Avoiding k-Rainbow Graphs in Edge Colorings of K_n and other Families of Graphs

by

Isabel Harris

A dissertation submitted to the Graduate Faculty of
Auburn University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

Auburn, Alabama
May 4, 2024

Keywords: Graph Theory, Discrete Math, Edge Coloring, k-Rainbows

Approved by

Pete Johnson, Chair, Professor of Mathematics
Jessica McDonald, Associate Professor of Mathematics
Melinda Lanius, Assistant Professor of Mathematics
Joseph Briggs, Assistant Professor of Mathematics

Abstract

A simple graph, G, avoids a k-rainbow edge coloring if any color appears on at least $k+1$ edges of G. For any positive integer k, the k-Anti-Ramsey Number, $AR_k(G,H)$, is the maximum number of colors in an edge coloring of the graph H such that no k-rainbow edge colored copy of G is a subgraph of H. This work will discuss $AR_k(G,H)$ where H is various types of graphs. In particular, this work will focus on $AR_k(G,K_n)$ and define G as AR_k-bounded if $AR_k(G,K_n)$ is bounded by some positive integer c for all n sufficiently large. Additionally, we will say G is AR_k-unbounded is no such positive integer exists. In this work we will determine which simple graphs are AR_k-bounded for any k. We will provide a lower bound for $AR_k(G,K_n)$ if G is AR_k-unbounded and an upper bound for $AR_k(G,K_n)$ if G is AR_k-bounded. We will also determine $AR_k(G,H)$ for various graphs G, H where H is not a complete graph.

Acknowledgments

I would like to thank my mentors at Auburn University, Drs. Pete Johnson, Melinda Lanius, and Lora Merchant and the late Drs. Dean Hoffman and Jennifer Stone. Thank you for your encouragement, faith, and gentleness and I learned. Thanks to Drs. Jessica McDonald and Joseph Briggs for serving on my committee and providing insights.

I am grateful to all the teachers in my life. You inspired a love of learning I hope to pass on to many more and taught me lessons more important than those listed in the course objectives. In particular, thank you to Dr. Jenny Johnson, Dr. Mike Johnson, Dr. Pat Evans, Dr. Jared Collins, and Dr. Charles Tucker for their confidence in me and encouragement to pursue graduate school. A special thanks to Mr. Jerry Stelmaszak, Ms. Jan Little, and Mr. Tim Frizzell for providing opportunities for growth and learning and pushing me to grow outside of what I thought I wanted.

I would like to thank my family and friends for their patience and encouragement. Thanks to my parents, Paul and Elizabeth Harris, and my sister, Rosalee, for their support through the stress and celebrations and for teaching by example. I am grateful to Roger for his confidence in me and gentleness.

Table of Contents

Abstract ii

Acknowledgments iii

List of Tables vi

List of Figures vii

1 Introduction 1

1.1 Common Definitions and Notation 1

1.2 History 2

1.3 Early Results 7

1.4 Outline of Work 8

2 AR_k-Bounded Graphs 9

2.1 Introduction 9

2.2 AR_k-Unbounded Graphs 9

2.3 AR_k-Bounded Graphs 12

2.4 Example: AR_1-bounded Graphs 32

3 AR_k on Various Graph Families 34

3.1 General 34

3.2 Complete Bipartite Graphs 34

3.3 Cycles 38

3.4 Paths 39

3.5 Complete r-Partite Graphs 42

4 Future Work 45

Bibliography 47

Appendices 49

A Dean Hoffman's Proof for $k = 1$ 50

List of Tables

A.1 AR_1 values for P_3 52

A.2 AR_1 values for $2K_2$ 52

A.3 AR_1 values for P_4 53

A.4 AR_1 values for $P_3 \cup P_2$ 57

List of Figures

2.1 Star coloring example 10

2.2 Max Matching coloring example 10

2.3 Case 1 Coloring with a Rainbow P_4 13

2.4 Case 2 Coloring that admits a Rainbow P_4 13

2.5 Case 1 Coloring that admits a Rainbow $P_3 + P_2$ 14

2.6 Case 2 Coloring 14

2.7 Case 2 Coloring that admits a Rainbow $P_3 + P_2$ 15

2.8 Lineage of 4-edged graphs 15

2.9 Necessary Subgraphs 16

2.10 G, with edges colored as any copy of G in K_n must be colored, in this case. . . . 20

2.11 List of graphs G cannot be for Lemma 10 21

2.12 List of exempted graphs 25

2.13 Partial list of exempted graphs from Lemma 10 26

2.14 List of Possibly Excepted Graphs from Corollary 2.4.5 27

2.15 Rainbow K_3 . . . 28

2.16 Rainbow $K_{1,3}$ . . . 29

2.17 Rainbow P_4 . . . 29

2.18 Coloring of K_4 . . . 30

2.19 Rainbow $P_3 + K_2$ . . . 30

2.20 Rainbow $3K_2$ . . . 31

2.21 Four AR_1-bounded graphs . . . 32

2.22 Three Edged Graphs . . . 32

3.1 $K_{1,k+1}$ k-rainbow avoiding coloring of $K_{n,m}$ for $1 \leq n \leq k \leq m$ . . . 35

3.2 $K_{1,r}$ k-rainbow avoiding coloring of $K_{n,m}$ for $\min\{n,m\} \geq r \geq k+2$ . . . 36

3.3 $2K_2$ k-rainbow avoiding coloring of C_4 . . . 37

3.4 rK_2 k-rainbow avoiding coloring of $K_{n,m}$ for $\min\{n,m\} \geq r \geq k+2$ . . . 37

3.5 $(k+1)K_2$ k-rainbow avoiding coloring of $C_{2(k+1)}$ . . . 38

3.6 $(k+2)K_2$ k-rainbow avoiding coloring of $C_{2(k+2)}$ . . . 39

3.7 $(k+1)K_2$ k-rainbow avoiding coloring of P_{2k+3} . . . 40

3.8 $(k+1)K_2$ k-rainbow avoiding coloring of P_{2k+2} . . . 40

3.9 $(k+2)K_2$ k-rainbow avoiding coloring of P_{2k+5} . . . 41

3.10 rK_2 k-rainbow avoiding coloring of P_{2r+1} for $r \geq k+1$ . . . 41

3.11 P_{k+3} k-rainbow avoiding coloring of P_{k+4} 42

3.12 P_{k+3} k-rainbow avoiding coloring of P_{k+5} 42

3.13 K_3 k-rainbow avoiding coloring of $K_{r,s,t}$ 43

3.14 K_3 k-rainbow avoiding coloring of $K_{r,s,t}$ 44

A.1 Four AR-bounded graphs 50

A.2 Coloring of K_4 that avoids rainbow copies of $2K_2$ 52

A.3 Coloring of $P_3 \cup P_2$ that cannot avoid a rainbow copy of $2K_2$ 53

A.4 Coloring of K_4 that avoids rainbow copies of P_4 53

A.5 Example of all colors at one vertex being the same color 54

A.6 Coloring of a vertex with monochromatic incident edges that permits a rainbow copy of P_4 Version 1 54

A.7 Coloring of a vertex with monochromatic incident edges that permits a rainbow copy of P_4 Version 2 54

A.8 Coloring of a vertex with three colors incident to a single vertex 55

A.9 Coloring of a vertex with three colors incident to a single vertex that permits a rainbow P_4 55

A.10 Grouping of vertices by the two colors incident 56

A.11 Groups of two sets of vertices with two colors incident 56

A.12 Extension of rainbow P_4 that permits a rainbow $P_3 \cup P_2$ 57

Table of Notation

G	A graph. In this work, a simple graph with no isolates.
$V(G)$	The vertices of graph G.
$E(G)$	The edges of graph G.
K_n	A complete graph on n vertices.
$K_{n,m}$	A complete bipartite graph with vertex sets of size n and m.
P_n	A path on n vertices.
C_n	A cycle on n vertices.
$[A, B]$	A complete bipartite graph with vertex sets A and B.
mH	m disjoint copies of graph H.
$G + H$	A disjoint copy of G and H.
$d(v)$	Degree of vertex v.
$\Delta(G)$	The maximum degree of G. That is, $\max\{d(v) \mid v \in V(G)\}$.
$\alpha'(G)$	The size of the maximum matching in a graph G. That is, the maximum number of mutually disjoint edges in G.
$\chi'(G)$	The chromatic index of G. That is, the minimum number of colors that may properly edge-color a graph.
$AR(G, H)$	The maximum number of colors that may be used on an edge coloring of H so that every copy of subgraph G has some color appearing on at least two edges.
$AR_k(G, H)$	The maximum number of colors that may be used on an edge coloring of H so that every copy of subgraph G has some color appearing on at least $k + 1$ edges.

Chapter 1

Introduction

1.1 Common Definitions and Notation

Throughout this work we will be using some established definitions from graph theory and their typical notations. We will also include some less well known definitions and introduce new definitions and their notations. We have included a list of terms and their definitions that will be helpful to know.

Graph: A graph $G = (V, E)$ is a collection of vertices, denoted $V(G) = V$, and a collection of edges, denoted $E(G) = E$, such that each edge connects two vertices. An edge e that connects vertices u and v is *incident* to u and v. We say that vertices u and v are *adjacent* if some edge connects u and v. In this work we will consider only simple graphs, so if $v \neq u$ then v and u may be connected by at most one edge, denoted vu or uv, and no vertex is connected to itself by an edge, i.e. uu can not be an edge.

Complete Graph: The complete graph K_n is a graph with n vertices such that every vertex is adjacent to every other vertex. These may also be referred to as cliques.

Subgraph: A subgraph $G' = (V', E')$ of graph $G = (V, E)$ is a graph such that $V' \subseteq V$ and $E' \subseteq E$.

Degree: The degree of a vertex $v \in V(G)$ is the number of edges of G to which v is incident.

Maximum Degree: The maximum degree of a graph G, denoted $\Delta(G)$, is the greatest of the degrees of vertices of G.

Matching: A matching in a graph G is a set of edges in G such that no two distinct edges in the set are incident to the same vertex.

Maximum Matching: A maximum matching in a graph G is a largest matching in G. We will denote the size of the maximum matching in a graph G by $\alpha'(G)$. In this paper we will commonly use the abbreviation "max matching" for a maximum matching.

Edge Coloring: An edge coloring of a graph G is the assignment of colors $c_1, ...c_n$ to the edges of the graph G. In this work we will only discuss edge colorings of graphs and thus references to colored graphs refer to a graph with edges that have been assigned colors.

1.2 History

Ramsey Theory

In the 1930 paper "On a Problem in Formal Logic" [15] British mathematician F. P. Ramsey proved the following theorem that has inspired many questions and the field known as Ramsey Theory [11]. We will state Ramsey's specific theorem and then discuss its meaning. [15]:

Theorem 1.1 *Given any $r, n,$ and μ we can find an m_0 such that, if $m \geq m_0$ and the r-combinations of any Γ_m are divided in any manner into μ mutually exclusive classes C_i $(i = 1, 2, ..., \mu)$, then Γ_m must contain a sub-class Δ_n such that all the r-combinations of members of Δ_n belong to the same C_i.*

In more broad terms, he introduced the question "How large a structure must be to admit a certain trait?" and concluded that the solution would be finite (although potentially very large) [11] [6].

