# On Translative Packing Densities in $E^2$ and $E^3$

by

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Abstract

The theory of packing and covering is an essential part of discrete geometry. In this dissertation we focus on and contribute to the knowledge on the densities of translative and lattice packings in $E^2$ and $E^3$. $d_T(C)$ and $d_L(C)$ will be used to denote the largest translative packing density and the largest lattice packing density of a planar disc or three dimensional body $C$, and for short, we will call them the translative packing density and the lattice packing density, respectively.

In 1892, Thue solved the problem of the densest packing of congruent circular discs in the plane. In 1950s, Rogers proved that for any convex disc $C$, $d_T(C) = d_L(C)$. This result was generalized by L. Fejes Tóth in 1985 to limited semi-convex domains. Besides, Fejes Tóth posed the question whether Rogers's equality remains true for non-convex domains. A. Bezdek answered this question negatively by providing a nonconvex disc, resembling a wrench. Bezdek determined the lattice packing density of his wrench and showed a non lattice-like translative packing of the wrench with a larger density. Note that Bezdek did not have to prove that the later packing has the largest density among translative packings, and this is the point where I joined this research area and proved the following:

**A** First, I proved what Bezdek already conjectured. Specifically, I showed that the translative packing, which Bezdek included in his paper, is in fact a densest translative packing of his wrench.

**B** Once A) was proved I could complete a new proof of Bezdek's result. This time all I had to prove was that lattice packings of the wrench cannot have a density equal to the largest translative packing density of the wrench.

**C** As a preparation for studying lattice packings in $E^3$, I proved a geometric property of point lattices. The one I proved could be interesting on its own. Let us assume that a point lattice contains all points whose position vectors are integer linear combinations

of three independent vectors. One cannot expect that $8$ of these lattice points form the vertices of a cube whose faces are parallel to coordinate planes. But for every $\varepsilon$, we can guarantee the existence of 8 lattice points which are vertices of a large parallelepiped, so that after proper scaling it is in the $\varepsilon$-neighborhood of a unit cube. We call such parallelepipeds $\varepsilon$-cubes.

**D** It would be interesting to explore translative packing densities in $E^3$, so I revisited Rogers's equality $d_T(C) = d_L(C)$, where $C$ denotes a convex disc in the plane. The question whether the same equality holds in $E^3$ is still open today for convex bodies. I proved that the equality holds for cylinders with convex base.

**E** Naturally, we would like to determine the largest translative packing density of cylinders whose base is Bezdek's wrench (called 3D-wrench). It was conjectured that stacking 3D-wrenches vertically over the densest planar lattice will give the densest 3D lattice. Surprisingly, this was not the case. It turned out that there is a different lattice packing of a single 3D-wrench, whose density is equal to the 3D-wrench's translative packing density.

# Acknowledgments

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Table of Contents

List of Figures

List of Tables

# Chapter 1

# Historical Background and Terminology

## 1.1 Definitions

The following two results of Thue and Kerschner planted the seeds of a new branch of geometry, which later, mainly through the work of László Fejes Tóth, began to grow and now is known as Discrete Geometry. Thue studied arrangements of nonoverlapping congruent circular discs and proved in 1892 that the natural "honeycomb" arrangement maximizes the percent of the plane covered by the discs. The honeycomb pattern refers to the tiling of the plane by regular hexagons. The incircles of the hexagons form an arrangement which we call "honeycomb" arrangement of the circles. The analogous result for circle coverings (i.e., the discs are not only allowed to overlap each other, but it is required that every point of the plane belongs to the closure of at least one disc) was considered and solved by Kershner in 1939. He showed that circumcircles of the same hexagonal tiling attain the minimal percent of the multiple covered parts of the plane.

The above questions motivated an array of general type questions each considering packings and coverings. Discrete geometry by its nature deals with questions which are easy to state and explain to young students, but whose solutions often require new ideas and also advanced techniques. The appeal of such problems attracted many young mathematicians around L. Fejes Tóth, including I. Bárány, A. and K. Bezdek, K. Böröczky, G. Fejes Tóth, Z. Füredi, E. Makai, J. Pach and the Hungarian school of discrete geometry born. For references about these results and about the general state of discrete geometry, we refer to the book [20] by G. Fejes Tóth and W. Kuperberg. This book is the English translation of the classic monograph of L. Fejes Tóth [18] and is supplemented by extensive surveys on recent progress of the field.

The packing problem for translates of a centrally symmetric convex body, especially in a lattice arrangement, has strong connections to the Geometry of Numbers, a theory initiated by H. Minkowski.

We already used couple of key terms of Discrete Geometry without formal definitions, like packing, covering, tiling and lattice. At the same time we avoided others like density, and instead used more intuitive phrases like "percent covered" or "most economical arrangements". Before going further we provide a list of precise definitions of terms we use in this dissertation.

**Definition 1.1.** Let $\mathcal{C} = \{C_1, C_2, ...\}$ be a collection of discs ( A disc is compact, connected and has nonempty interior.) in the plane (bodies in space) and let $D$ be a domain. If $\bigcup_i C_i \supseteq D$, $\mathcal{C}$ is called a *covering* of $D$. If $\bigcup_i C_i \subseteq D$ and no two of them have an interior point in common, then $\mathcal{C}$ is called a *packing* in $D$. If $\bigcup_i C_i = D$ and the members in $\mathcal{C}$ have mutually disjoint interiors, then $\mathcal{C}$ is called a *tiling* in $D$, and any set in $\mathcal{C}$ is called a *tile*.

**Definition 1.2.** If each member of $\mathcal{C}$ is congruent to the same disc $C$, then we say that $\mathcal{C}$ is a *packing with congruent copies* of $C$.

**Definition 1.3.** If all copies of $C$ are translates of each other, then we say that $\mathcal{C}$ is a *packing with translates* of $C$, or *translative packing* of $C$.

**Definition 1.4.** Given two linearly independent vectors $u_1$ and $u_2$ in the plane $E^2$, the $lattice$ $\Lambda$ generated by them is defined as

$$\Lambda(u_1, u_2) = \{m_1u_1 + m_2u_2 |\ m_1, m_2 \in Z\},$$

where $Z$ is the set of integers.

The set $\{u_1, u_2\}$ is called a *basis* of $\Lambda$. The parallelogram induced by vertices of the form $m_1u_1 + m_2u_2$, where $m_1, m_2 \in \{0, 1\}$, is called the *fundamental parallelogram* of $\Lambda$.

Similarly, $\Lambda(u_1, u_2, u_3) = \{m_1u_1 + m_2u_2 + m_3u_3 |\ m_1, m_2, m_3 \in Z\}$ is called a *lattice* in the 3-dimensional Euclidean space $E^3$, where $u_1$, $u_2$, and $u_3$ are linearly independent vectors in $E^3$. The parallelepiped induced by vertices of the form $m_1u_1+m_2u_2+m_3u_3$, where $m_i \in \{0, 1\}$ for every $i$, is called the *fundamental parallelepiped* of $\Lambda$.

**Definition 1.5.** Given a disc $C$ and a lattice $\Lambda$, the collection of translates $\mathcal{C} = \{C + u | u \in \Lambda\}$ is called a *lattice arrangement*.

In addition, if $\mathcal{C}$ is a packing, then it is called a *lattice packing*.

**Definition 1.6.** Let $\mathcal{C} = \{C_1, C_2, ...\}$ be a collection of discs in the plane and let $D$ be a domain. If $D$ is a bounded domain, then the *density* of the collection $\mathcal{C}$ with respect to $D$ is defined as

$$d(\mathcal{C}, D) = \frac{\sum_i A(C_i)}{A(D)},$$

where the sum is taken over all $i$ for which $C_i \cap D \neq \emptyset$, and $A(C_i)$ as well as $A(D)$ represents the area of $C_i$ and the area of $D$, respectively.

If $D$ is the whole plane, then we define the $upper$ and $lower\ densities$, denoted by $\overline{d}$ and $\underline{d}$ as follows.

$$\overline{d}(\mathcal{C}, E^2) = \limsup_{r\to\infty} d(\mathcal{C}, D(r)) \quad \text{and}$$

$$\underline{d}(\mathcal{C}, E^2) = \liminf_{r\to\infty} d(\mathcal{C}, D(r)),$$

where $D(r)$ denotes the circular disc of radius $r$ centered at the origin $O$. Notice that $\overline{d}(\mathcal{C}, E^2)$ and $\underline{d}(\mathcal{C}, E^2)$ are independent of the choice of the origin.

If $\overline{d}(\mathcal{C}, E^2) = \underline{d}(\mathcal{C}, E^2)$, then the common value is called the *density* of the collection $\mathcal{C}$ in the plane, and is denoted by $d(\mathcal{C}, E^2)$.

Completely analogously, we can define the density in $E^3$. The main difference is that the density in $E^3$ means the percent of the volume of the space occupied.

In the dissertation, I mainly talk about the maximum density of a translative packing of some disc (body resp.) $C$ and the maximum density of a lattice packing with some disc (body resp.) $C$. Both densest packings exist, which can be known from a general result of H. Groemer [6]. Let us denote them by $d_T(C)$ and $d_L(C)$, and call them the *translative packing density* and *lattice packing density*, respectively. Evidently, $d_T(C) \geq d_L(C)$.

**Definition 1.7.** A set is called *convex* if for any two of its points, it contains the entire line segment connecting them.

**Definition 1.8.** Given a convex disc $C$, let $S_1$ and $S_2$ be opposite parallel straight lines that support $C$ at the points $P_1$ and $P_2$. Then the arc $\widehat{P_1P_2}$ is one of the two arcs into which $P_1$ and $P_2$ divide the boundary of $C$.

A *semi-convex* domain is the region bounded by $\widehat{P_1P_2}$ and an arbitrary Jordan arc $\widehat{P_2P_1}$ within the disc $C$, see the region $S$ in the following figure 1.1.

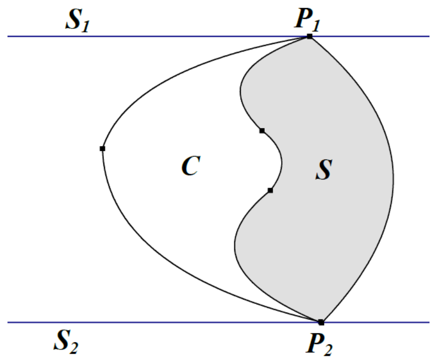



Figure 1.1: A semi-convex region $S$.

Choose any point $P_3$ on $\widehat{P_1P_2}$ and translate $\widehat{P_1P_2}$ through the vectors $\overrightarrow{P_3P_1}$ and $\overrightarrow{P_3P_2}$. The region enclosed by the original arc $\widehat{P_1P_2}$ and its translates is denoted by $R$. A domain bounded by $\widehat{P_1P_2}$ and any Jordan arc $\widehat{P_2P_1}$ which lies in $R$ instead of $C$ is called a *limited semi-convex*, see the domain $L$ in Figure 1.2.

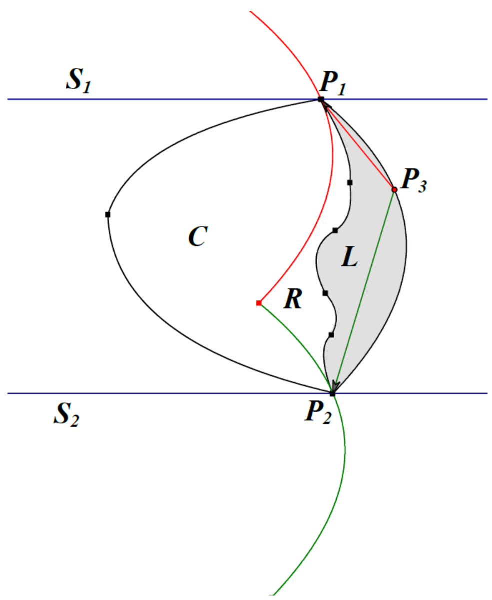



Figure 1.2: A limited semi-convex domain $L$.

**Definition 1.9.** A set $C$ is called *star-shaped* if for some point $p \in C$ and all points $q \in C$, the entire line segment $pq$ is contained in $C$.

**Definition 1.10.** Let $\mathcal{C} = \{C_1, C_2, \ldots\}$ be a packing in the plane, and let $v$ be a nonzero vector. For every $i$, let $S_i$ be defined as the set of those points $x \in E^2$, which are either in $C_i$ or for which the first intersection point of the ray parallel to $v$ and starting at $x$ with the set $C_1 \cup C_2 \cup \ldots$ belongs to $C_i$. $S_i$ is called the *shadow cell* of $C_i$, relative to $v$. See Figure 1.3.

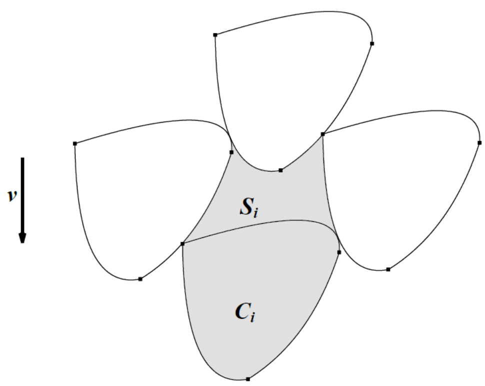



Figure 1.3: The shadow cell $S_i$ of $C_i$, relative to the vector $v$.

**Definition 1.11.** Given a disc $C$ in the plane (body in $E^3$), the *diameter* of $C$ is the maximum distance between two points from the disc (body resp), which is denoted by $\operatorname{diam}(C)$.

## 1.2 Introduction to the Research Problem of the Dissertation

In 1892, Thue [16] claimed that the density of any packing of equal circular discs is at most $\frac{\pi}{\sqrt{12}}$, which is the maximum density of any lattice packing of the circular disc.

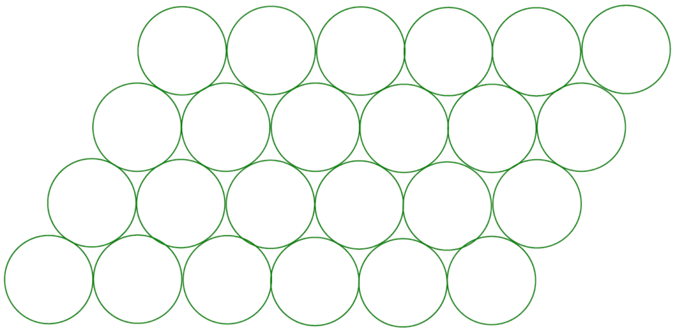


Figure 1.4: The densest packing of equal circular discs.

Thue's first proof had an error, but later he returned to the problem and solved it [17].

