square$

Proposition 3.2. *$nw(X_M) = nw(C_p(X)_M)$.*

Proof. We know $nw(X_M) = nw(C_p(X_M))$. By Lemma 3.5 part (2), it follows that $nw(C_p(X)_M) \leq nw(C_p(X_M))$. So we have $nw(C_p(X)_M) \leq nw(X_M)$.

On the other hand, this inequality together with Lemma 3.5 part (3) gives $nw(X_M) \leq nw((C_pC_p(X))_M) \leq nw(C_p(X)_M)$. $\square$

Lemma 3.7. *Let $Y \subseteq X_M$. If $Y \in M$, then $\overline{Y}^{X_M} = \overline{Y}^X \cap M$.*

Proof. Take $x \in \overline{Y}^X \cap M$. Then for all open V containing x, we have $V \cap Y \neq \emptyset$, in particular for such $V \in M$. Now suppose that $x \in \overline{Y}^{X_M}$. Then

$$M \models \forall V \in \tau \, (x \in V \implies V \cap Y \neq \emptyset)$$

which depends only on elements in M, so it is true in the universe. □

Lemma 3.8. *The following are modifications of well-known theorems found in [Ark92].*

(a) $iw(C_p(X)_M) \leq d(X_M)$.

(b) If $\kappa = \psi(C_p(X)_M)$ and $[M]^\kappa \subseteq M$ then $d(X_M) \leq \psi(C_p(X)_M)$. Moreover, $d(X_M) = iw(C_p(X)_M) = \psi(C_p(X)_M)$.

Proof. For (1), let $\lambda = d(X_M)$ and take Y such that $\overline{Y} = X_M$ and $|Y| \leq \lambda$. Then $w(C_p(Y)) \leq \mathbb{R}^Y \leq \lambda$ and the restriction map $\pi_Y : C_p(X_M) \to Z \subseteq C_p(Y)$ is a condensation. By Lemma 3.5 part 2, there is an embedding $\phi : C_p(X)_M \to C_p(X_M)$. So the composition $\pi_Y \circ \phi : C_p(X)_M \to Z_1 \subseteq C_p(Y)$ is also a condensation. Then we have,

$$iw(C_p(X)_M) \leq w(Z_1) \leq w(C_p(Y)) \leq \lambda.$$

For (2), let $\kappa = \psi(C_p(X)_M)$ and let $f_0 \in C_p(X)$ be the constantly zero function. Then $f_0 \in C_p(X)_M$. Fix a family γ of basic open sets of $C_p(X)_M$ such that $\bigcap \gamma = \{f_0\}$ and $|\gamma| \leq \kappa$. Since each $B(f_0, F, n) \in \gamma$ is an element of M, we get that $F \subseteq M$ as well. So, the set $Y = \bigcup_{B(f_0,F,n)\in\gamma} F \subseteq M \cap X$ and $|Y| \leq \kappa$. Since $M^\kappa \subseteq M$ we have $Y \in M$.

Next we show that $\overline{Y}^{X_M} = X_M$. Suppose not. Then by Lemma 3.7 we can pick $x \in X_M \backslash (\overline{Y}^X \cap M)$, and since $X_M \subseteq M$, we get that $x \notin \overline{Y}^X$. Since $X, Y \in M$ we have that $\overline{Y}^X \in M$ as well[1], and therefore there exists $g \in C_p(X) \cap M = C_p(X)_M$ such that $g(x) = 1$ and $g(Y) = \{0\}$. So, $g \in \bigcap \gamma$, $g \in C_p(X)_M$ and $g \neq f$ – a contradiction.

[1] $\overline{Y}^X$ is definable

The equality $d(X_M) = iw(C_p(X)_M) = \psi(C_p(X)_M)$ follows from Lemma 3.6 applied to the space $C_p(X)_M$ $\square$

Theorem 3.1. *From [JT98]:*

(a) Let $f \in \{c, hL, hd, \chi, \psi, s, w\}$. Then $f(X_M) \leq f(X)$.

(b) Let $f \in \{L, t\}$. Then both of the inequalities $f(X) < f(X_M)$ and $f(X) > f(X_M)$ are possible.

Our first goal will be to exhibit an example of a function space witnessing number (2). The following can be found in chapter 2 of [Ark92].