In Ramsey's theorem, by "the r-combinations of any Γ_m" Ramsey means "the r-subsets of any m-set." (For a non-negative integer k, a k-set is simply a set with k elements). By "divided in any manner into μ mutually exclusive classes C_i $(i = 1, 2, ...\mu)$" he means "partitioned into classes $C_1, ..., C_\mu$." What shocked and amazed the mathematicians of the 1930's - that made this theorem something really new - is that, given $r, n,$ and μ, the conclusion is not about μ-partitions of m-sets, but about μ-partitions of the collection of

r-subsets of a given m-set. The conclusion is that for m sufficiently large, no mater how the set of r-subsets of an m-set are partitioned into μ parts, there must be an n-subset of the given m-set of all of whose r-subsets are elements of one of those parts.

This result is powerful and concludes that these finite numbers exists, although they are difficult to find. Hungarian mathematician Paul Erdős was only 17 years old in 1930 [2]. He and others of that era quickly derived the corollary of Ramsey's theorem that is the foundation of Ramsey Theory in Graph Theory. To understand this corollary, observe that an edge in a simple graph can be considered to be a 2-subset of the set of vertices, and that in all of combinatorics, partitions are equivalent to colorings. Thus we have the following corollary of Ramsey's Theorem.

Corollary 1.1.1 *Given positive integers n and μ, for all positive integers m, sufficiently large, for every edge coloring of K_m with μ or fewer colors, there must be a monochromatic K_n subgraph in the K_m.*

The last part of the conclusion is, in other words, that for some color there are n vertices of the K_m such that all edges among those n vertices are that color.

Anti-Ramsey Theory

Ramsey theory has inspired many directions of research. In 1975, Erdős, Simonovits, and Sós introduced the idea of the anti-Ramsey number where the goal is to avoid a certain trait and established some preliminary results [5].

Definition 1 *A rainbow subgraph R of an edge colored graph G is a subgraph such that no two different edges of R bear the same color.*

Over time, results have been found for cycles, trees, bipartite graphs, and, most commonly, complete graphs.

Definition 2 *The Anti-Ramsey number of a graph G on graph H, $AR(G,H)$, is the maximum number of colors that can be used on an edge coloring of H such that no rainbow copy of G occurs as a subgraph of H.*

Many mathematicians have worked on exploring the anti-Ramsey number of graphs on complete graphs, $AR(G, K_n)$. These authors include Erdős [5] and Simonovits and Sós [16], Alon [1], Chen [4], Fujita [6], Jiang [9] and West [10], Manoussakis [12], Montellano-Ballesteros [13] and Neumann-Lara [14]. Many more have worked on this problem for various families of graphs other than complete graphs.

Rainbow-Subgraph Avoiding Edge Colorings

In the complete graph version of rainbow-subgraph avoiding edge coloring problem, we look for the maximum number of colors we can use in an edge coloring of K_n such that no copy of a given graph G is rainbow in the coloring of K_n. That is, each copy of G in K_n has at least two edges colored using the same color. We notate the maximum number of colors allowed on a copy of K_n that omits no rainbow copy of G using $AR(G, K_n)$, consistent with the notation and definition given previously.

In an interesting example, we learn that K_n can be edge-colored with $n-1$ or fewer colors so that no rainbow K_3 is present, but not with n colors. Gyárfás and Simonyi proved $AR(K_3, K_n) < n$ [8]. To illustrate Gyárfás and Simonyi's results we can use two colorings:
Coloring 1:[Gyárfás and Simonyi [8]] Partition $V(K_n)$ into two parts, A and B. Color all $[A, B]$ edges green, that is the edges between A and B are colored green. Iterate this process using a new color each time you iterate. For instance, at the first iteration, partition each $X \in \{A, B\}$ such that $|X| > 1$ into two parts X_1, X_2, and color the $[X_1, X_2]$ edges with a new color. This partitioning process, down to the unpartitionable singletons in $V(K_n)$, is encodable as the formation of a full binary tree with n leafs. The colors are in one-to-one correspondence with the acts of partition, and thus with the non-leafs of the tree. Therefore, there are exactly $n-1$ colors appearing.

For each 3-set $T \subseteq V(K_n)$, as the partitioning proceeds, there will be a "last" partition set $U \subseteq V(K_n)$ such that $T \subseteq U$; U is partitioned into U_1, U_2, neither containing T. Therefore one of the elements of T is in one of U_1, U_2, and the other two are in the other. Suppose

$T = \{u, v, w\}$ and $u \in U_1$, $v, w \in U_2$. Then the edges uv, uw will not bear the color assigned to vw in a subsequent partition that will separate v and w. Thus no K_3 in K_n is rainbow in such a coloring - and, although the result will have no application in this dissertation, it is worth noting that, also, no K_3 is monochromatic.

Coloring 2:[Hoffman and Johnson] Order the vertices of K_n as $\{v_1, v_2, ..., v_n\}$. Color each edge $v_j v_i$ using color c_{j-1} for $j > i$. Now let us show that every copy of K_3 in K_n will have some color on two edges, avoiding rainbow copies of K_3. Every copy of K_3 must have a vertex with a largest label, v_k, that is adjacent to two other vertices v_i and v_j. Without loss of generality, $i < j < k$. Notice edge $v_i v_j$ receives color c_{j-1} while edges $v_i v_k$ and $v_j v_k$ receives color c_{k-1}. Thus, two edges have the same color and no rainbow copy of K_3 occurs and $n-1$ colors were used. Therefore, $AR(K_3, K_n) \geq n-1$.

Theorem 1.2 (Gyárfás and Simonyi (2004)) *$AR(K_3, K_n) = n-1$.*

Proof: Since $AR(K_3, K_n) < n$ is proven in [8], it suffices to show that there is some coloring of $E(K_n)$ with exactly $n-1$ colors appearing such that no rainbow copy of K_3 exists. Using either Coloring 1 or Coloring 2 will show that $AR(K_3, K_n) \geq n-1$. ■

Using either of these two coloring methods, we see that $AR(K_3, K_n) = n-1$.

Edge colorings of type 2 are known as lexicographic colorings. They are a special case of type 1 colorings; if, in each partition in a type 1 coloring, one of the partition sets is a singleton, then the result will be a lexicographic coloring.

Although the result will have no application in this dissertation, it is worth noting that it is proven in [7] that every edge coloring of K_n which forbids rainbow K_3's and in which $n-1$ colors appear is of type 1.

D.G. Hoffman defined the following to initiate a new way of studying the anti-Ramsey numbers of graphs.

Definition 3 *We call a graph AR-bounded if there exists some fixed integer d such that $AR(G, K_n) \leq d$ for all n.*

Definition 4 *We call a graph AR-unbounded if it is not AR-bounded.*

K_3 is a nice example of an AR-unbounded graph since the maximum number of colors that can be used without permitting a rainbow copy of K_3 increases as n increases. We do not know if this holds, for n increasing from $|V(G)|$, for all AR-unbounded graphs G.

D.G. Hoffman worked on the question of which graphs are AR-unbounded. Though some of the results on Hoffman's question were known by authors mentioned previously, such as Erdős [5] and Simonovits and Sós [16], Alon [1], Chen [4], Fujita [6], Jiang [9] and West [10], Manoussakis [12], Montellano-Ballesteros [13] and Neumann-Lara [14], he found a clever proof that solved the problem completely. This proof can be found in Appendix A.

We will extend this question into one of our own, concerning the avoidance of k-rainbow copies of G on K_n.

We define the following to follow the extension of this problem:

Definition 5 *Suppose k is a positive integer and G is an edge-colored graph. G is k-rainbow in, or with, the coloring if and only if no color appears on more than k edges of G.*

That is, G is not k-rainbow in a coloring of $E(G)$ if and only if some color appears on at lease $k+1$ edges of G.

Definition 6 *Suppose that $k \in \mathbb{Z}^+$, G and H are graphs, and G has no isolated vertices and at least $k+1$ edges. The k-Anti-Ramsey Number $AR_k(G,H)$ is the largest number of colors that can appear in a coloring of $E(H)$ such that no subgraph of H isomorphic to G is k-rainbow in the restriction of the coloring to its edges. We will say that such a coloring avoids, or forbids, k-rainbow (copies of) G. When $H = K_n$ for some integer n, we abbreviate: $AR_k(G,K_n) = AR_k(G,n)$.*

By this definition, if H contains no copy of G then $AR_k(G,H) = |E(H)|$.

Definition 7 *We call a graph G AR_k-bounded if there exists some integer d such that $AR_k(G,n) \le d$ for all n.*

Definition 8 *We call a graph AR_k-unbounded if it is not AR_k-bounded.*

1.3 Early Results

Some results follow fairly simply from the definitions.

Proposition 1 *For $n < |V(G)|$, $AR_k(G, K_n) = \binom{n}{2}$.*

PROOF: Since K_n contains fewer vertices than G, there can be no copy of G in K_n. Thus, the edges may be colored with different colors, that is, by using $\binom{n}{2}$ colors, and there will be no k-rainbow copy of G in K_n because there is no copy of G at all. ■

Proposition 2 *For a graph G and $k \in \mathbb{Z}^+$,*

(a) If $|E(G)| \leq k$ then $AR_k(G, K_n)$ is undefined, for $n \geq |V(G)|$.

(b) If $|E(G)| = k + 1$ and $n \geq |V(G)|$ then $AR_k(G, K_n) = 1$.

(c) If $|E(G)| \geq k + 2$ and $n \geq |V(G)|$ then $AR_k(G, K_n) > 1$.

PROOF:

(a) Suppose that $|E(G)| \leq k$. Because $n \geq |V(G)|$, K_n contains copies of G. For any edge coloring of K_n, every copy of G will be k-rainbow with respect to the coloring, as no color can appear $k+1$ or more times on a set of k edges. Therefore there is no number of colors with which the edges of K_n can be colored so that k-rainbow copies of G are forbidden.

(b) Again, whatever $|E(G)|$ may be, $n \geq |V(G)|$ implies that there are copies of G in K_n. Since $|E(G)| = k + 1$, coloring $E(K_n)$ with one color will forbid k-rainbow copies of G. If $E(K_n)$ is colored with more than one color then a copy of G can be found in K_n with at least 2 colors on its edges. But then such a copy of G is k-rainbow, since none of the colors can appear $k+1$ times on that copy of G. Therefore, a coloring with more than 1 color cannot forbid k-rainbow copies of G.

(c) If $|E(G)| \geq k+2$ (and $n \geq |V(G)|$), color $E(K_n)$ with red and blue, with blue appearing on only one edge. Any copy of G in K_n will have all red edges, or one blue edge and the rest red. In either case, the copy is not k-rainbow. Thus $AR_k(G, K_n) \geq 2$.

■

In the remainder of this dissertation we will be mainly concerned with $AR_k(G, K_n)$, $n \geq |V(G)|$. The following is an exception.

Proposition 3 *If* $|E(G)| \geq k+1$ *then* $AR_k(G,G) = |E(G)| - k$

Proof: Color G so that $|E(G)| - (k+1)$ colors appear on one edge each and so that the remaining $k+1$ edges are colored with some new color, c. Then, since every edge is used in every copy of G, there is no k-rainbow copy of G since $k+1$ edges are all colored the same. Thus $AR_k(G,G) \geq |E(G)| - k$.