In 1951, Rogers [13] [14] proved that the densest translative packing of convex two-dimensional domains can be attained by a lattice packing. L. Fejes Tóth [4] generalized this result in 1985 to limited semi-convex domains (See Definition 1.8), with the proof based on an idea used in a new proof of Rogers's theorem. Later in 1986, Fejes Tóth [5] posed a conjecture claiming that for the union of two convex domains with a point in common, the translative packing density is the lattice packing density, and he proved the special case for the union of two unit circles in this paper, based on an idea in a proof of Thue's theorem. Heppes [9] generalized this result to more complicated regions bounded by circular arcs. And Kertész [10] proved for the union of two translates of a convex disc, the translative packing density is the lattice packing density.

When Fejes Tóth generalized Rogers's result to limited semi-convex domains, he posed the question that whether the property can be further generalized to non-convex domains. Bezdek [1] constructed a non-convex disc in 1985, resembling a wrench and consisting of five convex domains, illustrated in Figure 1.5. In order to prove the counterexample, Bezdek proved two lemmas. He proved the lattice packing in Figure 1.6 is the densest lattice packing of the wrench, while the translative packing in Figure 1.7 is denser than the densest lattice packing.

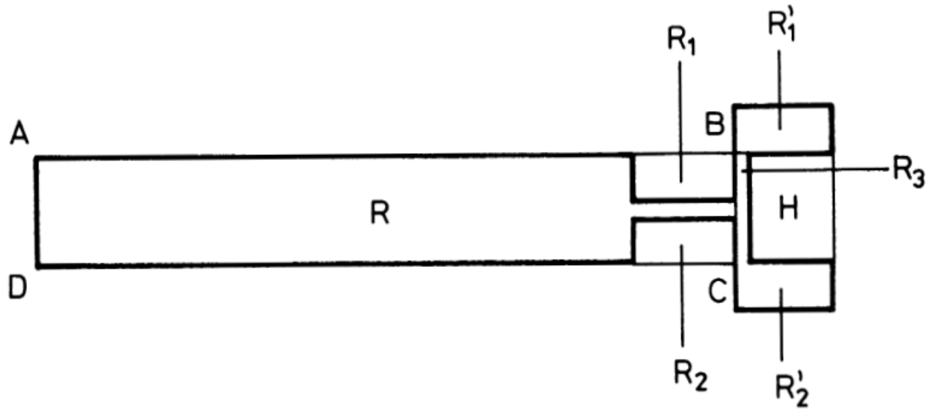



Figure 1.5: A counterexample of L. Fejes Tóth's question.

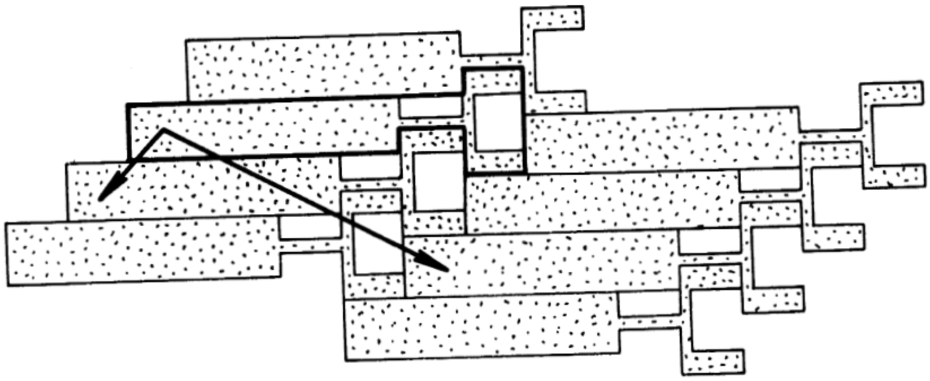


Figure 1.6: The densest lattice packing of the wrench.

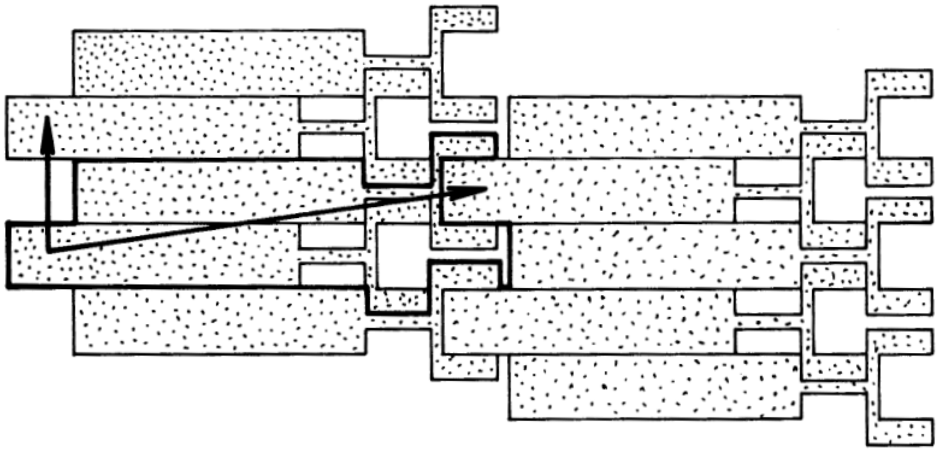


Figure 1.7: A denser translative packing of the wrench.

Later on, Kertész [1] modified the construction of Bezdek to obtain a semi-convex domain (Definition 1.8) such that the lattice packing density is less than the translative packing density. This counterexample can be seen in Figure 1.8, and the proof came from the proof of the wrench.

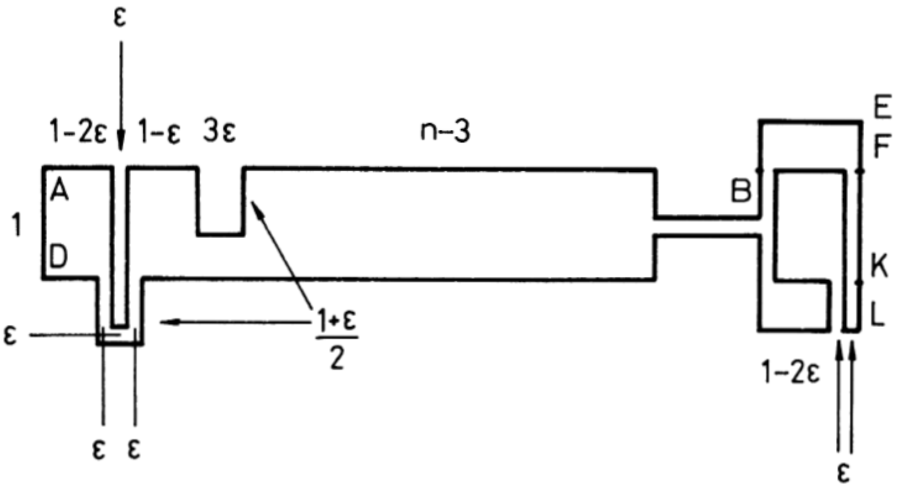



Figure 1.8: A semi-convex counterexample of L. Fejes Tóth's question.

By modifying the wrench in Figure 1.5, Heppes [7] [8] constructed a star-shaped domain $u$ (Definition 1.9), shown in Figure 1.9, which is the union of three convex domains, such that $d_T(u) > d_L(u)$. See the two packings in Figures 1.10 and 1.11.

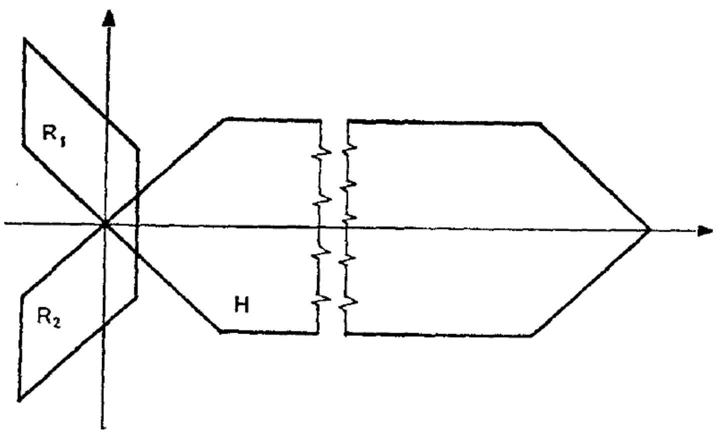



Figure 1.9: A star-shaped counterexample $u$ of L. Fejes Tóth's question.

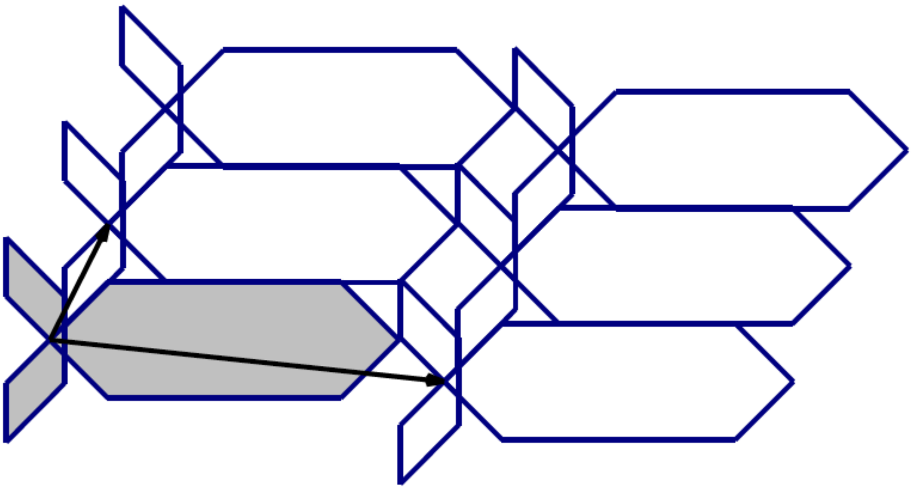


Figure 1.10: The densest lattice packing of $u$.

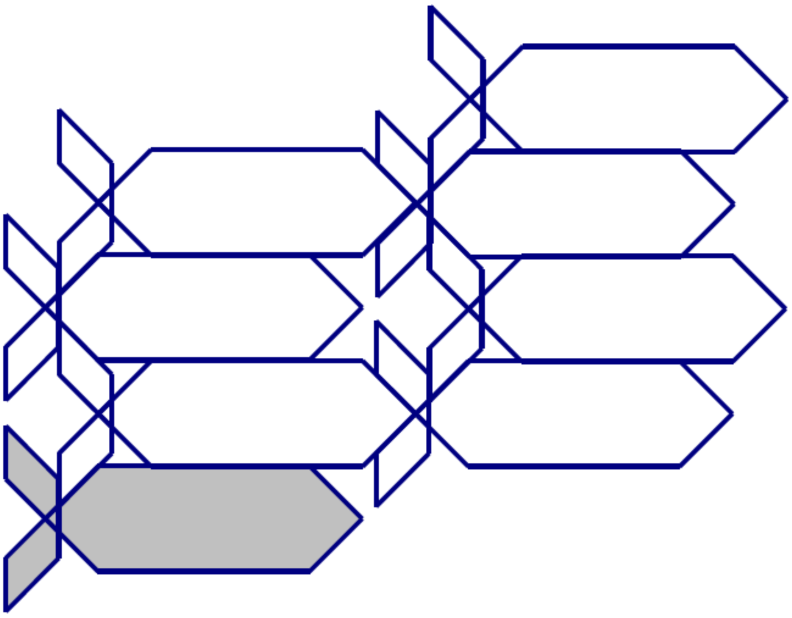


Figure 1.11: A denser translative packing of the single $u$.

For counterexamples in $E^3$, S. Szabó [15] in 1985 constructed a counterexample in space, and A. Bezdek and W. Kuperberg [2] constructed another one, which was based on a different idea, see Figure 1.12.

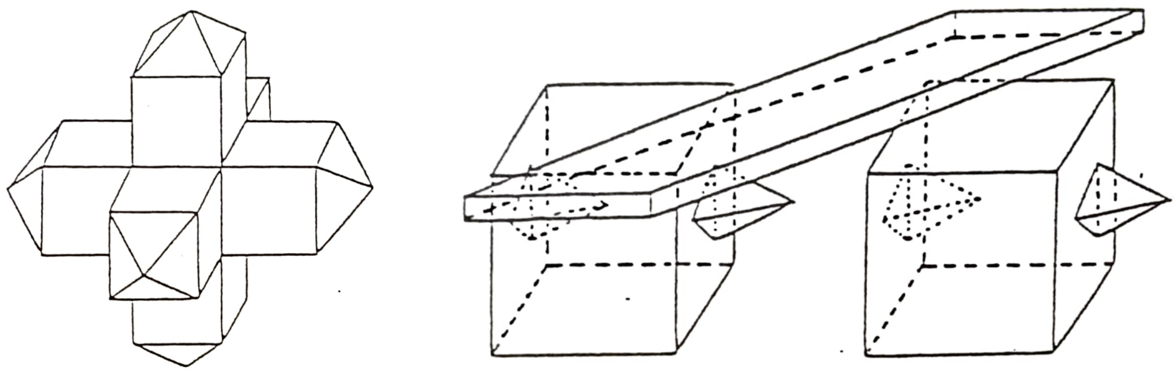


Figure 1.12: Szabó, 1985, Bezdek and Kuperberg, 1990

The above results concern the connections between the translative packing density and the lattice packing density raised in several natural questions. For comprehensive surveys of the theory of packing and covering, there are some books good to read, like Pach and Agarwal [12], L. Fejes Tóth [19], Brass, Moser and Pach [3], and Pach [11].

When Bezdek explained his counterexample, he used a concrete packing which contained translates of his planar wrench. All he needed was that its density is larger than its lattice packing density. It was conjectured that this particular packing is the densest among packings of translates. I will prove this conjecture in section 2.1. After we get the densest translative packing, a new idea to explain Bezdek's counterexample will be clear, and we will show it in section 2.2.

In chapter 3 I will prove in point lattices how to find 8 lattice points as vertices of a large parallelepiped based on a given $\varepsilon$, so that the parallelepiped is in the $\varepsilon$-neighbourhood of a given cube. The result can be applied to prove the inequality in Remark 2 in Chapter 4, and we have it as a separate chapter since it is interesting on its own.

Can one find a family of convex bodies in $E^3$ for which a theorem analogous to Rogers's planar theorem holds? With other words we need a subfamily of convex bodies for which the translative packing density is equal to the lattice packing density. In Theorem 4.1 I show that the family of cylinders with convex base is such a family.

It is natural to study cylinders whose base is Bezdek's planar wrench. Does this cylinder, called a 3D-wrench, have similar properties as its planar base? In other words, is it true that the translative packing density of the 3D-wrench is larger than its lattice packing density? It was expected that the answer is yes, based on the intuitive observation that stacking the 3D-wrenches vertically, the density of the bases remains equal to the density of the 3D-wrenches. I will answer this question negatively in Theorem 4.2. First I determine the translative packing density of the 3D-wrench, and then I show that there is a lattice packing which has density equal to the translative packing density.