Theorem 3.2 (Pytkeev-Arkhangel'skii). $t(C_p(X)) = \sup\{L(X^n) : n \in \omega\}$

Note that this immediately implies that if X is compact, then $C_p(X)$ has countable tightness. Now, we prove the following "submodel version" of a specific case:

Theorem 3.3. *If $t(C_p(X)_M) = \omega$, then $L(X_M) = \omega$.*

Proof. The proof follows the proof of Theorem 3.2 with appropriate modifications. Let $\mathcal{U}$ be an open cover of X_M. We say $\mu \in [\tau_M]^{<\omega}$ is *$\mathcal{U}$-small* if for every $V \in \mu$ there exists $U \in \mathcal{U}$ such that $V \subseteq U$. Note that each such μ is an element of M while $\mathcal{U}$ might not necessarily be. Every open set in X_M is a trace down of an open set in X, so by $V^{\uparrow}$, we mean the set that traces down to V. Denote by $\mathcal{E}$ the set of all $\mathcal{U}$-small families. For $\mu \in \mathcal{E}$, let $A_\mu = \{f \in C_p(X)_M : f(X \setminus \cup\mu^{\uparrow}) = 0\}$, and let $A = \cup_{\mu \in \mathcal{E}} A_\mu$. We want to show that $\overline{A}^{X_M} = C_p(X)_M$.

Let $f \in C_p(X)_M$ and $K \subseteq X_M$ finite. By previous lemma, these sets form a basis of $C_p(X)_M$, and we can just consider $\overline{A}^X \cap M$. Let θ_K be a finite subset of $\mathcal{U}$ covering K. For each $x \in K$, let $W_x = \cap\{V \in \theta_K : x \in V\}$, and consider the family $\mu_K := \{W_x : x \in K\}$. Take a function $g \in C_p(X)_M$ such that $g \upharpoonright K = f \upharpoonright K$ and $g(X \setminus \cup\mu_K^{\uparrow}) = 0$. Since θ_K is finite and $K \in M$, then $\mu_K^{\uparrow} \in M$, and so we can pick such $g \in M$ by elementarity. Then $g \in A_{\mu_K}$, so $f \in \overline{A}$.

Consider the function $f_1 \equiv 1 \in C_p(X)_M$. Then there is a subset $B \subseteq A$ such that B is countable and $f_1 \in \overline{B}^{X_M}$. Then there is a countable $\mathcal{E}_0 \subseteq \mathcal{E}$ such that $B \subseteq \cup_{\mu\in\mathcal{E}_0} A_\mu$. Fix $\mu \in \mathcal{E}_0$. For each $V \in \mu$, fix $U_V \in \mathcal{U}$ such that $V \subseteq U_V$, and let $\mathcal{V}_\mu := \{U_V : V \in \mu\}$. Then $\mathcal{V}_\mu$ is finite, and so $\mathcal{V} := \cup_{\mu\in\mathcal{E}_0}\mathcal{V}_\mu$ is countable. We will show that $\mathcal{V}$ covers X_M.

Fix $x \in X_M$, and let $U = \{f \in C_p(X)_M : f(x) > 0\}$. Then $f_1 \in U$, and since $f_1 \in \overline{B}$, we get that $U \cap B \neq \emptyset$. Thus, there exists $\mu \in \mathcal{E}_0$ such that there exists $g \in U \cap A_\mu$. Since $g(x) \neq 0$, we get that $x \in \cup\mu^{\uparrow}$, and thus there exists $U \in \mathcal{V}_\mu$ containing x. □

Our example makes use of the following theorem:

Theorem 3.4. *If M is countably closed, and X is compact, then X_M is countably compact.*

Proof. Let $\mathcal{U} = \{U_i \cap M : i \in \omega\}$ be an open cover of $X \cap M$ such that each $U_i \in M$. By countable closure, the set $\mathcal{U}' = \{U_i : i \in \omega\}$ is an element of M. Thus,

$$M \models \forall x \in X\ \exists U \in \mathcal{U}'\ (x \in U).$$

By elementarity, this is true in the universe, and we can find a finite subcover which traces down to the submodel. □

Example 3.1 (7.6 from [JT98]). *Let $X = 2^{\mathfrak{c}}$ with the usual topology. Let M be an elementary submodel of $H(\theta)$ such that $[M]^\omega \subseteq M$, $|M| = \mathfrak{c}$, $X \in M$, $\mathfrak{c} \subseteq M$, and also such that M includes a dense subset of X. Then X_M is a subspace of X. However, $|X \cap M| < |X|$, and so X_M cannot be closed (since it is dense). Therefore it is not compact, but by Theorem 3.4, X_M is countably compact and thus not Lindelöf.*

By the converse of the theorem above, we get that $t(C_p(X)) = \omega$. However, since $L(X_M) > \omega$, certainly $t(C_p(X)_M) > \omega$.

The following theorem of Tkachuk [Tka95] shows that, consistently, T_6 is downwards preserved for function spaces.

Theorem 3.5 (PFA). *If $C_p(X)$ is T_6, then $(C_p(X))^\omega$ is hereditarily Lindelöf.*

Tkachuk's theorem only relies on the statement "There are no S-spaces", and by a famous theorem of Todorcevic, PFA implies this statement. Since $hL(X_M) \leq hL(X)$ for all X and M, we get that $C_p(X)_M$ is hereditarily Lindelöf, and thus T_6. This is in contrast with the situation for arbitrary spaces, where there is a ZFC example of a T_5 space X and a submodel M such that X_M is not even T_4.

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