On the other hand, if $E(G)$ is colored with $|E(G)| - k + 1$ colors or more appearing, then the greatest number of edges that any one color can appear on is $|E(G)| - (|E(G)| - k) = k$, and thus G itself is k-rainbow. ■

1.4 Outline of Work

In the remainder of this work we will show the following results. In Chapter 2 we will prove the primary result of this dissertation by characterizing AR_k-bounded graphs for every positive integer k. The main results are in Theorem 2.1, Corollary 2.1.3, Corollary 2.1.4, Proposition 4, Theorem 2.3. In Chapter 3 we will discuss anti-Ramsey numbers, $AR_k(G, H)$, for H some graph that is not a complete graph. Finally, in Chapter 4 we will discuss future directions of this work.

Chapter 2

AR_k-Bounded Graphs

2.1 Introduction

In this chapter we will find for any given $k \in \mathbb{Z}^+$ the finite graphs G with no isolates that are AR_k-bounded. When we can, we will also find $AR_k(G, n)$. In order to avoid a k-rainbow edge coloring, at least $k+1$ edges must all be colored with the same color, see Proposition 2. Therefore, we will assume all graphs G have at least $k+1$ edges. Additionally, since isolates do not change any edge colorings, we will assume all graphs are isolate-free.

2.2 AR_k-Unbounded Graphs

Definition 9 *In a **star** coloring of K_n $(n > 1)$, single out a single vertex and color the edges incident to that vertex with $n-1$ colors appearing - i.e., make a rainbow $K_{1,n-1}$. Then color all other edges of K_n with a different color, c. Number of colors appearing: $n-1+1 = n$. See example below.*

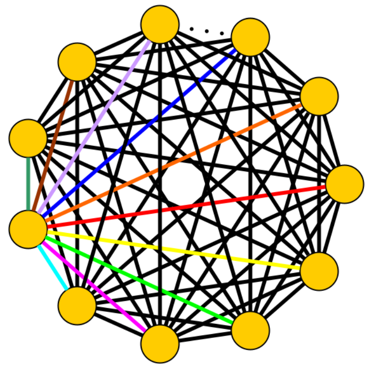

Figure 2.1: Star coloring example

Definition 10 *In a* ***max matching*** *coloring of* K_n*, take a maximum matching,* M*, in* K_n*, with* $\lfloor \frac{n}{2} \rfloor$ *edges, and make it rainbow. Then color all other edges with a new color,* c*. Number of colors appearing:* $\lfloor \frac{n}{2} \rfloor + 1$*. See example below.*

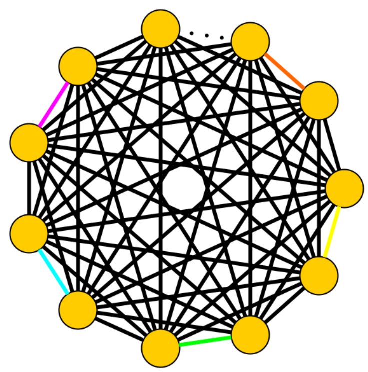

Figure 2.2: Max Matching coloring example

Theorem 2.1 *Suppose that* n, k *are integers,* $n \geq |V(G)|$*,* $k > 0$*, and* G *is an isolate-free graph with at least* $k+1$ *edges.*

1. *If* $\Delta(G) \geq k+2$*, then* $AR_k(G,n) \geq \lfloor \frac{n}{2} \rfloor + 1$*.*

2. *If* $|E(G)| - \Delta(G) \geq k+1$*, the* $AR_k(G,n) \geq n$*.*

3. *If* $\alpha'(G) \geq k+2$*, then* $AR_k(G,n) \geq n$*.*

4. *If* $|E(G)| - \alpha'(G) \geq k+1$*, then* $AR_k(G,n) \geq \lfloor \frac{n}{2} \rfloor + 1$*.*

Proof: In 1 and 4, consider a max matching coloring of K_n. In 2 and 3, consider a star coloring of K_n. ■

Corollary 2.1.2 *If any of the following conditions hold for G a finite graph with no isolated vertices, then G is AR_k-unbounded.*

1. $\Delta(G) \geq k+2$
2. $|E(G)| - \Delta(G) \geq k+1$
3. $\alpha'(G) \geq k+2$
4. $|E(G)| - \alpha'(G) \geq k+1$

Lemma 1 *If G is an isolate-free graph and $|E(G)| \geq 2k+2$, then either $|E(G)|-\alpha'(G) \geq k+1$ or $|E(G)| - \Delta(G) \geq k+1$.*

Proof: We shall prove the contrapositive. Suppose that $|E(G)| - \alpha'(G) \leq k$ and $|E(G)| - \Delta(G) \leq k$. Then $\alpha'(G), \Delta(G) \geq |E(G)| - k$. Let M be a matching in G with $|E(M)| = \alpha'(G)$ and let $v \in V(G)$ be a vertex of degree $d(v) = \Delta(G)$ in G. Clearly at most one edge of M can be incident to v. Therefore $|E(G)| \geq |E(M)| + d(v) - 1 = \alpha'(G) + \Delta(G) - 1 \geq 2(|E(G)| - k) - 1 \implies |E(G)| \leq 2k+1$. ■

Corollary 2.1.3 *If $|E(G)| \geq 2k+2$, then G is AR_k-unbounded.*

Corollary 2.1.4 *For an isolate-free graph G, with $k+2 \leq |E(G)| \leq 2k+1$, a necessary condition for G to be AR_k-bounded is $|E(G)| - k \leq X(G) \leq k+1$ for $X \in \{\Delta, \alpha'\}$.*

Proof: Follows from Theorem 2.1 and the corollaries above that for a graph to be AR_k-bounded, the following must be true:

1. $k+1 \leq |E(G)| \leq 2k+1$,
2. $|E(G)| - X(G) \leq k$, and
3. $X(G) \leq k+1$

for $X \in \{\Delta, \alpha'\}$. ■

2.3 AR_k-Bounded Graphs

Proposition 4 *Graphs with exactly $k+1$ edges are AR_k-bounded.*

See Proposition 2 in Chapter 1.

Proposition 5 *If G is an isolate-free graph, has exactly $k+2$ edges, and $n \geq |V(G)|$, then $AR_k(G, K_n) \geq 2$.*

Proof: See Proposition 2 in Chapter 1. ∎

Proposition 6 *If G is an isolate-free graph, has $k+2$ edges, and contains P_4 as a subgraph, then for $n \geq \max\{|V(G)|, 5\}$, $AR_k(G, n) = 2$.*

Proof: By Proposition 5, $AR_k(G, K_n) \geq 2$. It remains to be seen that if $E(K_n)$ is colored with exactly 3 colors appearing, then in some copy of G in K_n none of the 3 colors appear more than k times on its edges. Let the colors be red, blue, and green, and suppose that $E(K_n)$ is colored with these colors, with no k-rainbow copy of G. Under the assumption that there is no k-rainbow copy of G, we shall show the contrary, which will finish the proof. Let uv and vw be adjacent edges of different colors in some copy of G. Without loss of generality, suppose that uv is red and vw is blue. If all three colors appear on the edges of some P_4 in any copy of G, then each color can appear at most $k-1$ times on the remaining $k-1$ edges of that copy of G and thus at most k times in that copy of G. Since every P_4 in K_n can be considered to be a subgraph in copies of G, it follows that there are no rainbow P_4's in K_n. Therefore, each edge ux, $x \neq v$, is either red or blue, and the same holds for edges wy, $y \neq v$.

But the color green appears somewhere.

Case 1: vx is green for some x.

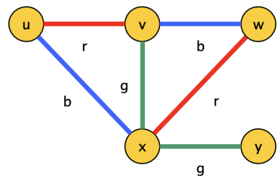

Figure 2.3: Case 1 Coloring with a Rainbow P_4

Edge ux is either red or blue; if red, then the path $uxvw$ is a rainbow P_4. Therefore, ux is blue. Symmetrically, wx is red. Since $n \geq 5$, there is a vertex $y \notin \{u, v, x, w\}$. If wy is red, then $xvwy$ is a rainbow P_4. If wy is blue, then $vxwy$ a rainbow P_4. But wy must be either red or blue so we have a contradiction.

Case 2: All edges incident to v are either red or blue, and some edge xy, $\{x, y\} \cap \{u, v, w\} = \varnothing$, is green.

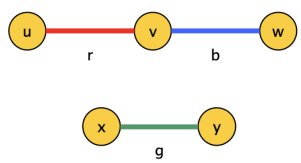

Figure 2.4: Case 2 Coloring that admits a Rainbow P_4

Then vx is either red or blue; whichever, we see a rainbow P_4, either $yxvu$ or $yxvw$.

Case 3: No edge incident to v is green, and uw is green.

Then, by the reasoning originally applied to uvw, every edge vx, $x \notin \{u, w\}$, must be red or green, and must be blue or green. Therefore, every edge must be green, and, since there are such edges, we are back in Subcase 1.1. ∎

Proposition 7 *If G is an isolate-free graph, has $k+2$ edges, and contains $P_3 \cup P_2$ as a subgraph, then for $n \geq \max\{|V(G)|, 8\}$, $AR_k(G, n) = 2$.*

Proof: By Proposition 6, $AR_k(G, K_n) \geq 2$. It remains to be seen that if $E(K_n)$ is colored with exactly 3 colors appearing, then in some copy of G in K_n none of the 3 colors appear more than k times on its edges. Let the colors be red, blue, and green, and suppose that $E(K_n)$ is colored with these colors, with no k-rainbow copy of G. Under the assumption that this cannot happen, we shall show that it must happen, which will finish the proof.

Suppose that the edges of K_n ($n \geq \max\{|V(G)|, 8\}$) are colored with red, blue, and green so that no copy of G in K_n is k-rainbow. As in the previous proof, because $P_3 \cup P_2$ in K_n will be a subgraph of copies of G in K_n, no copy of $P_3 \cup P_2$ in K_n is rainbow.

Let u, v, w and the coloring of the path uvw be as in Case 1. Since there can be no rainbow $P_2 + P_3$ in K_n, all edges xy, $\{x, y\} \cap \{u, v, w\} = \varnothing$, are either red or blue. Therefore, each green edge must be incident to at least one of u, v, w.

Case 1 For some vertex $x \notin \{u, v, w\}$, vx is green.

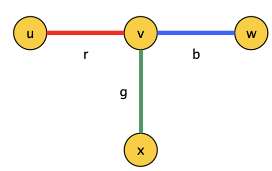

Figure 2.5: Case 1 Coloring that admits a Rainbow $P_3 + P_2$

For all edges yz, $y, z \notin \{u, v, x, w\}$, the edge yz must be colored one of red, blue, but also one of blue, green, and also one of red, green. Since $n \geq 8$, such edges exist so this case is impossible.

Case 2: There is no rainbow $K_{1,3}$.

Returning to u, v, w, now we have that each green edge must be incident to either u or w, and all edges xy, $x, y \notin \{u, w\}$are either red or blue. The edge uw cannot be green:

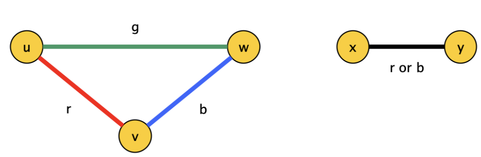

Figure 2.6: Case 2 Coloring

If uw were green, as in Figure 2.6, then there would be a rainbow $P_3 + P_2$ in K_n, either $xy + vuw$ or $xy + vwu$.

Without loss of generality, suppose that ux is green, for some $x \notin \{u, v, w\}$.