# Chapter 2

# On Translative Packing Densities in $E^2$

In this chapter, we explore the densest translative packing of the non-convex disc constructed by Bezdek. I will prove the densest translative packing of the wrench is the denser one shown in the paper [1]. Then I will explain a new proof of Bezdek's counterexample.

## 2.1 The Translative Packing Density of the Non-convex Disc Introduced by Bezdek

Bezdek conjectured that the denser translative packing (see Figure 1.7) is the densest translative packing of the single wrench, and I will prove this is true. The main idea is like this. Given any translative packing of the wrench, we first divide it into separate clusters. Then we will get the shadow cell (see Definition 1.10) of each cluster along the vertical direction. For shadow cells containing more than $5$ pieces, we will subdivide them orderly until all clusters have length at most $5$. Therefore, there will be at most $5$ kinds of shadow cells based on different lengths, and the diameters of all shadow cells have a common upper bound. Next we find out the densest shadow cell among all shadow cells, which is the one in Figure 2.9. Notice that from the denser packing in Figure 2.22, we can get a tiling of the densest shadow cell. Therefore, this translative packing is a densest translative packing of the wrench.

**Theorem 2.1.** *The packing in Figure 2.2 is a densest packing of translates of the disc $P$ shown in Figure 2.1.*

*Proof of Theorem 2.1.* Notice that each edge of $P$ has lengths, which will be helpful when we talk about densities, and these numbers came from the model 2.2 constructed by Bezdek. Also notice that the packings in Figures 1.7, 2.2 and 2.22 are the same one.

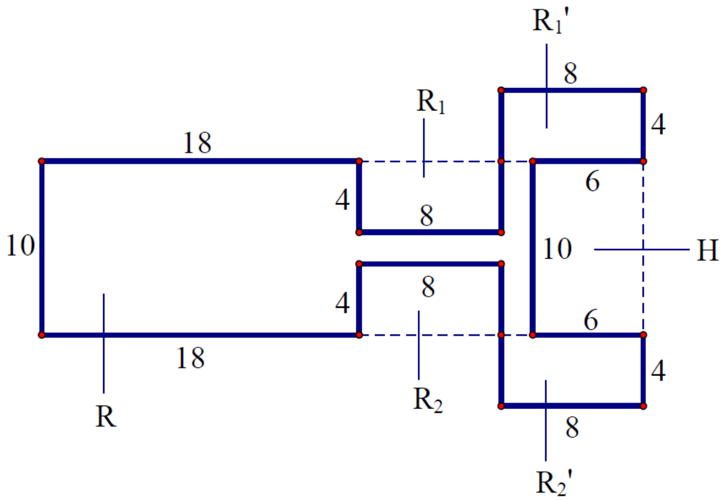



Figure 2.1: The piece $P$.

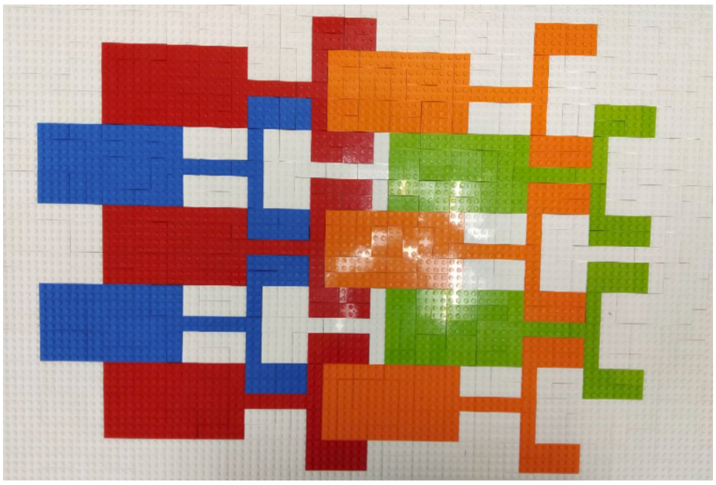


Figure 2.2: A densest packing with translates of $P$.

We will prove the theorem with one lemma and two steps. Let's first look at the lemma.

**Lemma 2.2.** *Given a packing* $\mathcal{C} = \{C_1, C_2, \ldots\}$ *in the plane. If there is a cell decomposition so that each disc is contained in one cell, and the diameters of all cells have a common upper bound, then the upper bound of densities in the cells is an upper bound of the upper density in the plane. Furthermore, if the density in each cell achieves the upper bound, and the cells tile the plane, then the density in the plane exists, which is the upper bound.*

Lemma 2.2 is a known result, and the proof can be found in Appendix A. From the proof we find that the lemma is also compatible with packings in $E^3$.

Now let us work on a translative packing of the disc $P$ with two steps. With it in mind that that we will say one piece is overlapped by another one if they are arranged in the way shown in the following figure.

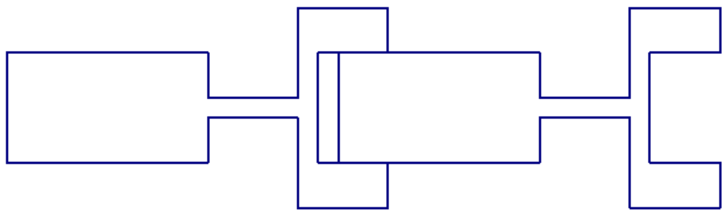


Figure 2.3: One piece is overlapped by another one.

**Step 1.** Shadow cell decomposition.

We will use the shadow cell decomposition method on a translative packing. Before this, let's get clusters in the packing. Cluster is an ordered list of pieces so that for any two adjacent pieces in the cluster, one piece is overlapped by another one. See the following figure 2.4.

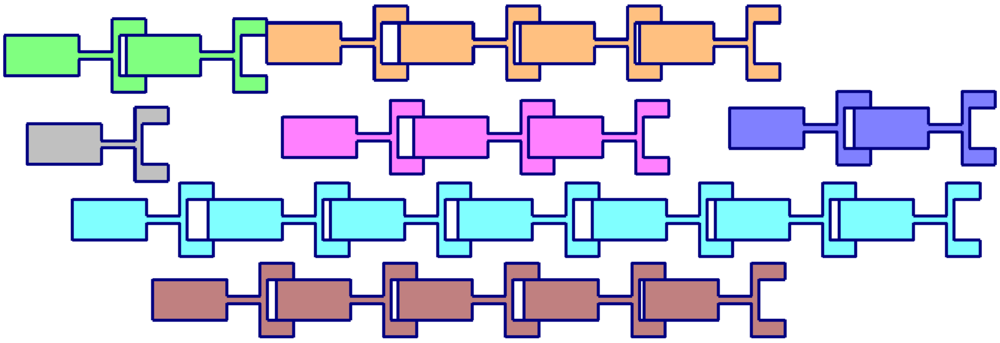


Figure 2.4: Clusters in the packing.

Then we get the shadow cell for each cluster along the vertical direction, which is illustrated in Figure 2.5.

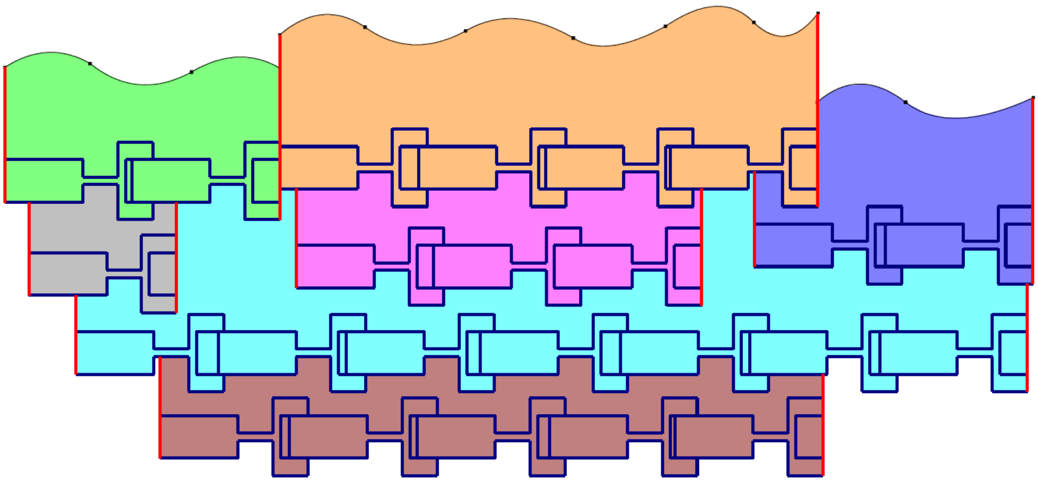


Figure 2.5: The shadow cell of every cluster in the packing.

As we can see, there can be shadow cells containing infinitely many pieces. And we will subdivide long shadow cells so that all shadow cells share the same upper bound for their diameters.

After calculation we find that we can subdivide shadow cells containing more than $5$ pieces. Therefore, for the cell containing more than $5$ pieces, we count the pieces from left to right, and subdivide the shadow area between the $(5n)$-th piece and the $(5n+1)$-th piece in the way shown in Figure 2.6, until the remaining shadow contains no more than $5$ pieces. Thus, there are at most five classes of shadow cells. Let's call them $G_1$, $G_2$, $G_3$, $G_4$, and $G_5$, so the shadow cells in $G_i$ have the same length $i$, which means they all contain the same number of pieces $P$.

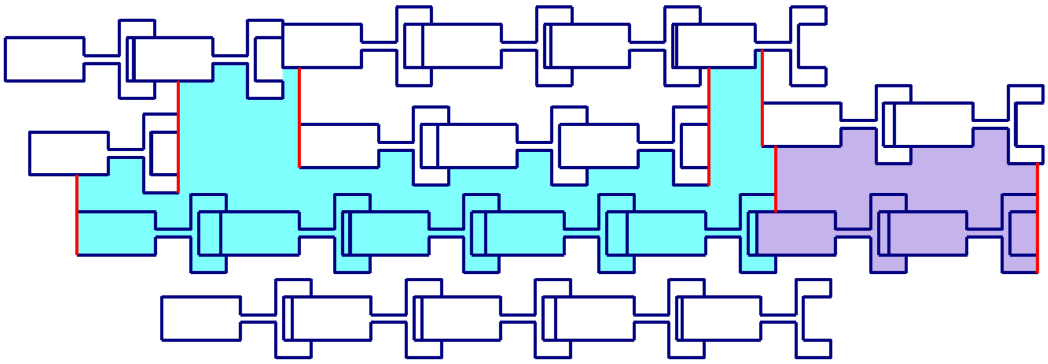


Figure 2.6: Subdivision on shadow cells with more than $5$ pieces contained.

**Step 2.** Show that the maximum density among densities in all cells comes from the cell including two pieces where one is "completely" overlapped by another one. The cell is illustrated in Figure 2.9 and it is the shaded domain, where $P_1$ and $P_2$ are the pieces in the cell.

Before we start, let's evaluate some areas. It's easy to see from Figure 2.1 that the area of the piece is $A(P) = 280$, the area of the region $H$ is $A(H) = 60$, and $A(R_1) = A(R_1') = A(R_2) = A(R_2') = 32$.

I will prove the upper bound of the densities in each class. For each class, the discussion will start with a table showing the results, and then we will talk about the detailed calculations.

Case 1: Shadow cells of subclusters of a single wrench.

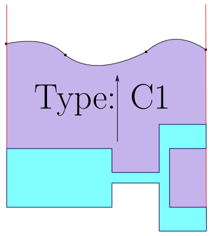
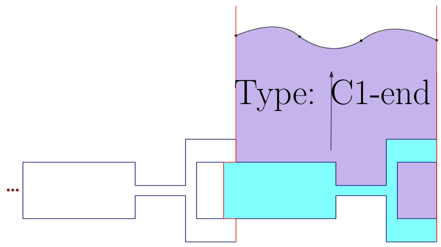
| 2 types of subclusters of length 1 and their shadow cells. | Upper bounds for the maximum cell density d (name of the type) |
|---|---|
|   The single piece forms a complete cluster. | $\mathrm{d(C1)} = \frac{280}{280+92} < \mathrm{d(C2)}$ |
|   The end subcluster of a cluster longer than 5. | $\mathrm{d(C1\text{-}end)} = \frac{280}{280+100} < \mathrm{d(C2)}$ |

Table 2.1: The maximum densities in shadow cells from $G_1$.

Notice there are two types of the shadow cells in $G_1$, $C1$ and $C1$-end. $C1$ is obtained directly from the cell partition, and $C1$-end is the remaining region after subdivision which contains one piece.

We first work with $C1$ and find the smallest shadow area $A(C1)$. As seen in Figure 2.7, let's assume $P_1$ is in $C1$. Since $C1$ is from $G_1$, $A(H)$ must be included in $A(C1)$. Then we have two cases based on whether $R_1$ is overlapped or not. If not, then at least $A(R_1)$ is added to $A(C1)$. If it is overlapped, only $R_2'$ of another piece $P_2$ can cover it, and then $A(R_2$ of $P_2)$ will be added to the area $A(C1)$, because $R_2$ of $P_2$ can not be covered by any piece if $P_2$ is invading $R_1$ of $P_1$. Therefore, the shadow area, $A(C1)$, shown in Figure 2.7 is the smallest, the density of $P_1$, depicted by $d(P_1) = \frac{A(P_1)}{A(C_1)} = \frac{280}{280+32+60} = \frac{280}{280+92}$, is the largest.

Now let's look at the cell $C1$-end containing the piece $P_1'$ and try to minimize the corresponding shadow area $A(C1\text{-end})$. Let's assume $P_1'$ belongs to some original cell $Ci$ with more than $5$ pieces, and it is the last one. Besides, the last subdivision should occur between $P_1'$ and

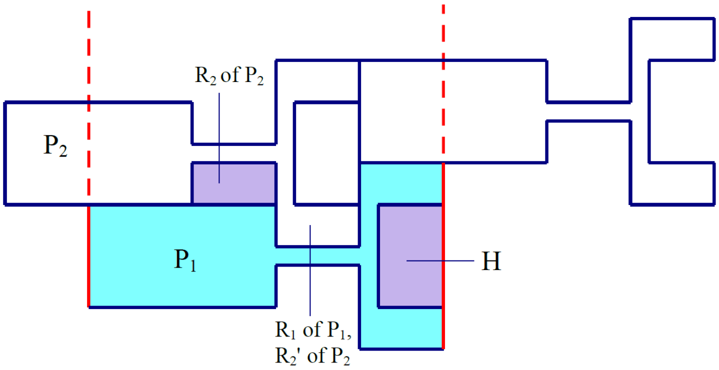



Figure 2.7: The smallest shadow cell in type $C1$.

the preceding one $P'_{5n}$. Firstly, we have $A(H)$ included in $A(C1\text{-end})$. What's more, $A(R_1$ of $P'_1)$ is in $A(C1\text{-end})$. The reason is that $R_1$ can only be covered by $R'_2$ of another piece $P'_2$, see Figure 2.8(1), but $P'_2$ and $P'_{5n}$ will overlap when $R'_2$ of $P'_2$ lies in $R_1$ of $P'_1$, so $R_1$ can not be covered and $A(R_1)$ has to be added to $A(C1\text{-end})$. Obviously, there are other areas added to $A(C1\text{-end})$. Actually, the smallest $C1$-end is the shaded region in Figure 2.8(2), and

$$A(C1\text{-end}) = 280 + 60 + 32 + 8 > A(C1).$$

Therefore, the optimal density in the cell from $G_1$ is obtained by the cell in Figure 2.7, and it is $d(P_1) = \frac{280}{280+92}$.