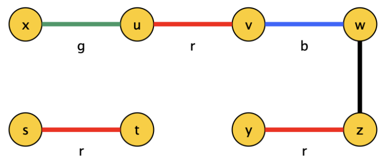

Figure 2.7: Case 2 Coloring that admits a Rainbow $P_3 + P_2$

Because $n \geq 8$, there are independent edges st, yz with $s, t, y, z \notin \{u, v, w, x\}$; both must be colored red, because, if not, we have a rainbow $P_2 + P_3$, $P_2 = yz$ or st and $P_3 = xuv$ or uvw. But, whatever wz is colored, there will be a rainbow $P_2 + P_3$ in K_n. ■

Lemma 2 *If G has $k+2$ edges, is isolate-free, and is none of K_3 (when $k = 1$), $K_{1,k+2}$, or $(k+2)K_2$, then G has either P_4 or $P_3 \cup P_2$ as a subgraph.*

Proof: Observe that all the four-edged graphs with no isolated vertices except $4K_2$ and $K_{1,4}$ have one of the two graphs P_4, $P_3 \cup P_2$ as a subgraph and extend this principle: if $k > 1$ and G has $k+2 \geq 4$ edges then unless $G \in \{K_{1,k+2}, (k+2)K_2\}$, G has a subgraph with 4 edges other than $4K_2, K_{1,4}$ and, therefore either P_4 or $P_3 \cup P_2$ as a subgraph. ■

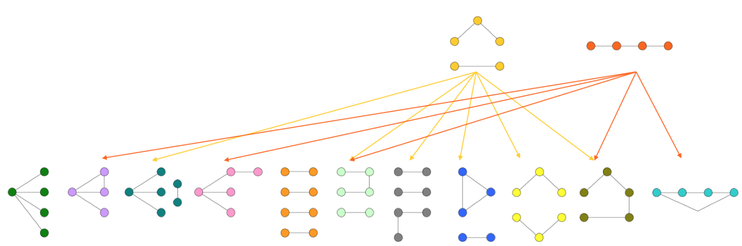

Figure 2.8: Lineage of 4-edged graphs

Theorem 2.2 *If $|E(G)| = k + 2$, is isolate-free, and G is none of K_3 (when $k = 1$), $K_{1,k+2}$, or $(k+2)K_2$, then G is AR_k-bounded.*

Proof: This is a corollary of Propositions 6 and 7, and Lemma 2. ■

Lemma 3 *For an isolate-free graph G with $k + 2 \leq |E(G)| \leq 2k + 1$, such that $|E(G)| - k \leq X(G) \leq k + 1$ for $X \in \{\Delta, \alpha'\}$, at least one of the following graphs from each class must be a subgraph. Class 1 subgraphs have $\Delta(G) + 1$ edges and Class 2 subgraphs have $\alpha'(G) + 1$ edges.*

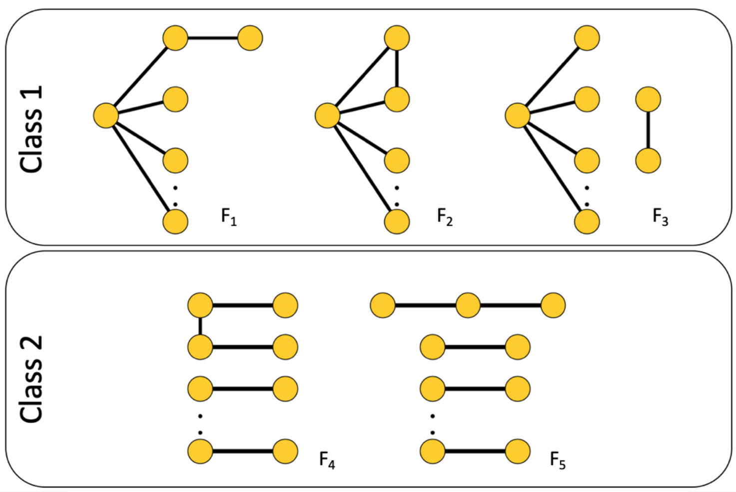

Figure 2.9: Necessary Subgraphs

Proof: Consider a vertex of maximum degree in G. It is adjacent to exactly $\Delta(G)$ edges. Since $\Delta(G) \leq k+1$ and $|E(G)| \geq k+2$, there must be some edge e not adjacent to the vertex of maximum degree. This edge must go somewhere and the three graphs in Class 1 represent the only possible configurations.

Likewise, consider a maximum matching in G, call it M; $|E(M)| = \alpha'(G)$. Since $\alpha'(G) \leq k + 1$ and $|E(G)| \geq k + 2$, there must be some edge e of G not in the maximum matching. This edge must go somewhere and the two graphs in Class 2 represent the only possible configurations. ■

Lemma 4 *Under the hypothesis of Lemma 3, if G is edge-colored and any one of the five graphs in Lemma 3 is a rainbow subgraph of G, then G is k-rainbow.*

PROOF: Since $|E(G)| - k \leq X(G)$, for $X \in \{\alpha', \Delta\}$, it follows that $|E(G)| \leq X(G) + k$. The subgraphs all have edge sets of size $X(G) + 1$. Thus, if any of these subgraphs is rainbow, then the maximum number of edges remaining to be colored is $k - 1$ and even if they all receive some color already on the subgraph, no color appears in $E(G)$ more than k times, so G is k-rainbow. ■

Lemma 5 *Suppose that $E(K_n)$ is colored and H is a rainbow subgraph of K_n with a maximum number of edges. [Equivalently, H is formed by taking one edge of each color class.] Then any $e \in E(K_n) \backslash E(H)$ bears a color appearing in H.*

The proof is left to the reader.

Lemma 6 *Suppose that the hypotheses of Lemmas 3 and 5 hold, and F_1, F_2, and F_3 are the Class 1 graphs in Figure 2.9,reading left to right.*

(a) If $\Delta(H) > \Delta(G)$ and $n \geq \Delta(G) + 4$ then K_n contains a rainbow F_3.

(b) If $\Delta(H) > \Delta(G)$, $n \geq \Delta(G) + 3$, and K_n contains no rainbow F_1, then K_n contains a rainbow F_3.

PROOF: (a) Since $\Delta(H) > \Delta(G)$, H contains a $D = K_{1,\Delta(G)+1}$; let w be the central vertex and $x_1, ..., x_{\Delta(G)+1}$ be the leafs. Since $n \geq \Delta(G)+4$ there are two vertices $y, z \in V(K_n) \backslash V(D)$. If $yz \in E(H)$ then we have our rainbow $F_3 = (D - x_{\Delta(G)+1}) \cup yz$. Otherwise, if $yz \notin E(H)$, then by Lemma 5 yz must bear the same color as some $e \in E(H)$. Then $H' = (H - e) \cup yz$ is rainbow, and, whether $e \in \{wx_i | i = 1, ..., \Delta(G) + 1\}$ or not, H' contains an $F_3 = K_{1,\Delta(G)} + yz$. (b) Because $\Delta(H) > \Delta(G)$, H contains a $D = K_{1,\Delta(G)+1}$ subgraph as in (a). Because $n \geq \Delta(G) + 3$, there is a vertex $y \in V(K_n) \backslash \{w, x_1, ..., x_{\Delta(G)+1}\}$. If $x_i y \in E(H)$ for some $i \in \{1, ..., \Delta(G) + 1\}$ then K_n would contain a rainbow F_1. Therefore, we may assume that

$x_iy \in E(K_n)\backslash E(H)$ for each i. Let $x = x_1$. Then xy bears a color appearing on some edge $e \in E(H)$. Then $H' = (H - e) \cup xy$ is rainbow. Unless $e = wx_1$, we will have an F_1 subgraph in H. Therefore, $e = wx_1$ and H' contains an F_3, $(K_{1,\Delta(G)+1} - x_1) \cup x_1y$. ■

Lemma 7 *Suppose that the hypotheses of Lemmas 3 and 5 hold, and* $\Delta(H) > \Delta(G) + 1$. *Then* K_n *contains a rainbow* F_1.

Proof: Since $\Delta(H) > \Delta(G) + 1$, H contains a subgraph $D = K_{1,\Delta(G)+2}$, with vertex w of degree $\Delta(G) + 2$ and leafs $x_1, ..., x_{\Delta(G)+2}$. If $x_1x_2 \in E(H)$ then $((D - wx_1) - x_{\Delta(G)+2}) \cup x_1x_2$ is a rainbow F_1 in K_n.

Otherwise, if $x_1x_2 \notin E(H)$ then x_1x_2 bears a color appearing on an edge $e \in E(H)$. Then $H' = (H - e) \cup x_1x_2$ is rainbow.

If $e \notin E(D) \cup \{x_ix_j | 1 \le i < j \le \Delta(G) + 2\}$ then $((D - wx_1) - x_{\Delta(G)+2}) \cup x_1x_2$ is an F_1 subgraph of H', which is, therefore, rainbow. If $e \in \{x_ix_j | 1 \le i < j \le \Delta(G)+2\}$ then a rainbow F_1 in H can be found by repeating the first part of this proof with the edge x_ix_j playing the role played by x_1x_2 there. If $e \in \{wx_i | 3 \le i \le \Delta(G) + 2\}$ then $((D - wx_1) - x_i) \cup x_1x_2$ is a rainbow F_1 subgraph of H'. If $e \in \{wx_1, wx_2\}$ – say $e = wx_1$ – then $((D-wx_1)-x_{\Delta(G)+2}) \cup x_1x_2$ is a rainbow F_1 subgraph of H'. ■

Lemma 8 *Suppose that* $k > 1$. *Suppose that the hypotheses of Lemmas 3 and 5 hold, and, in addition,* $n \ge \max\{\Delta(G) + 4, |V(G)|\}$ *and the hypothetical coloring of* $E(K_n)$ *forbids* k*-rainbow copies of* G. *Then* $\Delta(H) \le \Delta(G) + 1$.

Proof: Since $n \ge |V(G)|$, any subgraph of G in K_n can be embedded as a subgraph of a copy of G (possibly many different copies of G) in K_n. Since the coloring of $E(K_n)$ forbids k-rainbow copies of G, by Lemma 4 it follows that for $i \in \{1, 2, 3\}$, if F_i is a subgraph of G, then no F_i in K_n can be rainbow. Since $n \ge \Delta(G) + 4$ and $\Delta(H) > \Delta(G) + 1 > \Delta(G)$ then by Lemma 6(a), K_n contains a rainbow F_3. Therefore, F_3 is not a subgraph of G.

By Lemma 3, then either F_1 or F_2, or both, are subgraphs of G. By Lemma 7, F_1 is ruled out. This leaves us with one possibility: G contains F_2 as a subgraph but neither F_1

nor F_3.

In this case, we will need only the assumption that $\Delta(H) > \Delta(G)$. Assuming this, let $D = K_{1,\Delta(G)+1}$ be a subgraph of H with, as before, vertex w of degree $\Delta(G) + 1$ and leafs $x_1, ..., x_{\Delta(G)+1}$. Because G contains no F_1 or F_3 as a subgraph, and G has no isolated vertices, every $K_{1,\Delta(G)}$ subgraph of G is spanning, in G. Therefore $|V(G)| = 1 + \Delta(G)$ and every $K_{1,\Delta(G)}$ subgraph of D is a spanning subgraph of a copy of G in K_n.

For every pair i, j satisfying $1 \le i < j \le k+1$, $D \cup x_i x_j$ contains subgraphs isomorphic to F_2, none of which can be rainbow because k-rainbow copies of G are forbidden in the coloring of K_n. Therefore $x_i x_j \notin E(H)$; therefore $x_i x_j$ bears a color on some $e \in E(H)$. Then $H_{ij} = (H - e) \cup x_i x_j$ is rainbow. By arguments deployed previously, the non-existence of rainbow F_2's in K_n forces $e \in \{wx_i, wx_j\}$.

For every copy of G in K_n, some color must appear at least $k+1$ times on the edges of that copy, because the coloring of $E(K_n)$ forbids k-rainbow copies of G. Because D is rainbow and the only colors that could possibly appear on any leaf-to-leaf edge $x_i x_j$ are the colors on wx_i, wx_j, it follows that for each subgraph G of the graph $D' = D \cup \{x_i x_j | 1 \le i < j \le \Delta(G)+1\}$ with w having degree $\Delta(G)$ in that copy of G, for some $x_i \in V(G)$ there are k values of $j \in \{1, ..., \Delta(G) + 1\} \backslash \{i\}$ such that $x_i x_j \in E(G)$ and the edge $x_i x_j$ is colored with the color on wx_i.

Then x_i has degree at least $k+1$ in that copy of G (taking into account its adjacency to w) so $\Delta(G) \ge k+1$. By hypothesis, $\Delta(G) \le k+1$. Therefore, $\Delta(G) = k+1$. Also, counting just the edges of (this copy of) G that we know of, that are incident to w or x_i, we have $|E(G)| \ge \Delta(G) + k = k + 1 + k = 2k + 1$.

On the other hand, by hypothesis, $|E(G)| \leq 2k+1$. Therefore, $|E(G)| = 2k+1$, and G is the graph depicted in Figure 2.10.

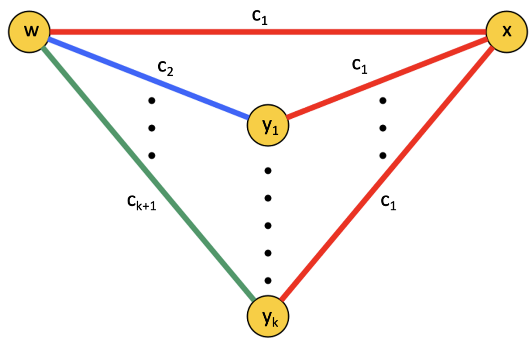

Figure 2.10: G, with edges colored as any copy of G in K_n must be colored, in this case.

In Figure 2.10 we have indicated the necessary coloring of the edges of any copy of G in D', under the assumptions of this case. We can now derive a contradiction by considering just the edge-colored graph in Figure 2.10. Consider another copy of G on the vertices $w, x, y_1, ..., y_k$, obtained by leaving all the edges incident to w, deleting all the edges incident to x except wx, and adding the edges y_1x and y_1y_j, $j \in \{2, ..., k\}$. That is, we demote x to the role of y_1 and promote y_1 to the role of x. But in this copy of G, the edge y_1x is required to bear the color on wy_1, c_2. However, because of y_1's role in the copy of G in Figure 2.10, the color of y_1x has already been determined to be $c_1 \neq c_2$. ■

Comment on the requirement that $k > 1$ in Lemma 8:

When $k = 1$, the only graphs G satisfying $3 = k+2 \leq |E(G)| \leq 2k+1 = 3$ and $2 = |E(G)| - k \leq \Delta(G) \leq k+1 = 2$ are K_3, P_4, and $P_3 + K_2$. Of these K_3 is an F_2, P_4 is an F_1, and $P_3 + K_2$ is an F_3. The proof of Theorem 1.2 shows that for all $n \geq 3$, $E(K_n)$ can be colored with $n-1$ colors appearing so as to forbid a rainbow K_3 but with a rainbow $K_1, n-1$ present. Therefore, the conclusion of Lemma 8 cannot be extended to $k = 1$ when $G = K_3$; but Propositions 6 and 7 affirm that the conclusion does hold when $k = 1$ in all other cases.

Lemma 9 *Suppose that the hypotheses of Lemmas 3 and 5 are satisfied, and* $\alpha'(H) > \alpha(G) + 1$. *Then* K_n *contains a rainbow* F_5.

Proof: Let M be a maximum matching in H, with edges $x_i y_i$, $i = 1, ..., \alpha'(H)$. Note that by the hypotheses of Lemma 3, $\alpha'(H) \geq \alpha'(G) + 2 \geq 4$. Consider the edge $x_1 x_2$. If $x_1 x_2 \in E(H)$, then the graph F with edges $\{x_1 x_2\} \cup \{x_2 y_2, ..., x_{\alpha'(G)+1} y_{\alpha'(G)+1}\}$ is a rainbow F_5 in K_n. Otherwise, if $x_1 x_2 \notin E(H)$, then $x_1 x_2$ bears the same color as some $e \in E(H)$. Then $H' = (H - e) \cup x_1 x_2$ is rainbow. If $e \notin E(M)$ then H' contains the graph $F \simeq F_5$ described above; the same holds if $e \in \{x_1 y_1\} \cup \{x_i y_i | \alpha'(G) + 2 \leq i \leq \alpha'(H)\}$. If $e = x_i y_i$ for some $i \in \{3, ..., \alpha'(G) + 1\}$, then H' contains $(F - \{x_i, y_i\}) \cup x_{\alpha'(G)+2} y_{\alpha'(G)+2} \simeq F_5$. Finally, if $e = x_2 y_2$ then H' contains $(F - y_2) \cup x_1 y_1 \simeq F_5$. ■

Lemma 10 *Suppose that the hypotheses of Lemmas 3 and 5 are satisfied, and, in addition,* $n \geq |V(G)|$ *and the hypothesized coloring of* $E(K_n)$ *forbids* k*-rainbow copies of* G. *Suppose that, for* $k \in \{1, 2, 3, 4\}$, G *is not among the following graphs:*

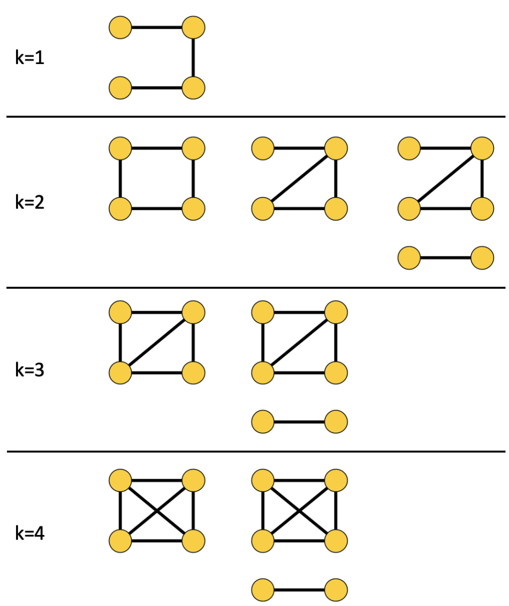

Figure 2.11: List of graphs G cannot be for Lemma 10

Then $\alpha'(H) \leq \alpha'(G) + 1$.

Proof: As in the proof of Lemma 8, because $n \geq |V(G)|$ and the coloring of $E(K_n)$ forbids k-rainbow copies of G, Lemma 4 decrees that for $i \in \{4, 5\}$, if F_i is a subgraph of G then there can be no rainbow F_i in K_n with the hypothetical coloring.

Suppose that $\alpha'(H) > \alpha'(G) + 1$. It follows from Lemma 9 that F_5 is not a subgraph of G. Therefore F_4 must be, by Lemma 3. We shall finish the proof by showing that the assumption that $\alpha'(H) > \alpha'(G)_1$ together with F_5 not being a subgraph of G implies the existence of a rainbow F_4 in K_n, unless G is one of the excluded graphs listed in the lemma statement.

Remarks: We do not claim that the lemma's conclusion fails for these graphs, but this proof does not work for these particular graphs. Additionally, for the final part of the proof we only need the assumption that $\alpha'(H) \geq \alpha'(G) + 1$.

Since G contains no F_5 and G has no isolated vertices, it follows that $|V(G)| = 2\alpha'(G)$ and G has a perfect matching. Supposing that $\alpha'(H) \geq \alpha'(G) + 1$, let M be a maximum matching in H, as in the proof of Lemma 9. Let $E(M) = \{x_1y_1, ..., x_{\alpha'(H)}y_{\alpha'(H)}\}$. Any matching N with edges $E(N) \subseteq E(M)$, and $|E(N)| = \alpha'(G)$ can be a spanning subgraph of a copy of G in K_n - possibly, in fact, of several different copies of G - and, therefore, can contain a copy of an F_4, a subgraph of the copy of G. Let us examine one of these F_4's: for convenience, and without loss of generality, let it be the F_4 with edge set $\{x_1x_2\} \cup \{x_iy_i | 1 \leq i \leq \alpha'(G)\}$. If $x_1x_2 \in E(H)$ then this F_4 is rainbow. Therefore $x_1x_2 \notin E(H)$ and x_1x_2 bears the same color as some $e \in E(H)$. By arguments previously involved in the proof of Lemma 9, the non-existence of a rainbow F_4 in K_n forces $e \in \{x_1y_1, x_2y_2\}$.

Thus, for all $1 \leq i < j \leq \alpha'(H)$, because the edges x_iy_i, x_jy_j can be part of a submatching N of M such that $|E(N)| = \alpha'(G)$, each of the four edges x_ix_j, x_iy_j, y_ix_j, y_iy_j must bear either the color on x_iy_i or the color on x_jy_j.

For every copy of G containing such a submatching N of M, some color must appear on the edges of G at least $k+1$ times. N itself is rainbow and the only colors on the other edges of G are among the colors on N, with the color on an edge with one end in $\{x_i, y_i\}$ and the other in $\{x_j, y_j\}$, $i \neq j$, forced to be either the color on x_iy_i or the color on x_jy_j. Therefore, a color c that appears at least $k+1$ times on G must be the color on some $x_iy_i \in E(N)$ and the edges on which it appears are x_iy_i and at least k other edges of N, each with one end in $\{x_i, y_i\}$. Let us call such an edge $x_iy_i \in E(N)$ a k-splendid edge of the copy of G. Let us call edges with one end in $\{x_i, y_i\}$ and the other in $\{x_j, y_j\}$, $i \neq j$, cross-edges.

No such G can have two different k-splendid edges: suppose x_iy_i, $x_jy_j \in E(N)$, $i \neq j$, bearing colors c_i, c_j, are both k-splendid in G. The sets of edges in G colored c_i, c_j are disjoint, so $|E(G)| \geq (k+1)+(k+1) = 2k+2$; but, by hypothesis, $|E(G)| \leq 2k+1$.

Now suppose that we have, without loss of generality, a copy of G containing the edges x_iy_i, $i = 1, ..., \alpha'(G)$, with x_1y_1 as its unique k-splendid edge.

Call this copy of G, G_1. G_1 has $\alpha'(G)$ matching edges, with x_iy_i colored, say, c_i, $i \in \{1, ..., \alpha'(G)\}$, and at least k cross-edges edge-adjacent to x_1y_1, all colored c_1. Let t be the number of cross-edges in G not colored c_1. Then

$$|E(G_1)| = |E(G)| \geq \alpha'(G) + k + t \implies |E(G)| - k \geq \alpha'(G) + t \geq \alpha'(G).$$

But, also, by hypothesis,

$$|E(G)| - k \leq \alpha'(G).$$

Therefore, $\alpha'(G) + k = |E(G)|$ and $t = 0$.