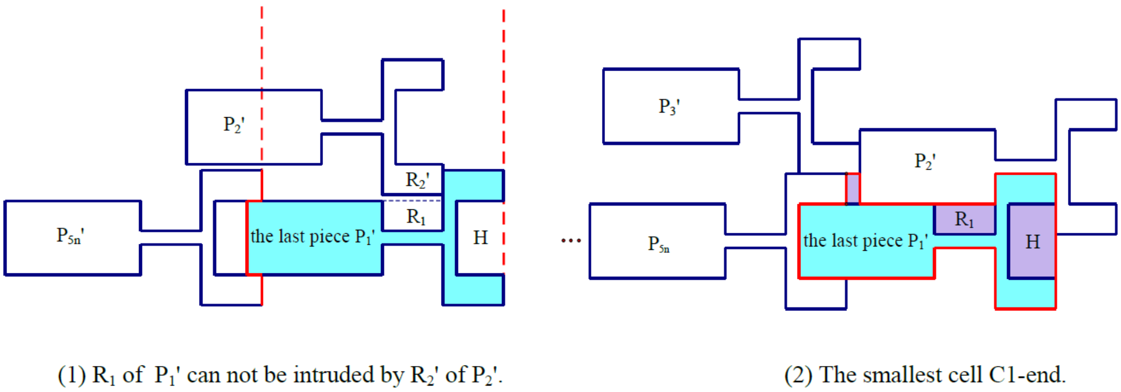



Figure 2.8: The shadow cell $C1$-end.

Case 2: Shadow cells of subclusters of two wrenches.

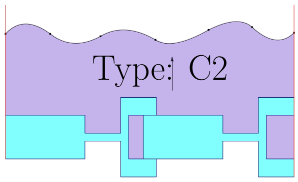
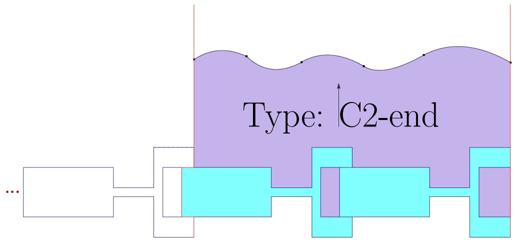
| 2 types of subclusters of length 2 and their shadow cells. | Upper bounds for the maximum cell density d (name of the type) |
|---|---|
|   The 2 pieces form a complete cluster. | $\mathrm{d(C2)} = \frac{280}{280+80}$ |
|   The end subcluster of a cluster longer than 5. | $\mathrm{d(C2\text{-}end)} < \frac{280}{280+86} < \mathrm{d(C2)}$ |

Table 2.2: The maximum densities in shadow cells from $G_2$.

Now let's move on to $G_2$, in which each cell contains two pieces. Figure 2.9 demonstrates a cell containing pieces $P_1$ and $P_2$. The area of this cell is

$$A = 280 \times 2 + 160,$$

so its density is

$$d = \frac{280 \times 2}{280 \times 2 + 160} = \frac{280}{280 + 80}.$$

We will show that this is the optimal density among all cells in $G_2$.

We understand there are two kinds of cells in $G_2$: $C2$ and $C2$-end. $C2$ is from the original shadow partition, while $C2$-end is obtained by subdivision and it is the remaining region where there are two pieces left. Let's talk about densities in $C2$ and $C2$-end separately and start with $C2$.

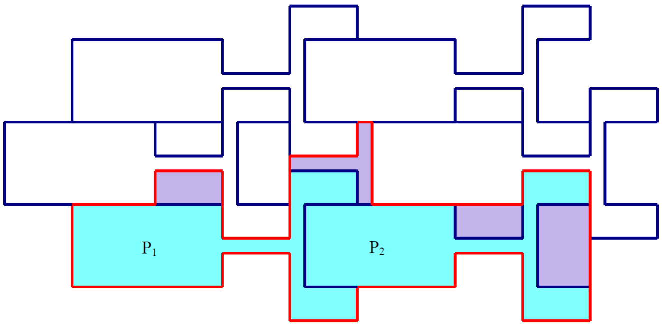



Figure 2.9: The density of the shaded cell is $\frac{280}{280+80}$.

Before we start, I'd like to make a note about "the distance between two pieces". It means how far it is when one piece is moved close to another one until they completely touch each other.

Figure 2.10 demonstrates a general cell $C2$ from $G_2$. Denote the distance between $P_1$ and $P_2$ by $a$, so $0 \leq a < 6$. The sum of areas of $P_1$, $P_2$ and the region between them is $280 \times 2 + 10a$, $A(S_1) = (18 + 8) \times 4 = 104$, and $A(S_2) = (20 + a) \times 4 = 80 + 4a$.

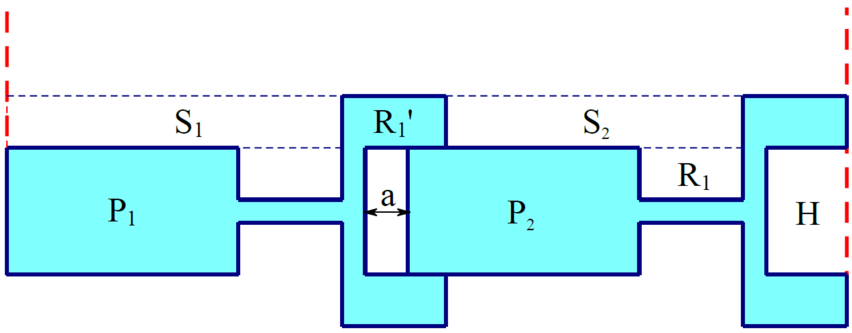



Figure 2.10: The shadow cell $C2$.

What we'll do is to pack the region right above $P_1$ and $P_2$ with translates of $P_1$, and try to get a cell with a smaller area than $A = 280 \times 2 + 160$.

Since $C2$ is from $G_2$, $A(H \text{ of } P_2) = 60$ is in $A(C2)$. $R_1$ of $P_2$ can not be covered, because it can only be covered by $R_2'$ of some other piece $P_3$, and the part $R$ of $P_3$ will overlap $R_1'$ of $P_1$ if it is covered. Therefore, $A(R_1 \text{ of } P_2) = 32$ is also part of $A(C2)$.

Let's move to $S_2$ of $P_2$. It needs to be overlapped by a piece $P_3$ in order to get a smaller cell area than $A$. To be specific, it should be overlapped by $R$ of $P_3$. Denote the distance between

$P_3$ and $P_2$ by $b$, so $0 \leq b < 4$, and the area added to $A(C2)$ after $S_2$ is overlapped by $R$ of $P_3$ is

$$4(2 + a) + 18b + 8b = 8 + 4a + 26b,$$

which is the blue area in Figure 2.11.

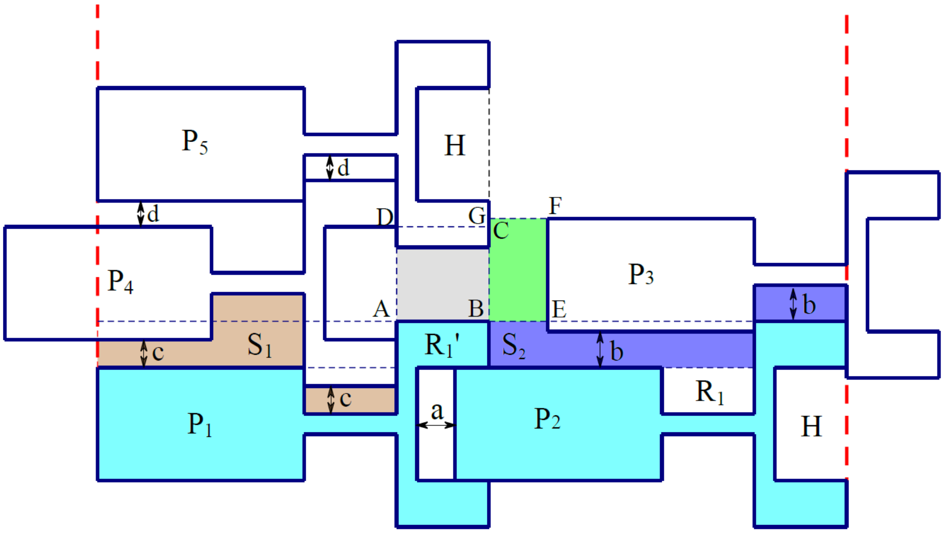



Figure 2.11: Investigation to obtain the optimal density of the cell $C2$.

Likewise, $R$ of $P_4$ overlaps $S_1$ such that $A(C2)$ can be as small as possible. Then $R_2'$ of $P_4$ covers $R_1$ of $P_1$. Suppose the distance between $P_1$ and $P_4$ is $c$, then the area added to $A(C2)$ will be

$$32 + 18c + 8c = 32 + 26c,$$

which is the area of the brown region in Figure 2.11.

Now let's focus on the region $ABCD$. The area is $8(6 + c) = 48 + 8c$, so it needs to be covered by some piece so that $A(C2)$ will not be greater than $A$. This can only be realised by $R_2'$ of $P_5$ in the way shown in Figure 2.11. Assume the distance between $P_4$ and $P_5$ is $d$, then the added area is

$$8(2 + c + d) = 16 + 8c + 8d,$$

and that's the area of the grey region in Figure 2.11.

Finally, we find the region BEFG can not be overlapped by any piece, and its area is

$$(2+a)(10-4+b) = 12+6a+2b+ab.$$

Adding these areas together, we have

$$A(C2) \geq (280\times2+10a)+60+32+(8+4a+26b)+(26c+32)+(16+8c+8d)+(12+6a+2b+ab),$$

i.e.,

$$A(C2) \geq 280\times 2+160+20a+28b+34c+8d+ab.$$

Luckily, when $a=b=c=d=0$, $H$ of $P_5$ can be overlapped by another piece such that the total area of the cell $C2$ is $280\times 2+160$, so this is the smallest area of the cell $C2$. The desired cell $C2$ is shown in Figure 2.9 as the shaded domain.

Let's now turn to $C2$-end and find the smallest area $A(C2\text{-end})$.

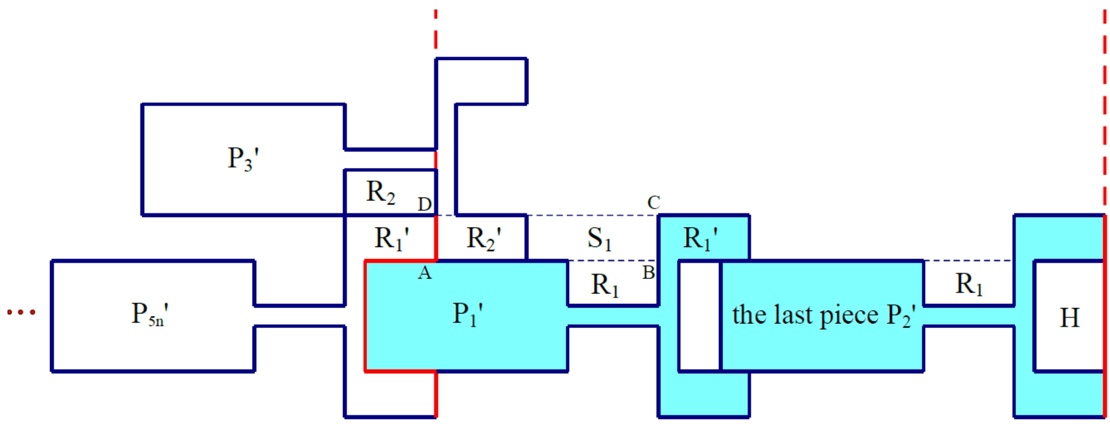



Figure 2.12: $P_1' \cup P_2' \in C2$-end.

As shown in Figure 2.12, there is a subdivision between the piece $P_{5n}'$ and $P_1'$. The cell $C2$-end is the remaining region containing $P_1'$ and $P_2'$.

Since $P_2'$ is the last piece, $H$ and $R_1$ of $P_2'$ can not be overlapped by any piece with the same reason as we talked about $C2$. $R_1$ of $P_1'$ can also not be covered so as to avoid overlapping on $R_1'$ of $P_{5n}'$.

Now let's look at the rectangle $ABCD$, which is also denoted by $S_1$. $A(S_1) \geq 4(18-6+8) = 80$, and $S_1$ can only be overlapped by $R_2'$ of another piece and at most once.

Thus,

$$A(C2\text{-end}) > A(P_1') + A(P_2') + A(H \text{of} P_2') + A(R_1 \text{of} P_2') + A(R_1 \text{of} P_1') + [A(S_1) - A(R_2' \text{of} P_3')].$$

That is,

$$A(C2\text{-end}) > 280 \times 2 + 60 + 32 + 32 + 48 = 280 \times 2 + 172 > A = 280 \times 2 + 160,$$

which means the optimal density of cells in $G_2$ is obtained by the cell $C2$, and it is shown in Figure 2.9.

Case 3: Shadow cells of subclusters of 3 wrenches.

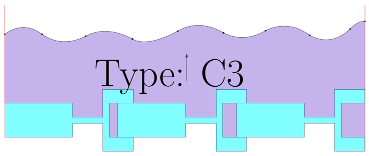
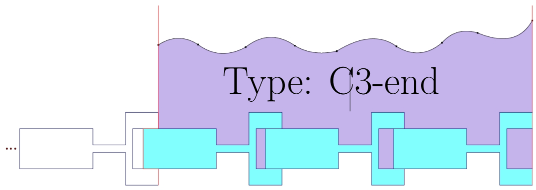
| 2 types of subclusters of length 3 and their shadow cells. | Upper bounds for the maximum cell density d (name of the type) |
|---|---|
|   The 3 pieces form a complete cluster. | $\mathrm{d(C3)} < \frac{280}{280+\frac{244}{3}} < \mathrm{d(C2)}$ |
|   The end subcluster of a cluster longer than 5. | $\mathrm{d(C3\text{-}end)} < \frac{280}{280+\frac{244}{3}} < \mathrm{d(C2)}$ |

Table 2.3: The upper bound of densities in shadow cells from $G_3$.

Now, let's find out the optimal density of the cell from $G_3$. There are two kinds of cells in $G_3$: $C3$ and $C3$-end. The cell $C3$ is from cell partition, and $C3$-end is obtained after subdivision if there are three pieces left in the remaining region, then that is $C3$-end.