Therefore, every copy of G consists of a matching of $\alpha'(G)$ edges, with exactly k cross-edges, all edge-adjacent to one of the matching edges.

Back to G_1: for $i \in \{1, ..., \alpha'(G)\}$ let f_i denote the number of cross-edges in G_1 adjacent to x_iy_i. Without loss of generality, $f_2 \geq ... \geq f_{\alpha'(G)}$. We have

$$\sum_{i=2}^{\alpha'(G)} f_i = k.$$

Let $s \in \{2, ..., \alpha'(G)\}$ be the largest index such that $f_s > 0$.

Suppose that $k > 1$ and $s > 2$.

For $2 \leq i \leq s$ consider the copy G_{1i} obtained by interchanging the roles of x_i and y_i in G_1. Thus, if $x_1x_i \in E(G_1)$ then $x_1y_i \in E(G_{1i})$ and similarly for x_1y_i, y_1x_i, and y_1y_i. All other edges of G_1 are as they were. Clearly the only candidate for k-splendid edge in G_{1i} other than x_1y_1 is x_iy_i. But since $s > 2$ there are cross-edges in $E(G_{1i})$ not adjacent to x_iy_i. Therefore, x_1y_1 is k-splendid in each graph G_{1i}, $i = 2, ..., s$. Therefore, each cross-edge in G_{1i} is colored c_1.

Now consider the copy of G obtained by interchanging roles of x_1 and y_1 in G_1. Again, only x_1y_1 can possibly be the k-splendid edge in this new copy of G, G_1'. Again considering the graphs G_{1i}', we find that for each i, $2 \leq i \leq s$, all four of the edges adjacent to both x_1y_1 and x_iy_i must be colored c_1.

Therefore, $s = 2$: for if $s \geq 3$ consider G_2, the copy of G obtained from G_1 by inverting the roles of x_1y_1 and x_2y_2 - this G will not have the same edges of G_1 but it will have cross-edges between x_1y_1 and x_2y_2, which will bear color c_1, not c_2, and it will have cross edges between x_2y_2 and x_3y_3. Thus, G_2 can have no k-splendid edge in the matching $x_1y_1, ..., x_{\alpha'(G)}y_{\alpha'(G)}$, so the coloring of K_n fails to forbid k-rainbow copies of G.

We have concluded that $s = 2$ under the assumption that $k > 1$. But if $k = 1$ then we have, as before,

$$3 = k + 2 \leq |E(G)| \leq 2k + 1 = 3,$$

so

$$2 = |E(G)| - k \leq \alpha'(G) \leq k + 1 = 2,$$

so $s = 2$ and G can only be P_4, the first of the excluded graphs in the Lemma statement.

For $k > 1$, the k cross-edges of G_1 are between x_1y_1 and x_2y_2 so $2 \leq k \leq 4$, and, for each k, $|E(G)| = \alpha'(G) + k$, so $2 \leq |E(G)| - k = \alpha'(G) \leq k + 1$, which gives us eleven more (besides P_4) possible exceptions to the Lemma's conclusion. However, four of these do not qualify as exceptions because they do not satisfy the hypotheses of Lemma 3. The four are as follows:

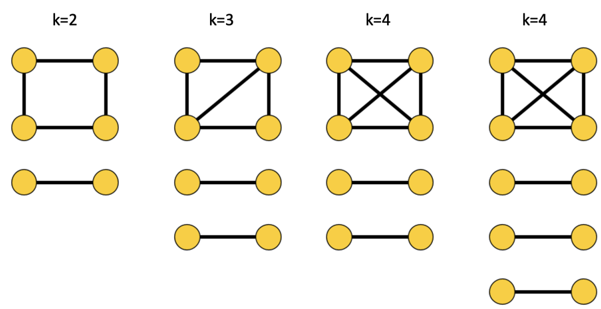

Figure 2.12: List of exempted graphs

in each case, $\Delta(G) < |E(G)| - k$. The seven of the eleven remaining graphs are excluded graphs other than P_4 in the statement of the lemma. ■

We shall soon deal with the eight exceptional graphs excluded from the conclusion of Lemma 10, after the statement and proof of what is our main result, Theorem 2.3, below. For the proof we need two well known results in graph theory, which are stated in Lemma 11. As elsewhere in this section, G is a finite simple graph and k is a positive integer.

Lemma 11 *(a) (Vizing's Theorem)* $\chi'(G) \leq \Delta(G) + 1$

(b) $|E(G)| \leq \chi'(G)\alpha'(G)$*.*

Theorem 2.3 *Suppose that* $k > 1$*,*

(i) $k + 2 \leq |E(G)| \leq 2k + 1$*, and*

(ii) for each $X \in \{\Delta, \alpha'\}$*,* $|E(G)| - k \leq X(G) \leq k + 1$*.*

Suppose that $n \geq \max\{\Delta(G) + 4, |V(G)|\}$ *and suppose that* G *is isolate-free and not one of the eight exceptions listed in Lemma 10. Then* $AR_k(G,n) \leq (\Delta(G) + 2)(\alpha'(G) + 1)$*.*

Proof: Let $E(K_n)$ be colored with exactly $AR_k(G,n)$ colors so that k-rainbow copies of G are forbidden, and let H be a rainbow subgraph of K_n such that $|E(H)| = AR_k(G,n)$. By Lemmas 8 and 10 we have that $\Delta(H) \leq \Delta(G) + 1$ and $\alpha'(H) \leq \alpha'(G) + 1$.

Therefore, by Lemma 11,

$$AR_k(G,n) = |E(H)| \leq \chi'(H)\alpha'(H)$$

$$\leq (\Delta(H)+1)\alpha'(H)$$

$$\leq (\Delta(G)+2)(\alpha'(G)+1).$$

■

From the proofs of the lemmas preceding it is easily seen that the inequality in Theorem 2.3 can be sharpened for graphs satisfying certain additional requirements. For instance, if G, k, and n satisfy the hypothesis of the theorem and, in addition, G contains no F_5 subgraph, then $AR_k(G,n) \leq (\Delta(G)+2)\alpha'(G)$.

Concerning the exceptional graphs listed in Lemma 10, we have the following.

Theorem 2.4 *If G and k are any of:*

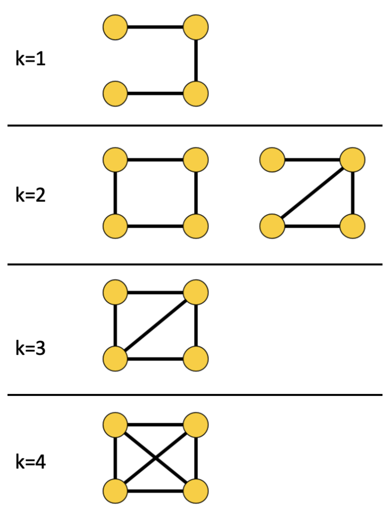

Figure 2.13: Partial list of exempted graphs from Lemma 10

then $AR_k(G,n) = 2$ for all $n \geq 5$.

PROOF: For each graph G listed, $|E(G)| = k + 2$ and G contains a P_4 as a subgraph. The conclusion follows from Proposition 6. ■

Corollary 2.4.5 *With the possible exceptions of*

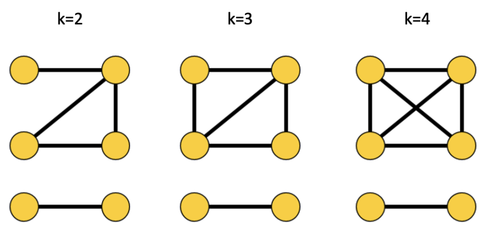

Figure 2.14: List of Possibly Excepted Graphs from Corollary 2.4.5

for every positive integer k and a graph G with no isolated vertices and more than $k+1$ edges, for G to be AR_k-bounded it is necessary and sufficient that $|E(G)| \leq 2k+1$ and $|E(G)| - k \leq X(G) \leq k+1$, for each $X \in \{\Delta, \alpha'\}$.

Let us call these three exceptions Z_i for $i \in \{2,3,4\}$ such that $i = k$ for the positive integer k values listed with each graph.

Theorem 2.5 *For $n \geq 6$, $AR_2(Z_2, n) = 3$.*

PROOF: To show that $AR_2(Z_2, n) \geq 3$, we will construct a coloring of K_n, $n \geq 6$, using 3 colors where no 2-rainbow copy of Z_2 exists. Make a K_3 in K_n rainbow with colors red, blue, and green. Let all other edges in K_n be green. Since no subgraph of K_n with this coloring has more than 2 non-green edges, each copy of Z_2 in K_n must have 3 green edges, and therefore is not 2-rainbow.

Now it suffices to show that no edge coloring of K_n with 4 colors appearing can forbid 2-rainbow copies of Z_2. Suppose the contrary, and assume that $E(K_n)$ is colored with colors red, blue, green, and yellow so that no copy of Z_2 is 2-rainbow.

Since $|E(Z_2)| = 5$, and $k = 2$, if any 3-edge subgraph of Z_2 is rainbow in this supposed 4-coloring, then in any copy of Z_2 containing those three edges, the other two edges must be colored with one of the 3 colors on the rainbow subgraph; this is the only way that the copy of Z_2 with a rainbow 3-edge subgraph can avoid being 2-rainbow.

We shall prove the theorem by showing that no 3-edge subgraph of Z_2 can be rainbow. However, every graph with 3 edges and no isolates is a subgraph of Z_2, and with 4 colors appearing, there is no difficulty in finding 3 edges in K_n of different colors.

Suppose there is a rainbow K_3 in K_n, say $uvwu$ with colors red, blue, and green on the edges, as depicted in Figure 2.15.

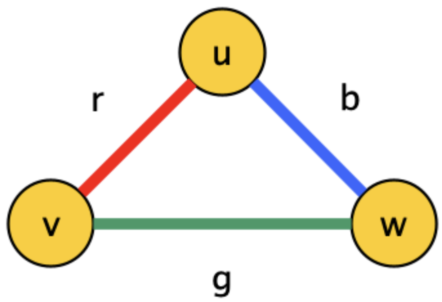

Figure 2.15: Rainbow K_3

Then, to avoid a 2-rainbow copy of Z_2, for any distinct $p, q, t \in V(K_n) \backslash \{u, v, w\}$ and $a \in \{u, v, w\}$, the edges pq and ta bear the same color, one of red, blue, or green. Note such p, q, t must exist since $n \geq 6$. Letting p, q, t, and a vary, we find that every edge of K_n is colored with one of the colors red, blue, or green, contradicting the assumption that $E(K_n)$ is colored with 4 colors appearing.

By similar arguments, we can conclude there is no rainbow $K_{1,3}$ in K_n, with the supposed edge coloring. Suppose there is a rainbow $K_{1,3} = [\{v\}, \{u, w, z\}]$ in K_n colored using red, blue, and green as depicted in Figure 2.16.

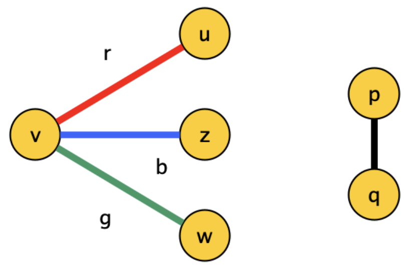

Figure 2.16: Rainbow $K_{1,3}$

Let $p, q \in V(K_n)\backslash\{v, u, w, z\}, p \neq q$. Then edges pq and uz must bear the same color, one of red, blue, green. Without loss of generality, we can suppose that both edges are colored red. Replacing uz by wz we conclude that wz must be red. Now, there exists a rainbow K_3, $vzwv$, which has already been excluded as a possibility.

Now, suppose there is a rainbow P_4 in K_n, as depicted in Figure 2.17.

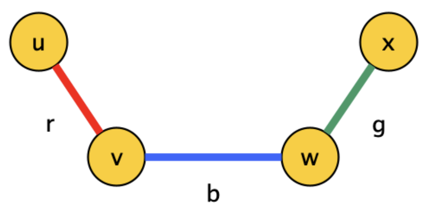

Figure 2.17: Rainbow P_4

Since there can be no rainbow K_3, the color on vx must be either blue or green and the color on uw must be either red or blue. Considering the triangles $uxvu$ and $uxwu$, and the $K_{1,3}$'s with central vertices v, w, u, x, we see that we are forced to color all 3 edges vx, uw, and ux with the color blue, as seen in Figure 2.18.

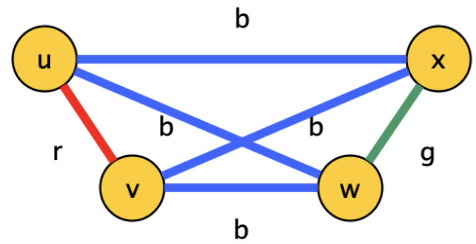

Figure 2.18: Coloring of K_4

Now we see that if there is any edge in K_n independent of the 6 edges above which bear a color other than blue, then there will be a 2-rainbow Z_2 in K_n. Therefore, every such edge is blue.

So the only edges that could possibly be colored yellow are edges with one end in $\{u, v, x, w\}$ and the other not. But if any such edge is colored yellow then there is a rainbow $K_{1,3}$ in K_n.

Now suppose there is a rainbow $P_3 + K_2$, as shown in Figure 2.19.

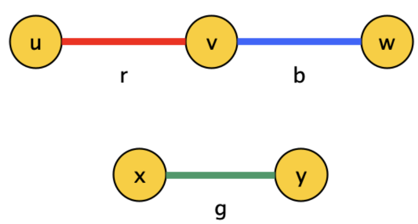

Figure 2.19: Rainbow $P_3 + K_2$

Notice that to avoid a rainbow P_4, edges xv and yv must receive color green. However, now edges ux and wy cannot be colored in any way to avoid a rainbow P_4. Thus, any coloring of K_n with a rainbow $P_3 + K_2$ will also have a rainbow P_4, a previously excluded possibility.

Last, suppose there is a rainbow $3K_2$, as shown in Figure 2.20.

Consider the edge ux. It must be colored red, blue, or green to avoid the existence of a 2-rainbow Z_2. However, if colored green or red, it permits a rainbow $P_3 + K_2$, a previously excluded possibility. If it is colored blue, it permits a rainbow P_4, also previously excluded.

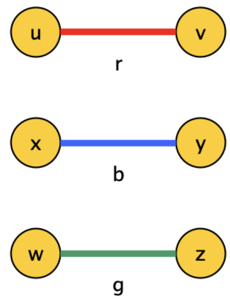

Figure 2.20: Rainbow $3K_2$

Thus, $E(K_n)$ may not be colored with 4 colors appearing so that 2-rainbow copies of Z_2 are avoided. ∎

Lemma 12 *Suppose that F is a subgraph of G and $|E(G)| - |E(F)| = |E(G) \backslash E(F)| = r$, $1 \le r < k$. Suppose that $n \ge |V(G)|$, $a \in \mathbb{Z}^+$, $a \le AR_k(G,n)$. Then $AR_{k-r}(F,n) \ge a$.*

PROOF: Let $\phi : V(K_n) \to \{1, \ldots, a\}$ be a coloring with a colors appearing such that no copy of G is k-rainbow with respect to ϕ. We shall see that with respect to ϕ, no copy of F is $(k-r)$-rainbow.

Let F' be a copy of F in K_n. Since F is a subgraph of G and $n \ge |V(G)|$, we can "complete" F' to a copy of G by adding some r edges of K_n. Because this copy of G is not k-rainbow, some color appears on $k+1$ edges of G. Then at least $k+1-r = (k-r)+1$ of these edges are in F. Therefore, F' is not $(k-r)$-rainbow. Since the copy of F was arbitrary, it follows that $a \le AR_{k-r}(F,n)$. ∎

Corollary 2.5.6 *For $n \ge 6$,*

(a) $AR_3(Z_3, n) = 3$

(b) $AR_4(Z_4, n) = 3$

PROOF: Let $k \in \{3, 4\}$. To show that $AR_k(Z_k, n) \ge 3$, we will construct a coloring of K_n, $n \ge 6$, using 3 colors where no k-rainbow copy of Z_k exists. Make a K_3 in K_n rainbow with colors red, blue, and green. Let all other edges in K_n be green. Since no subgraph of

K_n with this coloring has more than k non-green edges, each copy of Z_k in K_n must have $|E(Z_k)| - 2 = k + 1$ green edges, and therefore is not k-rainbow.

Now, let us show that $4 > AR_k(Z_k, n)$, $k \in \{3, 4\}$.

(a) Suppose $4 \leq AR_3(Z_3)$. Because $1 = |E(Z_3)| - |E(Z_2)|$, and Z_2 is a subgraph of Z_3, then $AR_{3-1}(Z_2) = AR_2(Z_2) \geq 4$, contradicting Theorem 2.5. Thus, $AR_3(Z_3, n) = 3$.

(b) Similarly, suppose $4 \leq AR_4(Z_4)$. Because $2 = |E(Z_4)| - |E(Z_2)|$, and Z_2 is a subgraph of Z_4, then $AR_{4-2}(Z_2) = AR_2(Z_2) \geq 4$, contradicting Theorem 2.5. Thus, $AR_4(Z_4, n) = 3$. ■

2.4 Example: AR_1-bounded Graphs

Dean Hoffman solved the question of which graphs are AR-bounded with a proof provided in Appendix A. His theorem is as follows:

Theorem 2.6 *If a graph is one of the following four, it is AR_1-bounded. Otherwise, it is AR_1-unbounded.*

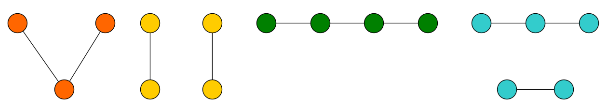

Figure 2.21: Four AR_1-bounded graphs

Using the results proven in this Chapter, a graph G is AR_1-bounded if and only if $2 \leq |E(G)| \leq 3$ and $|E(G)| - 1 \leq X(G) \leq 2$ for $X \in \{\Delta, \alpha'\}$.

For $|E(G)| = 2$, $1 \leq X(G) \leq 2$ permits both the two edged graphs: P_3 and $2K_2$.

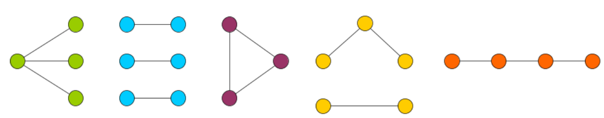

Figure 2.22: Three Edged Graphs

For $|E(G)| = 3$, $2 \leq X(G) \leq 2$ permits two of the three edged graphs: P_4 and $P_3 + K_2$.

These are exactly the graphs Hoffman found to be AR_1-bounded. His proof is detailed in Appendix A.

Chapter 3

AR_k on Various Graph Families

A natural extension of the question of which graphs are AR_k-bounded on complete graphs of sufficient size is the question of which graphs are AR_k-bounded on other families of graphs. In this section we will discuss some preliminary results on various families of graphs. Reminder G must have at least $k+1$ edges, otherwise we cannot avoid a k-rainbow copy of G. Additionally, recall that $AR_k(G,H)$ is the maximum number of colors that can be used in an edge coloring of H such that there is no k-rainbow copy of G in some edge coloring of H using $AR_k(G,H)$ colors.

3.1 General

Proposition 8 *For G with at least $k+1$ edges and no isolates,* $AR_k(G,G) = |E(G)| - k$

Proof: Color G so that $|E(G)| - (k+1)$ colors appear on one edge each and so that the remaining $k+1$ edges are colored with some new color, c. Clearly this is not a k-rainbow coloring of G.

If $E(G)$ were colored with $|E(G)| - r$ different colors appearing, with $r < k$, then the greatest number of appearance possible of any single color would be $r+1 < k+1$, so the coloring would be k-rainbow. ■

3.2 Complete Bipartite Graphs

Lemma 13 $AR_k(K_{1,k+1}, K_{n,m}) = 1$ *for* $\min\{n,m\} \geq k+1$.

Proof: In an edge coloring of $K_{n,m}$ in which no $K_{1,k+1}$ is k-rainbow, every $K_{1,k+1}$ is monochromatic. From this and the assumption that $n, m \geq k+1$ it follows that any two adjacent edges must be the same color, and from there it is plain that only one color can appear. ∎

Lemma 14 $AR_k(K_{1,k+1}, K_{n,m}) = n$ *for* $1 \leq n \leq k < m$.

Proof: In an edge coloring in which no $K_{1,k+1}$ is k-rainbow, every $K_{1,k+1}$ is monochromatic. Since $1 \leq n \leq k \leq m$ every copy of $K_{1,k+1}$ in $K_{n,m}$ must have its central vertex on the part of $K_{n,m}$ with n independent vertices. From these two facts we see that all edges adjacent to a vertex in the part of $K_{n,m}$ with n independent vertices must be monochromatic. With the n vertices, we get there can be n colors. See the coloring below for an example of how the edges may be colored.

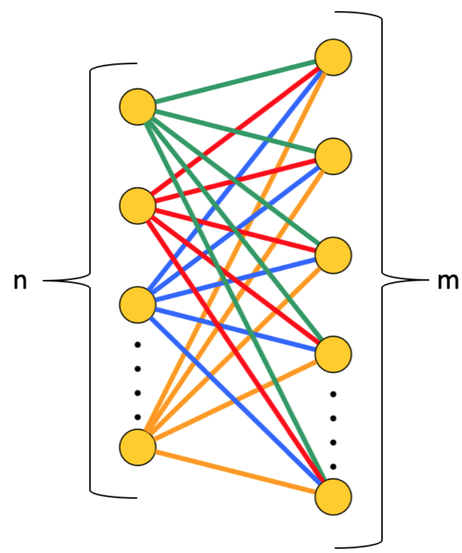

Figure 3.1: $K_{1,k+1}$ k-rainbow avoiding coloring of $K_{n,m}$ for $1 \leq n \leq k \leq m$

So, $AR_k(K_{1,k+1}, K_{n,m}) \geq n$. It remains to show that $AR_k(K_{1,k+1}, K_{n,m}) \leq n$. Let us assume for contradiction that there exists some coloring of $K_{n,m}$ with $n+1$ colors. Then by the pigeonhole principle at least two colors are incident to a vertex on the part of $K_{n,m}$ with n independent vertices, which contradicts what we have already stated.