We'll investigate whether the largest density in $G_3$ can be smaller than that in $G_2$, and it suffices to see whether the smallest area of $C3$ or $C3$-end is smaller than $280 \times 3 + 80 \times 3$.

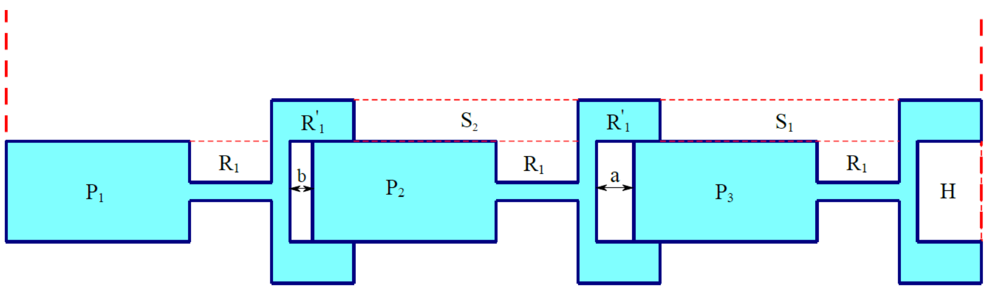



Figure 2.13: $P_1 \cup P_2 \cup P_3 \in C3$.

Let's first talk about $C3$, and Figure 2.13 gives a general version. It's clear that the region $H$ cannot be overlapped because $C3$ is from $G_3$ and $P_3$ is the last piece. The $R_1$ region of $P_2$

and $P_3$ can not be overlapped either, otherwise the invading pieces will overlap the $R_1'$ region of $P_1$ and $P_2$, respectively. So $A(H) + A(R_1 \text{ of } P_2) + A(R_1 \text{ of } P_3) = 60 + 32 + 32 = 124$ is added to $A(C3)$. For the $R_1$ region of $P_1$, if it's not overlapped, then we add $A(R_1 \text{ of } P_1) = 32$ to $A(C3)$. If it's overlapped, it can only be overlapped by $R_2'$ of $P_4$, see Figure 2.14. Notice $R_2$ of $P_4$ will be included in the shadow of $C3$, and it cannot be covered by any piece, so $A(R_2$ of $P_4)$ will be added to $A(C3)$. Therefore,

$$A(C3) > A(P_1) + A(P_2) + A(P_3) + 124 + 32 = 280 \times 3 + 156.$$

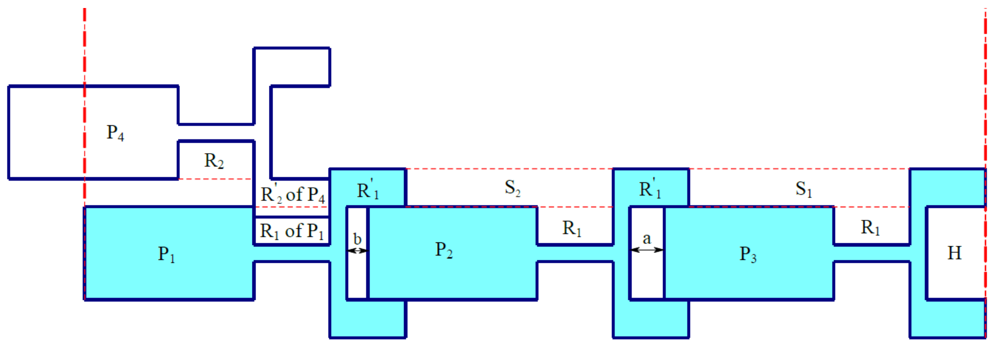



Figure 2.14: How $R_1$ of $P_1$ can be covered.

Thus, the extra area added to $A(C3)$ would be smaller than $(280\times3+80\times3)-(280\times3+156) = 84$ if the optimal density in $G_3$ is bigger than that in $G_2$.

Now let's look at Figure 2.15, and observe how the region $S_2$ can be overlapped. It's easy to see $A(S_2) = (20 + b)(4) = 80 + 4b \geq 80$, and $S_2$ can only be overlapped by $R_2'$ of $P_5$ and at most once. We also notice that $R_2$ of $P_5$ will be part of $C3$ if $R_2'$ of $P_5$ overlaps $S_2$, and it won't be overlapped by any piece. Therefore, at least $80 + 4b$ will be added to $A(C3)$.

Now let's turn to the region $S_1$. The biggest area that can be covered is shown in Figure 2.16, where $R$ of $P_6$ overlaps $S_1$, and the left part of $S_1$ cannot be covered by any piece, so the left area is at least $4(2 + a) = 8 + 4a$. Combined with the case of $S_2$, the area added to $A(C3)$ is at least $(80 + 4b) + (8 + 4a) = 88 + 4a + 4b$, which is bigger than $84$. Therefore, the optimal density of $C3$ cannot be greater than that in $G_2$.

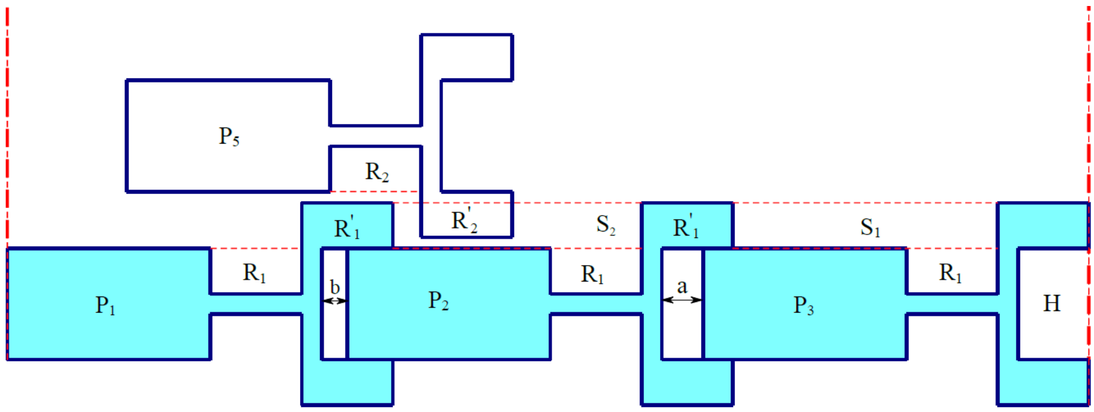



Figure 2.15: How $S_2$ can be covered.

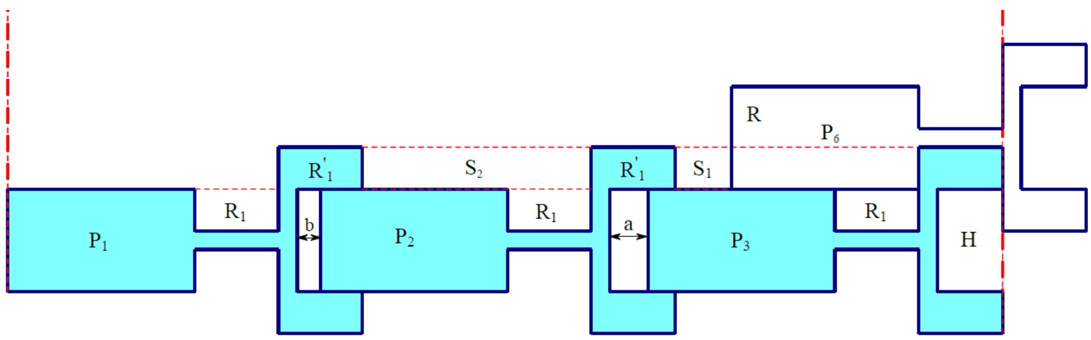



Figure 2.16: How $S_1$ can be covered.

Now let's look at $C3$-end and denote the area of $C3$-end by $A(C3\text{-end})$. In a similar manner as we talked about $A(C3)$, it's easy to see

$$A(C3\text{-end}) \geq 280 \times 3 + 60 + 32 \times 2 + 32 + (80 + 4b) + (8 + 4a) = 280 \times 3 + 244 + 4a + 4b.$$

The only difference between $C3$-end and $C3$ happens on the shadow areas of $P_1'$ and $P_1$. For $R_1$ of $P_1'$, it will never be overlapped due to the existence of the preceding piece $P_{5n}'$, see Figure 2.17, so $A(R_1 \text{ of } P_1') = 32$ will be added to $A(C3\text{-end})$, and this difference causes the same lower bound of $A(C3\text{-end})$ and $A(C3)$. Therefore, the best density of $C3$-end is also smaller than that in $G_2$. Thus, the optimal density in $G_3$ can not be greater than that in $G_2$.

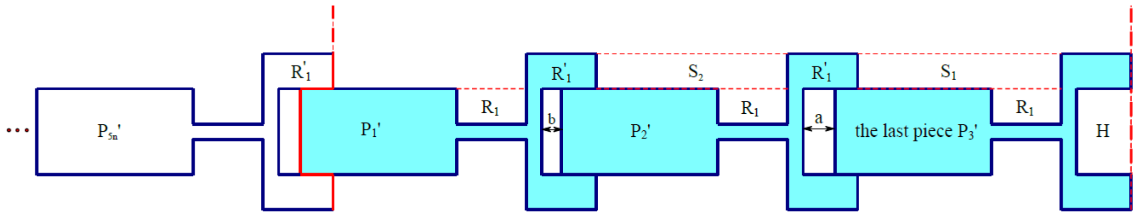



Figure 2.17: $P_1' \cup P_2' \cup P_3' \in C3$-end.

Case 4: Shadow cells of subclusters of 4 wrenches.

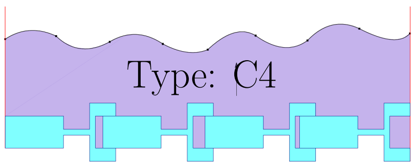
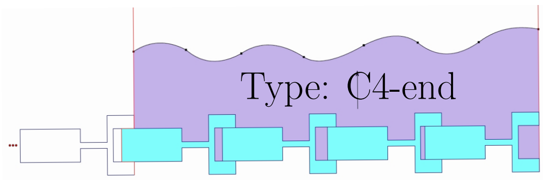
| 2 types of subclusters of length 4 and their shadow cells. | Upper bounds for the maximum cell density d (name of the type) |
|---|---|
|   The 4 pieces form a complete cluster. | $\mathrm{d(C4)} < \frac{280}{280+89} < \mathrm{d(C2)}$ |
|   The end subcluster of a cluster longer than 5. | $\mathrm{d(C4\text{-}end)} < \frac{280}{280+89} < \mathrm{d(C2)}$ |

Table 2.4: The upper bound of densities in shadow cells from $G_4$.

In a similar way, we can get the area of $C4$ and $C4$-end is bigger that

$$280 \times 4 + 60 + 32 \times 3 + 32 + 80 \times 2 + 8 = 280 \times 4 + 356,$$

which is greater than $280 \times 4 + 80 \times 4$, and it means the optimal density in $G_4$ is smaller than the optimal density in $G_2$.

Case 5: Shadow cells of subclusters of five wrenches.

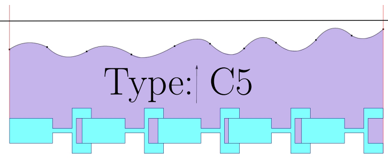
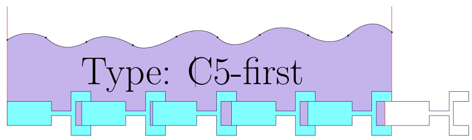
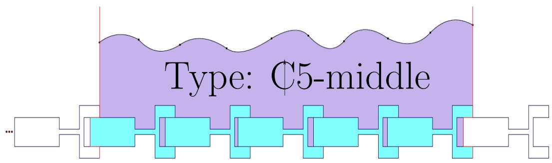
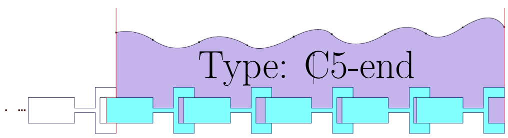
| 4 types of subclusters of length 5 and their shadow cells. | Upper bounds for the maximum cell density d (name of the type) |
|---|---|
|   The 5 pieces form a complete cluster. | $\mathrm{d(C5)} < \frac{280}{280+93.6} < \mathrm{d(C2)}$ |
|   1st subcluster of a cluster longer than 5. | $\mathrm{d(C5\text{-}first)} < \frac{280}{280+81.6} < \mathrm{d(C2)}$ |
|   Internal subcluster of a cluster longer than 10. | $\mathrm{d(C5\text{-}middle)} < \frac{280}{280+81.6} < \mathrm{d(C2)}$ |
|   The end subcluster of a cluster longer than 5. | $\mathrm{d(C5\text{-}end)} < \frac{280}{280+93.6} < \mathrm{d(C2)}$ |

Table 2.5: The upper bound of densities in shadow cells from $G_5$.

There are four kinds of cells in $G_5$, denoted by $C5$, $C5$-first, $C5$-middle, and $C5$-end. $C5$ is from the original cell partition, see Figure 2.18. $C5$-first, $C5$-middle, and $C5$-end are obtained from different positions after subdivision, and they are shown in Figures 2.19, 2.20, 2.21, respectively.

For $A(C5)$ and $A(C5\text{-end})$, both are greater than

$$280 \times 5 + 60 + 32 \times 4 + 32 + 80 \times 3 + 8 = 280 \times 5 + 468,$$

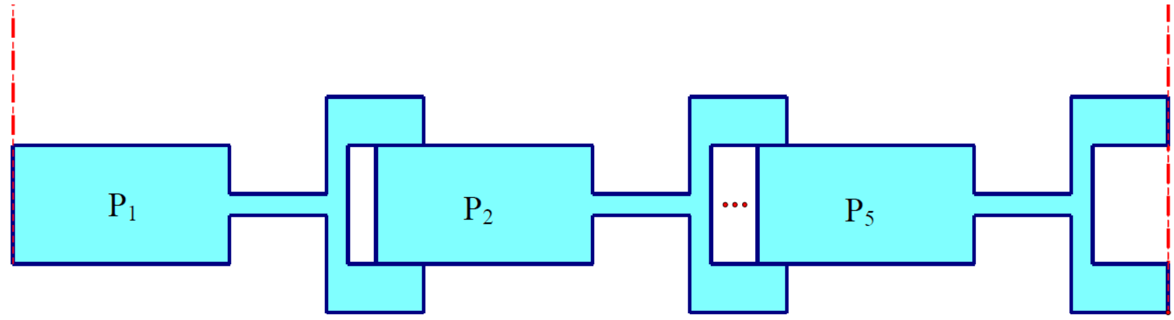



Figure 2.18: $\bigcup_{i=1}^{5} P_i$ is a $C5$ subcluster.

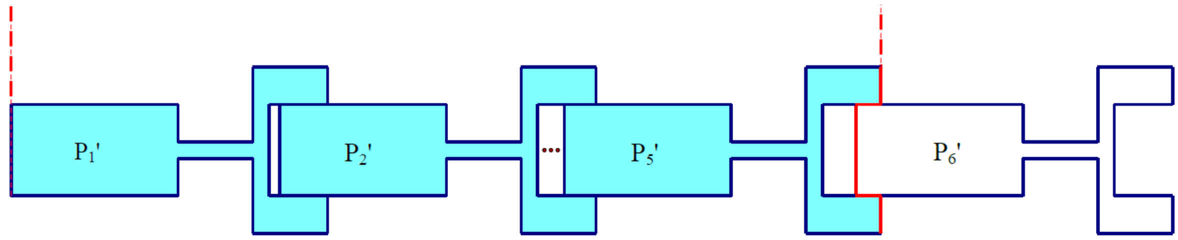



Figure 2.19: $\bigcup_{i=1}^{5} P_i'$ is a $C5$-first subcluster.

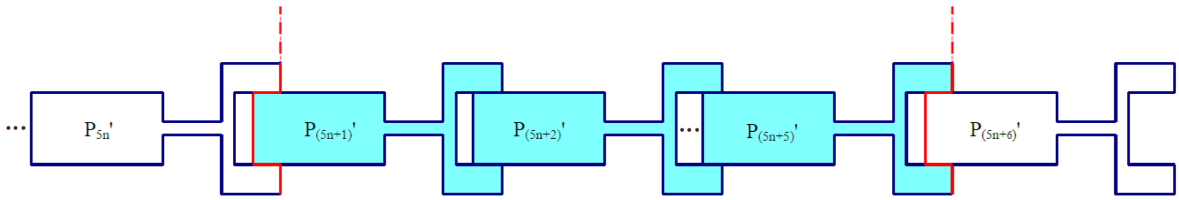



Figure 2.20: $\bigcup_{i=1}^{5} P_{(5n+i)}'$ is a $C5$-middle subcluster.

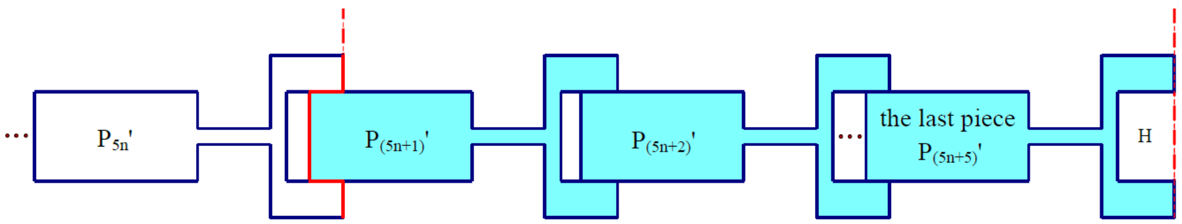



Figure 2.21: $\bigcup_{i=1}^{5} P_{(5n+i)}'$ is a $C5$-end subcluster.

which can be obtained in a similar manner as we talked about $C3$, and it is bigger than $280 \times 5 + 80 \times 5$.

For $A(C5\text{-first})$ and $A(C5\text{-middle})$, each is greater than

$$280 \times 5 + 60 + 32 \times 4 + 32 + 80 \times 3 + 8 - 60 = 280 \times 5 + 468 - 60 = 280 \times 5 + 408,$$

and the lower bound can be realised if $P'_6$ and $P'_{(5n+6)}$ overlap $P'_5$ and $P'_{(5n+5)}$, totally and respectively.

Since $280 \times 5 + 408$ is also bigger than $280 \times 5 + 80 \times 5$, the optimal density in $G_5$ is smaller than the optimal density in $G_2$, that is, $\frac{280}{280+80}$.

Therefore, the optimal density in cells among all $G_i$'s is realised by the pairs from $G_2$ and it is shown in Figure 2.9.

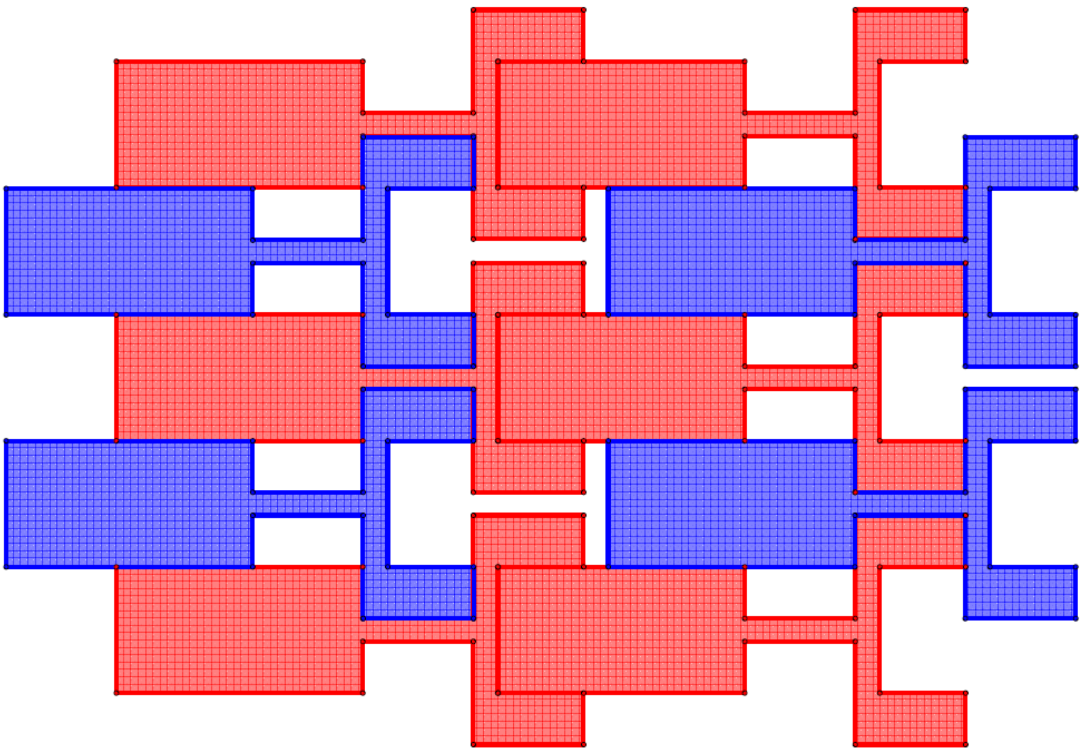


Figure 2.22: A densest packing with translates of the disc $P$.

Figure 2.22 (which is the same as Figure 2.2) illustrates a packing with translates of $P$. By the shadow cell partition method used in Step 1, we can obtain a tiling in the plane, where the tile is the shadow cell in Figure 2.9. In other words, all shadow cells in this packing has the optimal density. By Lemma 2.2, the density of this packing exists, which is the upper bound. To be specific, this is a densest translative packing of the disc $P$.

$\square$

One more observation is that the translative packing of the wrench in Figure 2.22 (also in Figure 2.2) is a lattice packing of the union of two wrenches.

Now that we have a densest translative packing of $P$, in order to prove $d_T(P) > d_L(P)$, it suffices to prove that the density of any lattice packing of $P$ is less than the density of the densest translative packing of $P$, $d(C2)$.

Notice that there are two classes of lattice packings of $P$ based on whether the region $H$ of $P$ (see Figure 2.1) is overlapped by another piece. Correspondingly, there are only two kinds of clusters. Furthermore, there are two kinds of shadow cells. Figure 2.23 illustrates the lattice packing where $H$ is not overlapped by any other pieces, so all clusters are single, which leads to only $C1$ as the cell.

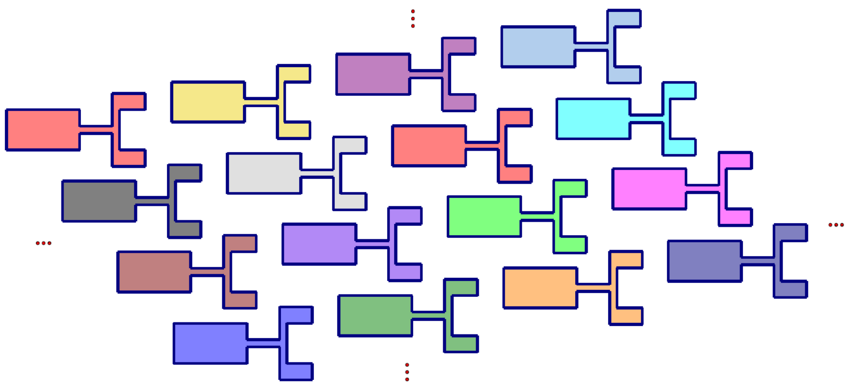


Figure 2.23: A lattice packing where the region $H$ is not overlapped.

If the region $H$ is overlapped in a lattice packing, then each cluster will be infinite long, see Figure 2.24. We then subdivide the corresponding shadow cells in the same way as we did in Theorem 2.1, and the resulting shadow cells are all $C5$-middle.

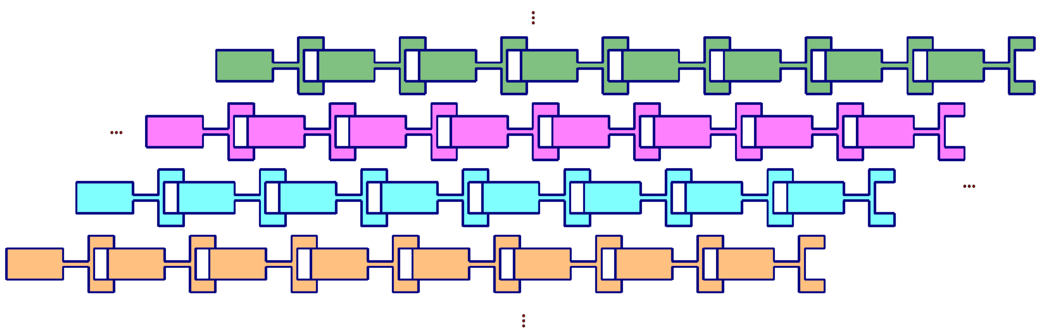


Figure 2.24: A lattice packing where the region $H$ is overlapped.

Since both maximum densities in $C1$ and $C5$-middle are less than $d(C2)$, there is no lattice packing density equal to $d(C2)$. $\square$

Chapter 3

On Special Parallelepipeds in Point Lattices in $E^3$

In this chapter I will prove a geometric property of point lattices. It is about how to get $8$ lattice points which are vertices of a large parallelepiped based on a given $\varepsilon$ $(> 0)$, so that the parallelepiped is in the $\varepsilon$-neighbourhood of a cube which is similar to the unit cube in the point lattice. This result turns out to be useful at studying lattice packings, but we find it interesting on its own, so we include it in a separate chapter.

Before we prove the theorem, let me first restate the large parallelepiped in a more technical form.

**Definition 3.1.** Given $\varepsilon > 0$, a cube $C$ with edge length $e$, and spheres $S_i (i = 1, 2, \ldots, 8)$ with each one centered at one vertex of $C$ and all with radii $r$ satisfying $\frac{r}{e} < \varepsilon$. If there is a parallelepiped $P$ such that each vertex lies in one sphere, then $P$ is called $\varepsilon$**-cube**, see Figure 3.1.

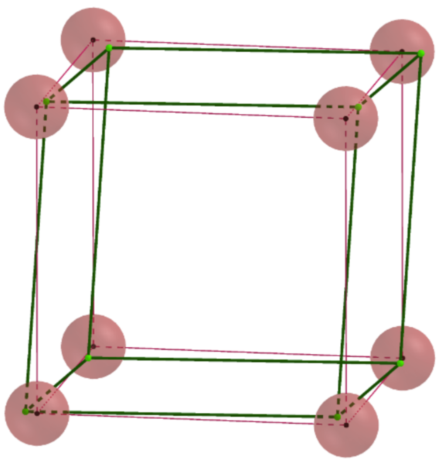


Figure 3.1: The parallelepiped is $\varepsilon$-cube.

Note that the faces of the parallelepiped may not be parallel to the faces of the cube.

**Theorem 3.1.** *Given $\varepsilon > 0$ and a point lattice. There exist $8$ lattice points as vertices of a large parallelepiped based on $\varepsilon$, so that after proper scaling, the parallelepiped is in the $\varepsilon$-neighbourhood of a cube which is similar to the unit cube. In other words, $\varepsilon$-cube exists.*

Before we prove the theorem, let me first prove two lemmas, which will be helpful in the proof of the theorem.

**Lemma 3.2.** *For any point inside a parallelepiped with edge lengths $a$, $b$, and $c$, there is a vertex of the parallelepiped such that the distance of the vertex and the point is less than $\frac{a+b+c}{2}$.*

*Proof of Lemma 3.2.* As shown in Figure 3.2, there is a point $P$ inside a parallelepiped with vertices $V_i$ $(i = 1, 2, \ldots, 8)$ whose edge lengths are $a$, $b$, and $c$. We will show that there is a vertex $V_i$ so that $|V_iP| < \frac{a+b+c}{2}$.

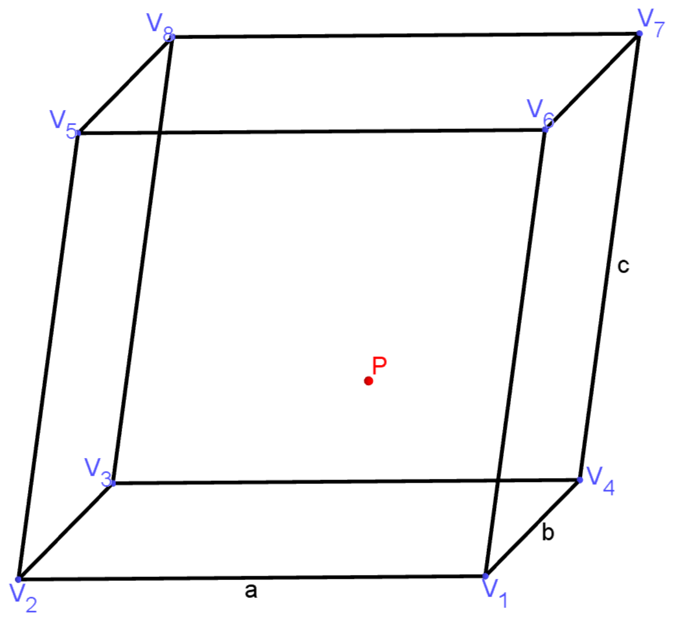



Figure 3.2: Notation for Lemma 3.2.

In order to prove it, we need the following statement.

**Statement 3.1.** Given a point $P$ inside a triangle $CAB$, either $|AP| < |AC|$ or $|BP| < |BC|$.

This can be proved by extending the perpendicular line segment $DP$. Since the intersection point $E$ can be on the edge $AC$ or edge $BC$, then $|AP| < |AC|$ or $|BP| < |BC|$ correspondingly.

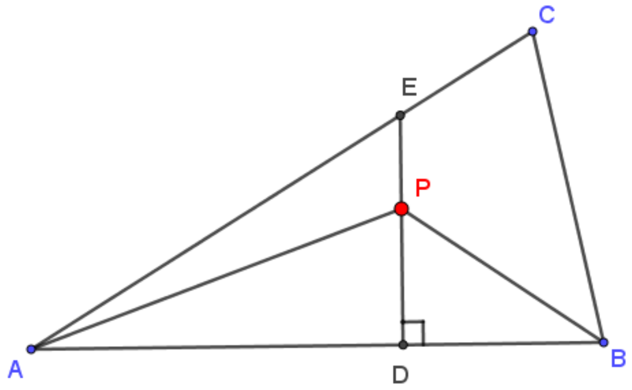



Figure 3.3: For $P$ inside a triangle $CAB$, $|AP| < |AC|$ or $|BP| < |BC|$.

Let's go back to the parallelepiped. As we can see from Figure 3.4, the point $O$ is the center of the parallelepiped, so $|OV_i| < \frac{a+b+c}{2}$, $i = 1, 2, \ldots, 8$. Since the point $P$ will lie in one of the pentahedra, let's take $O - V_1V_2V_3V_4$ as an example. The other situations are similar if $P$ is in one of the other pentahedra.

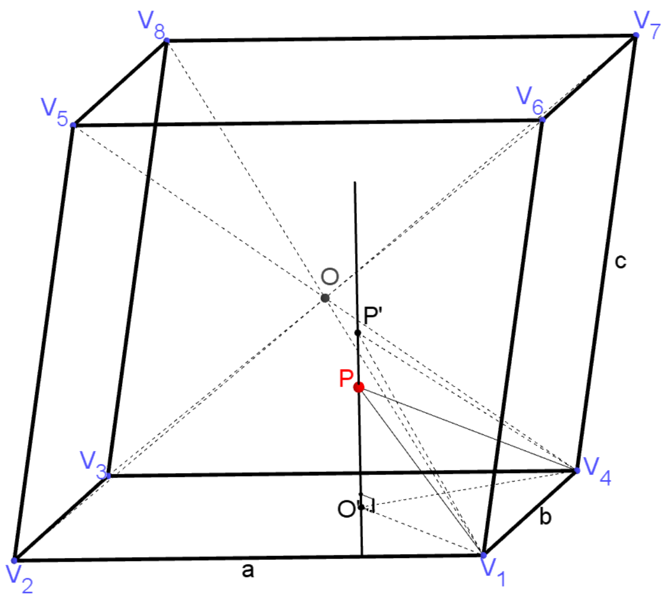



Figure 3.4: Either $|PV_1| < \frac{a+b+c}{2}$ or $|PV_4| < \frac{a+b+c}{2}$.

Illustrated in Figure 3.4, the line $l$ that passes through $P$ is perpendicular to the plane $V_1V_2V_3V_4$, and it intersects the parallelogram $V_1V_2V_3V_4$ and the triangle $OV_1V_4$ at points $O'$ and $P'$, respectively. Therefore, $|V_1P'| \geq |V_1P|$ and $|V_4P'| \geq |V_4P|$.

Let's now apply the statement 3.1 in the triangle $OV_1V_4$, so at least $|V_1P'| < |V_1O|$ or $|V_4P'| < |V_4O|$.

If $|V_1P'| < |V_1O|$, then $|V_1P| \leq |V_1P'| < |V_1O| < \frac{a+b+c}{2}$, which means the vertex $V_1$ is the desired vertex. Likewise, if $|V_4P'| < |V_4O|$, then by $|V_4P'| \geq |V_4P|$, we have $|V_4P| < |V_4O|$, so $V_4$ is the vertex such that $|V_4P| < \frac{a+b+c}{2}$.

Therefore, the vertex exists such that the distance between the vertex and any point in the parallelepiped is smaller than half of the sum of the lengths of three adjacent edges.

$\square$

**Lemma 3.3.** *Given a cube with vertices denoted by* $V_1, V_2, \ldots, V_8$ *such that vertices* $V_2$, $V_3$ *and* $V_4$ *are adjacent to the vertex* $V_1$, *and a parallelepiped with vertices* $V_1', V_2', \ldots, V_8'$ *satisfying* $V_2'$, $V_3'$ *and* $V_4'$ *are adjacent to the vertex* $V_1'$. *Besides, each* $V_i'$ *is in the neighborhood of* $V_i$. *If* $\max\{|V_iV_i'| : i = 1, 2, 3, 4\} < c$, *then* $\max\{|V_iV_i'| : i = 1, 2, 3, 4, 5, 6, 7, 8\} < 5c$.

*Proof of Lemma 3.3.* Let's first consider a square $ABDC$ and a parallelogram $A'B'D'C'$, as shown in Figure 3.5. We will show if $|AA'| < c$, $|BB'| < c$, and $|CC'| < c$, then $|DD'| < 3c$ by using vector computation.

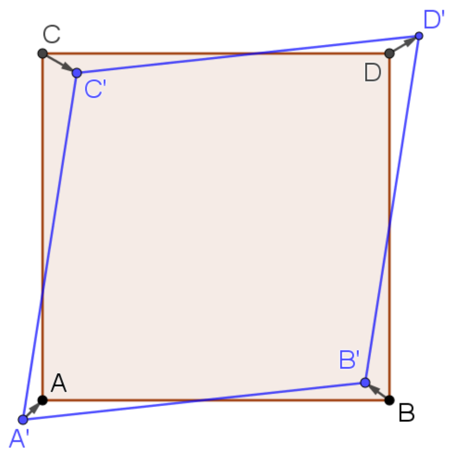



Figure 3.5: If $|AA'| < c$, $|BB'| < c$, and $|CC'| < c$, then $|DD'| < 3c$.

Since

$$\begin{aligned}
\overrightarrow{DD'} &= \overrightarrow{DC} + \overrightarrow{CC'} + \overrightarrow{C'D'} \\
&= \overrightarrow{BA} + \overrightarrow{CC'} + \overrightarrow{A'B'} \\
&= (\overrightarrow{BB'} + \overrightarrow{B'A'} + \overrightarrow{A'A}) + \overrightarrow{CC'} + \overrightarrow{A'B'} \\
&= \overrightarrow{BB'} + \overrightarrow{A'A} + \overrightarrow{CC'},
\end{aligned}$$

then

$$|\overrightarrow{DD'}| = |\overrightarrow{BB'} + \overrightarrow{A'A} + \overrightarrow{CC'}| \leq |\overrightarrow{BB'}| + |\overrightarrow{A'A}| + |\overrightarrow{CC'}| < 3c.$$

Let's now apply the planar conclusion on the cube and the parallelepiped in Figure 3.6.

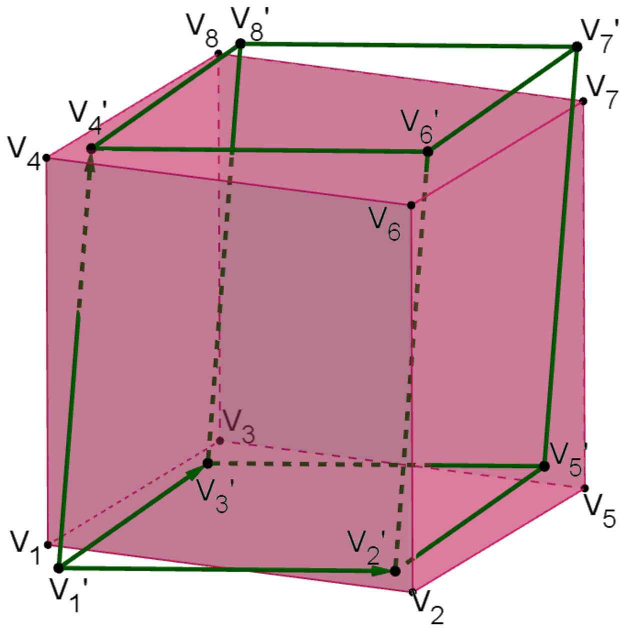



Figure 3.6: If $\max\{|V_iV_i'| : i = 1,2,3,4\} < c$, then $\max\{|V_iV_i'| : i = 1,2,3,4,5,6,7,8\} < 7c$.

First let's consider the square $V_1V_2V_5V_3$ and the parallelogram $V_1'V_2'V_5'V_3'$. Since $|V_iV_i'| < c$, where $i = 1,2,3$, $|V_5V_5'| < 3c$. Let's then look at the front square $V_1V_2V_6V_4$ and the parallelogram $V_1'V_2'V_6'V_4'$. Since $|V_jV_j'| < c$, where $j = 1,2,4$, $|V_6V_6'| < 3c$. Likewise, if we use the planar result on the square $V_1V_3V_8V_4$ and the parallelogram $V_1'V_3'V_8'V_4'$, immediately we will get $|V_8V_8'| < 3c$.

In order to obtain $|V_7V_7'|$, let's express $\overrightarrow{V_7V_7'}$ as follows.

$$
\begin{aligned}
\overrightarrow{V_7V_7'} &= \overrightarrow{V_7V_8} + \overrightarrow{V_8V_8'} + \overrightarrow{V_8'V_7'} \\
&= \overrightarrow{V_5V_3} + \overrightarrow{V_8V_8'} + \overrightarrow{V_3'V_5'} \\
&= (\overrightarrow{V_5V_5'} + \overrightarrow{V_5'V_3'} + \overrightarrow{V_3'V_3}) + \overrightarrow{V_8V_8'} + \overrightarrow{V_3'V_5'} \\
&= \overrightarrow{V_5V_5'} + \overrightarrow{V_3'V_3} + \overrightarrow{V_8V_8'}
\end{aligned}
$$

Since

$$\overrightarrow{V_5V_5'} = \overrightarrow{V_5V_3} + \overrightarrow{V_3V_3'} + \overrightarrow{V_3'V_5'}$$

$$
\begin{aligned}
&= \overrightarrow{V_2V_1} + \overrightarrow{V_3V_3'} + \overrightarrow{V_1'V_2'} \\
&= (\overrightarrow{V_2V_2'} + \overrightarrow{V_2'V_1'} + \overrightarrow{V_1'V_1}) + \overrightarrow{V_3V_3'} + \overrightarrow{V_1'V_2'} \\
&= \overrightarrow{V_2V_2'} + \overrightarrow{V_1'V_1} + \overrightarrow{V_3V_3'},
\end{aligned}
$$

then

$$\overrightarrow{V_7V_7'} = (\overrightarrow{V_2V_2'} + \overrightarrow{V_1'V_1} + \overrightarrow{V_3V_3'}) + \overrightarrow{V_3'V_3} + \overrightarrow{V_8V_8'} = \overrightarrow{V_2V_2'} + \overrightarrow{V_1'V_1} + \overrightarrow{V_8V_8'}.$$

Therefore,

$$|\overrightarrow{V_7V_7'}| = |\overrightarrow{V_2V_2'} + \overrightarrow{V_1'V_1} + \overrightarrow{V_8V_8'}| \leq |\overrightarrow{V_2V_2'}| + |\overrightarrow{V_1'V_1}| + |\overrightarrow{V_8V_8'}| < c + c + 3c = 5c.$$

$\square$

*Proof of Theorem 3.1.*

Assume the edge lengths of the fundamental parallelepiped are $a$, $b$, and $c$. Let $D$ denote a huge (that is, much bigger than $\max\{a, b, c\}$) cube with vertices $V_1, V_2, \ldots$, and $V_8$, which is obtained from the unit cube after proper scaling. Since each vertex of $D$ is inside one parallelepiped, we can select four of them, called $V_1, V_2, V_3$ and $V_4$, labelled in the order shown in Figure 3.6. By Lemma 3.2, there are vertices $V_i'$ from four fundamental parallelepipeds such that $|V_i'V_i| < \frac{a+b+c}{2}, i = 1, 2, 3, 4$.

Consider the parallelepiped $P$ generated by vectors $\overrightarrow{V_1'V_2'}, \overrightarrow{V_1'V_3'}$ and $\overrightarrow{V_1'V_4'}$. By Lemma 3.3, all distances $|V_i'V_i| < \frac{5}{2}(a + b + c), i = 1, 2, \ldots, 8$.

Let $\varepsilon = \frac{5}{2}(a+b+c)$, and draw spheres centered at each vertex of the cube with radius $\varepsilon$, so all vertices of the parallelepiped $P$ lie in these spheres. By the definition 3.1, the parallelepiped $P$ is $\varepsilon$-cube, relative to the cube $D$.

$\square$

# Chapter 4

## On Translative Packing Densities in $E^3$

In this chapter, we will talk about the translative packing densities in $E^3$. Rogers proved that a densest packing with translates of a convex disc can be achieved by the densest lattice packing. We will prove that this result can be generalized in $E^3$ to cylinders with convex bases. That is, the translative packing density of a cylinder with convex base is equal to the lattice packing density of such cylinders. Then we will look at the cylinder whose base is the planar wrench. We will refer to this cylinder as a 3D-wrench. Note that we have a complete understanding of the densest translative and the densest lattice packing of the planar wrench. Intuitively, it was expected that the densest 3-dimensional packings in each of the two categories are obtained by stacking the cylinders over the planar densest packings. The interesting thing is that stacking the cylinders over the densest translative packing of the base will result a 3D lattice packing. Besides the above results, I will introduce the relationships about the translative packing densities between the cylinder and its base.

**Theorem 4.1.** *Let $C$ be a cylinder. If the base of $C$ is a convex disc, then $d_T(C) = d_L(C)$.*

**Theorem 4.2.** *Let $C$ be a cylinder. If the base of $C$ is the planar wrench of Bezdek (see Figure 2.1), then $d_T(C) = d_L(C)$.*

*Proof of Theorem 4.1.* Since any lattice packing of $C$ is also a packing with translates of $C$, we have $d_T(C) \geq d_L(C)$. Now I will prove $d_T(C) \leq d_L(C)$ in three steps.

Let $B$ denote the convex base of the cylinder $C$.

**Step 1.** Let's prove $d_T(C) \leq d_T(B)$, that is, the translative packing density of the cylinder is no more than the translative packing density of the convex base.

Given a translative packing of the cylinder $C$ with density $d_T(C)$. At the same time, there is a tiling with copies of a huge cube such that the base of the cube is parallel to the base of the cylinder in the packing and also each rectangular face of the 3D-wrench is parallel to a face of the cube.

In view of the definition of the density, there exists one cube $U_i$, where the density of cylinders is at least $d_T(C)$. Otherwise, the density $d_T(C)$ couldn't be achieved in space. Let's denote the density of cylinders inside $U_i$ by $d_i$, so $d_i \geq d_T(C)$. If the cube is sufficiently large, the density of cylinders completely inside $U_i$ can be equal to $d_i - \varepsilon$, where $\varepsilon$ can be as small as we wish.

Now let's slice the cube $U_i$ parallel to the base. By the knowledge of integration, we understand there exists one cross-section where the density of the base $B$ is at least $d_i - \varepsilon$. Let's denote the cross-section by $S_j$, and denote the density of $B$ in $S_j$ by $d_{ij}$. So $d_{ij} \geq d_i - \varepsilon$.

Let's then get a packing of translates of the base $B$ by extending $S_j$ face to face. So the density of the translative packing is $d_{ij}$. Therefore, $d_{ij}$ is no more than the largest translative packing density, i.e., $d_{ij} \leq d_T(B)$.

Combined with the above inequalities, we have $d_T(C) - \varepsilon \leq d_T(B)$, where $\varepsilon$ can be arbitrarily small by letting the cube sufficiently large. Thus, $d_T(C) \leq d_T(B)$.

**Step 2.** $d_T(B) = d_L(B)$ can be immediately obtained from Rogers's result [13].

**Step 3.** Now let's prove $d_L(B) \leq d_L(C)$, i.e., the lattice packing density of the cylinder is an upper bound of the lattice packing density of its base.

Let's start with the densest lattice packing of $B$, so the density of $B$ in the plane is $d_L(B)$. Considering any fundamental parallelogram, we can have an edge-by-edge tiling with the fundamental parallelogram as the tile. All tiles are identical, so the density of the base in each tile is $d_L(B)$.

Now if we lift each tile up to the height of the cylinder $C$, we will get one layer of parallelepipeds. The density of the cylinder $C$ in each parallelepiped is $d_L(B)$.

Let's then translate congruent copies of the layer up and down such that there is no space between adjacent layers. The space can be tiled by translates of the parallelepipeds in this way, and there will also be a lattice packing of the cylinder $C$ with density $d_L(B)$, which can not exceed the maximum lattice packing density. That is, $d_L(B) \leq d_L(C)$.

Combined with 3 steps, we get $d_T(C) \leq d_T(B) = d_L(B) \leq d_L(C)$, and we already have $d_T(C) \geq d_L(C)$, so $d_T(C) = d_L(C)$.

□

**Remark 1.** From the proof we find the relationships about translative packing densities between the cylinder and its base, which are

$$d_T(C) = d_T(B),\ d_L(C) = d_L(B).$$

**Remark 2.** Notice that $d_L(C) \leq d_L(B)$ can also be proved by using the method in Step 1. We start with the densest lattice packing of $C$ and a tiling of an appropriate large cube, and then by the method in Step 1 we can get $d_L(C) \leq d_T(B)$. Since $B$ is a convex disc, $d_T(B) = d_L(B)$, therefore we have $d_L(C) \leq d_L(B)$.

Notice the large cube in Step 1 can be replaced by $\varepsilon$-cube introduced in Theorem 3.1 in Chapter 3. Since the density in each $\varepsilon$-cube is the same, we can use any one and work on it.

The method in Step 3 cannot be directly applied to prove $d_T(C) \geq d_T(B)$, because a bounded domain, which is the fundamental parallelepiped, is required in Step 3. For a densest translative packing of the convex disc $B$, it is not clear whether there is a cell decomposition such that the diameters of all cells have a common upper bound. But we believe $d_T(C) \geq d_T(B)$ can be proved with a method similar to the method in Step 3.

We also notice that the cylinder $C$ with convex base can be generalized to a cylinder with any shape as the base as long as the translative packing density of the shape is equal to the lattice packing density of the shape.

**Corollary 4.1.** *Let $B$ be a disc such that $d_T(B) = d_L(B)$, and let $C$ be a cylinder with $B$ as the base. Then $d_T(C) = d_L(C) = d_L(B) = d_T(B)$.*

*Proof of Theorem 4.2.*

Let's first have a look at the 3D-wrench, see Figure 4.1.

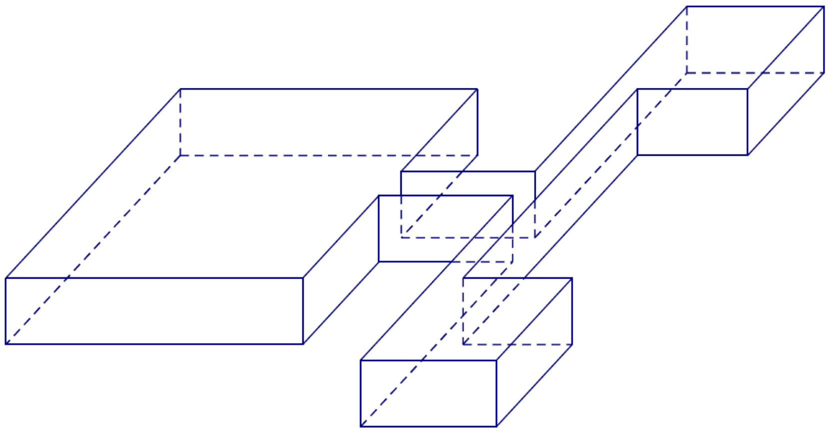


Figure 4.1: The 3D-wrench.

In order to prove $d_T(C) = d_L(C)$, we first have $d_T(C) \geq d_L(C)$, and I will prove $d_T(C) \leq d_L(C)$ in two steps.

The planar wrench is still denoted by $P$.

Step 1. I will prove $d_T(C) \leq d_T(P)$.

This can be done with the method in Step 1 in the proof of Theorem 4.1.

Step 2. I will describe a lattice packing of the 3D-wrench with density equal to $d_T(P)$.

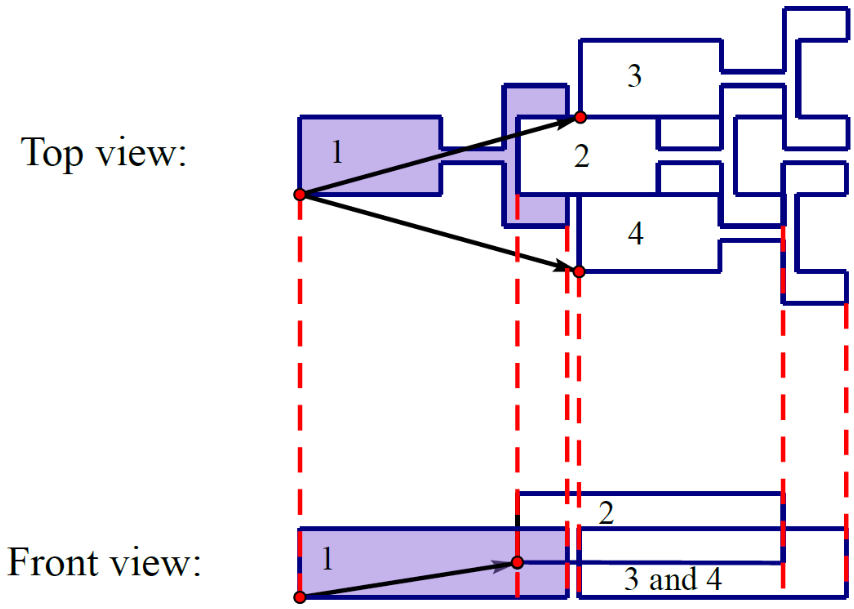



Figure 4.2: A lattice packing of the 3D-wrench.

Figure 4.2 depicts the top view and the front view of the desired lattice packing. The 3D-wrench labeled by $1$, $3$ and $4$ are at the same horizontal level, and they generate a horizontal layer of the lattice packing. Then the layer is shifted up non-vertically to a height equal to half of the height of a 3D-wrench, which can be seen from the front view. The desired lattice packing is constructed with the 3 independent vectors.

Notice that as a result exactly half of the mouth of each 3D-wrench belongs to another 3D-wrench. Therefore, each horizontal cross-section crosses exactly two layers. Furthermore, all top views of the cross-section are the same packing shown in Figure 2.22, which is a densest translative packing of the planar wrench $P$. Therefore, the density of this lattice packing of $C$ is $d_T(P)$.

Combined with Step 1 and Step 2, we have $d_T(C) \leq d_T(P) \leq d_L(C)$. Plus, $d_T(C) \geq d_L(C)$, we get $d_T(C) = d_L(C)$.

□

**Remark 3.** From the proof we can get the relationship between the translative packing density of the 3D-wrench and the translative packing density of the planar wrench, which is

$$d_T(C) = d_T(P).$$

**Remark 4.** Thanks to the densest translative packing of the planar wrench shown in Figure 2.22, the unproved part $d_T(C) \geq d_T(P)$ can be proved independently by the method in Step 3 in the proof of Theorem 4.1.

Appendix A

# Appendix

## A Proof of Lemma 2.2

*Proof of Lemma 2.2.* Figure A.1 demonstrates a packing $\mathcal{C} = \{C_1, C_2, \ldots\}$ in the plane. Suppose there is a cell decomposition $\mathcal{T} = \{T_1, T_2, \ldots\}$ so that each disc is contained in one cell. Without loss of generalization, we can assume $C_i \in T_i$. Furthermore, suppose $\text{diam}(T_i) \leq l$ and $\frac{A(C_i)}{A(T_i)} \leq d_0$ for all $i$.

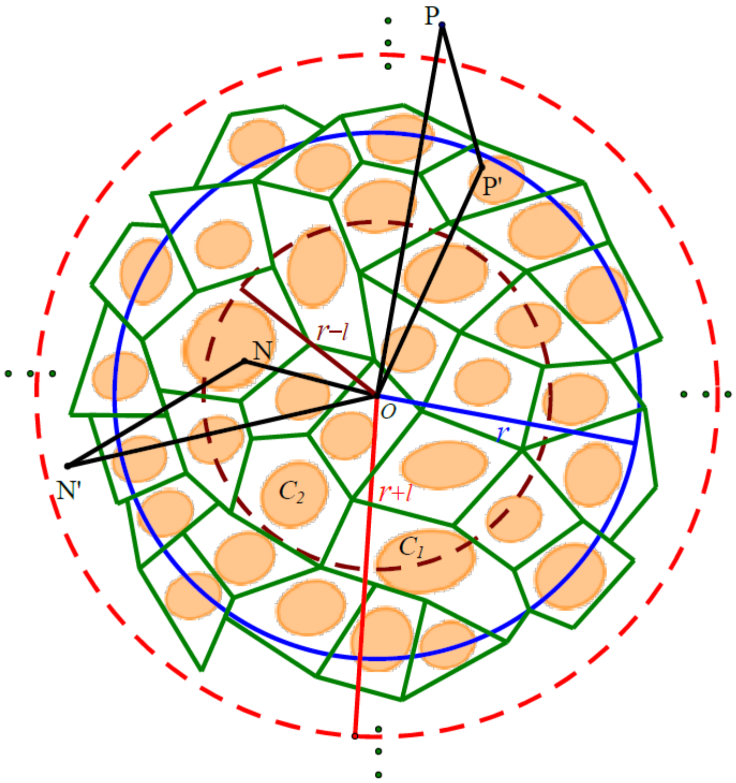



Figure A.1: Packing $\mathcal{C}_r = \{C_1, C_2, \ldots, C_{n_r}\}$ and the cell decomposition.

Since it it trivial if $d_0 = 1$, we will talk about the case when $d_0 < 1$.

Let $D(r)$ denote a circular disc with radius $r$, and $D(r+l)$ be the circular disc with radius $r+l$ and having the same center as $D(r)$. Suppose discs in $\mathcal{C}_r = \{C_1, C_2, \dots C_{n_r}\}$ have common interior points with $D(r)$. Then we have $\mathcal{T}_r \in D(r+l)$, where $\mathcal{T}_r = \{T_1, T_2, \dots T_{n_r}\}$ is the set of corresponding cells.

The reason why $\mathcal{T}_r \in D(r+l)$ is as follows.

If not, suppose $T_k$ $(\in \mathcal{T}_r)$ is not in $D(r+l)$, which means there is a point $P$ from $T_k$, but not in $D(r+l)$. Since $C_k \subset T_k$ and $C_k \cap D(r) \neq \emptyset$, there is a point $P' \in C_k \cap D(r)$ such that $|PP'| + |OP'| \geq |OP| > l + r$. Since $|OP'| \leq r$, $|PP'| > l$, which contradicts the condition that diam$(T_k) \leq l$, so $\mathcal{T}_r \subset D(r+l)$. Thus, we have $\sum_{i=1}^{n_r} A(T_i) \leq \pi(r+l)^2$.

Since each $\frac{A(C_i)}{A(T_i)} \leq d_0$, $\sum_{i=1}^{n_r} A(C_i) \leq d_0 \cdot \sum_{i=1}^{n_r} A(T_i)$. Therefore, $\sum_{i=1}^{n_r} A(C_i) \leq d_0 \cdot \pi(r+l)^2$. Furthermore, $\frac{\sum_{i=1}^{n_r} A(C_i)}{\pi r^2} \leq \frac{d_0 \cdot \pi(r+l)^2}{\pi r^2}$, and $\limsup\limits_{r\to\infty} \frac{\sum_{i=1}^{n_r} A(C_i)}{\pi r^2} \leq \limsup\limits_{r\to\infty} \frac{d_0 \cdot \pi(r+l)^2}{\pi r^2} = d_0$ which means $\overline{d} \leq d_0$.

Furthermore, if each $d_i = \frac{A(C_i)}{A(T_i)} = d_0$, then $\Sigma A(C_i) = d_0 \cdot \Sigma A(T_i)$.

Let $D(r-l)(r > l)$ denote the circular disc with radius $r - l$ and having the same center as $D(r)$. Suppose the cells tile the plane, and we have $\mathcal{T}_r \supset D(r-l)$.

If not, there would be a point $N$ from $D(r-l)$ and $N$ does not belong to any tile in $\mathcal{T}_r$. Suppose $N \in$ a tile $T_g$, so $T_g \notin \mathcal{T}_r$, and the disc $C_g(\subset T_g) \notin \mathcal{C}_r$, which means $C_g$ is totally outside $D(r)$. Let's choose any point $N'$ in $C_g$, and connect $O$ and $N'$, $O$ and $N$, and $N$ and $N'$. Then $|ON'| > r$ and $|ON| \leq r - l$, therefore $|NN'| \geq |ON'| - |ON| > l$, but diam$(T_g) \leq l$, a contradiction. Thus, $\mathcal{T}_r \supset D(r-l)$.

Therefore, $\sum_{i=1}^{n_r} A(T_i) \geq \pi(r-l)^2$ is obtained. And then $\sum_{i=1}^{n_r} A(C_i) \geq d_0 \cdot \pi(r-l)^2$, as well as $\liminf\limits_{r\to\infty} \frac{\sum_{i=1}^{n_r} A(C_i)}{\pi r^2} \geq \liminf\limits_{r\to\infty} \frac{d_0 \cdot \pi(r-l)^2}{\pi r^2}$. This gives us $\underline{d} \geq d_0$.

Now we have $d_0 \leq \underline{d} \leq \overline{d} \leq d_0$, which means $\underline{d} = \overline{d} = d_0$, and that's the density in the plane. Since it's an upper bound, it is the largest density in the plane.

□

**Remark 5.** If the density in each cell is strictly less than $d_0$, then the upper density in the plane is strictly less than $d_0$.

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