∎

Proposition 9 *$AR_k(K_{1,r}, K_{n,m}) \geq \min\{n,m\} + 1$ for $\min\{n,m\} \geq r \geq k+2$.*

PROOF: Color $K_{n,m}$ using one color, c. Find a maximum matching in K_n, call it M. Notice the size of the maximum matching in G is $\alpha'(G) = \min\{n,m\}$, so $|M| = \min\{n,m\}$. Recolor the edges in this matching using new colors, $c_1, c_2, ..., c_{\min\{n,m\}}$. See the image below for an example.

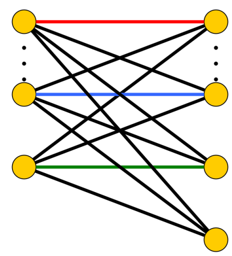

Figure 3.2: $K_{1,r}$ k-rainbow avoiding coloring of $K_{n,m}$ for $\min\{n,m\} \geq r \geq k+2$

Notice each vertex in $K_{n,m}$ has at most one adjacent edge not colored c. Thus, in each copy of $K_{1,k+2}$ there is at most one edge not colored c and at least $k+1$ edges colored c which ensures there is no k−rainbow copy of $K_{1,k+2}$ in $K_{n,m}$. There are $\min\{n,m\}+1$ colors used in this coloring of $K_{n,m}$ which is k−rainbow avoiding. ■

Lemma 15 *$AR_k((k+1)K_2, K_{n,m}) = 1$ for $\min\{n,m\} \geq k+1$ unless $k=1$ and $n=m=2$, in which case $AR_1(2K_2, K_{2,2}) = 2$.*

PROOF: Clearly, if $k=1$, $n=m=2$, we have $AR_1(2K_2, K_{2,2}) = AR_1(2K_2, C_4) = 2$. See Figure 4.3.

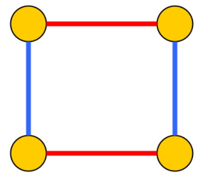

Figure 3.3: $2K_2$ k-rainbow avoiding coloring of C_4

Otherwise: Assume that $m \geq n \geq k+1$, $k \geq 1$, and we are not in the case above. Assume $k+1 \leq n \leq m$. In any edge coloring of $K_{n,m}$ with no k-rainbow $(k+1)K_2$, every $(k+1)K_2$ must be monochromatic. Suppose $K_{n,m}$ is so colored. Let the vertices on one side of $K_{n,m}$ be $v_1, ..., v_n$ and on the other side, $w_1, ..., w_m$. Consider the matching $M = \{v_1w_1, ..., v_{k+1}w_{k+1}\}$. Let c be the color on the edges of this matching. If $n > k+1$, and $n \geq i > k+1$, then $(M \backslash \{v_1w_1\}) \cup \{v_iw_i\}$ is also a matching with $k+1$ edges, all of whose edges except v_iw_i are colored c. Therefore v_iw_i must be colored c.

Given $1 \leq i < j \leq k+1$, we can replace v_iw_i, v_jw_j in M by v_iw_j, v_jw_i and conclude both of these are colored c, if $k > 1$. Otherwise, if $k = 1$ but, say $m > 2$ we can use the fact that v_iw_j, $i \leq k+1$, $j > i$ must be colored c, by the arguments above, we can still conclude that v_iw_j, v_jw_i are colored c. And similarly for v_iw_j, $i, j > k+1$. ■

Lemma 16 $AR_k(rK_2, K_{n,m}) \geq \max\{n, m\} + 1$ *for* $\min\{n, m\} \geq r \geq k+2$.

PROOF: Color $K_{n,m}$ using one color, c. Now color the edges adjacent to some vertex, v, with $d = d(v) = \max\{m, n\}$ colors such that each edge is one of the colors $c_1, c_2, ..., c_d$. See Figure 4.4 for an example.

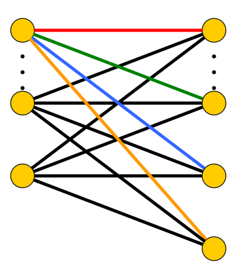

Figure 3.4: rK_2 k-rainbow avoiding coloring of $K_{n,m}$ for $\min\{n, m\} \geq r \geq k+2$

Notice at most one edge adjacent to v can be in a copy of rK_2. Thus, only one edge not colored c can be in any rK_2 and the remaining $r \geq k+1$ edges must be colored with c which ensures there is no $k-$rainbow copy of rK_2 in $K_{n,m}$. There are $\max\{n,m\}+1$ colors used in this coloring of $K_{n,m}$ which is $k-$rainbow avoiding. $\max\{n,m\}+1$ is an unbounded sequence so rK_2, $r > k+2$, is AR_k-unbounded. ■

3.3 Cycles

Since G must be a subgraph of H, notice that G can only be some collection of paths or the cycle itself, that is $G = H$.

Lemma 17 *$AR_k((k+1)K_2, C_{2(k+1)}) = 2$.*

PROOF: Find a maximum matching M in $C_{2(k+1)}$. Then $|M| = k+1$. Then if M is not k-rainbow, M must be colored a single color c_1. Notice $E - M = N$ is a second matching of size $k+1$, and so must be colored either with c_1 or a second color c_2, if the edge coloring of $C_{2(k+1)}$ is to avoid k-rainbow $(k+1)K_2$'s.

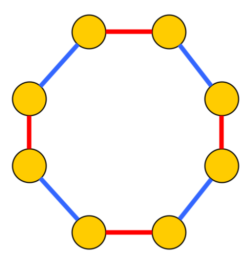

Figure 3.5: $(k+1)K_2$ k-rainbow avoiding coloring of $C_{2(k+1)}$

■

Lemma 18 *$AR_k((k+2)K_2, C_{2(k+2)}) \geq 4$*

PROOF: We will color the edges of $C_{2(k+2)}$ with four colors so that no subgraph $(k+2)K_2$ is k-rainbow. Clearly $E(C_{2(k+2)})$ can be partitioned into two matchings $(k+2)K_2$, call them M and N. Notice $|M| = k+2$. Then $k+1$ edges in M must be colored a single color c_1 and one edge can be colored c_2. Notice $E - M = N$, a second matching of size $k+2$. Color $k+1$ edges in N with color c_3 and one edge with color c_4.

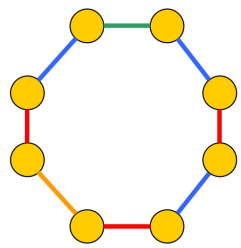

Figure 3.6: $(k+2)K_2$ k-rainbow avoiding coloring of $C_{2(k+2)}$

Notice that we cannot "mix and match" the edges between N and M to create other matchings of size $k+2$ and thus each may keep their own colors without allowing a k-rainbow copy of $(k+2)K_2$. ■

Lemma 19 $AR_k((k+2)K_2, C_{2(k+2)}) \leq 4$

Proof: Suppose that the edges of $C_{2(k+2)}$ are colored with v colors so that no matching $(k+2)K_2$ is k-rainbow. Clearly, for every such matching M in $C_{2(k+2)}$, $E(C_{2(k+2)})\backslash M = M^-$ is another such matching. Take any such M: $k+1$ of its edges must bear the same color, so at most 2 colors can appear on M, and the same goes for M^-. Thus, $v = 4$. ■

Corollary 3.0.7 $AR_k((k+2)K_2, C_{2(k+2)}) = 4$

3.4 Paths

Since G must be a subgraph of H, notice that G can only be some collection of paths.

Lemma 20 $AR_k((k+1)K_2, P_{2k+3}) = 2$

Proof: Notice $|E(P_{2k+3})| = 2(k+1)$. To see that $AR_k((k+1)K_2, P_{2k+3}) \geq 2$, find a maximal matching M in P_{2k+3}. $|M| = k+1$. Color every edge in it c to avoid a k-rainbow copy of $(k+1)K_2$. Notice $P_{2k+3} - M = N$, another maximal matching of size $k+1$. It must be colored all the same to avoid a k-rainbow copy of $(k+1)K_2$. However, it can be colored a new color,

Figure 3.7: $(k+1)K_2$ k-rainbow avoiding coloring of P_{2k+3}

c_1, since no edge can be in M and N.

To see that $AR_k((k+1)K_2, P_{2k+3}) \leq 2$ suppose that the edges of P_{2k+3} are colored with v colors so that no matching $(k+1)K_2$ is k-rainbow. Clearly, for every such matching M in P_{2k+3}, $E(P_{2k+3})\backslash M = M^-$ is another such matching. Take any such M: $k+1$ of its edges must bear the same color, so at most 1 colors can appear on M, and the same goes for M^-. Thus, $v = 2$. ∎

Lemma 21 $AR_k((k+1)K_2, P_{2k+2}) = k+1$.

Proof: Notice $|E(P_{2k+2}| = 2k+1$. There is only one copy of $(k+1)K_2$ in P_{2k+2}. Color it with color c to avoid a k-rainbow copy of $(k+1)K_2$. Notice no other edge in P_{2k+2} can be in a copy of $(k+1)K_2$ other than those colored c. Color these $2k+1-(k+1) = k$ edges with k colors. Thus, $k+1$ colors can be used but no more. ∎

Figure 3.8: $(k+1)K_2$ k-rainbow avoiding coloring of P_{2k+2}

Lemma 22 $AR_k((k+2)K_2, P_{2k+5}) = 4$.

Proof: Notice there are only two copies of $(k+2)K_2$ in P_{2k+5} and they are edge disjoint. Call one copy G and the other G'. Thus, each copy can be colored independently of the other. In each copy, $k+1$ edges must be colored the same leaving one edge to be colored using a new color. Thus, a coloring of G can use 2 colors and a coloring of G' can use 2 colors giving a total of 4 colors. ∎

Figure 3.9: $(k+2)K_2$ k-rainbow avoiding coloring of P_{2k+5}

Lemma 23 *$AR_k(rK_2, P_{2r+1}) = 2(r-k)$ for $r \geq k+1$.*

PROOF: Notice there are only two copies of rK_2 in P_{2r+1} and they are edge disjoint. Call one copy G and the other G'. Thus, each copy can be colored independently of the other. In each copy, $k+1$ edges must be colored the same leaving $r-(k+1)$ edge to be colored using new colors. Thus, a coloring of G can use $r-(k+1)+1 = r-k$ colors and a coloring of G' can use $r-k$ colors giving a total of $2(r-k)$ colors. ■

Figure 3.10: rK_2 k-rainbow avoiding coloring of P_{2r+1} for $r \geq k+1$

Lemma 24 *$AR_k(P_{k+2}, P_r) = 1$ for all $r \geq k+2$.*

PROOF: Notice $|E(P_{k+2})| = k+1$. All of these edges must be colored the same to avoid a k-rainbow copy of P_{k+2}. Suppose for contradiction 2 colors can be used on the edges of P_r that avoid a k-rainbow copy of P_{k+2}. Call these colors c_1 and c_2, respectively.

Then at some point an edge colored c_1 and an edge colored c_2 are adjacent. Then there exists at least one copy of P_{k+2} with edges colored both c_1 and c_2, which means at most k edges are colored the same and this copy of the P_{k+2} is k-rainbow.

Thus, only one color may be used to color P_r. ■

Lemma 25 $AR_k(P_{k+3}, P_{k+4}) = 3$

PROOF: Color a copy of P_{k+4} as shown in Figure 4.11. That is, so that the internal P_{k+2} are colored c and the end edges are colored c_1 and c_2.

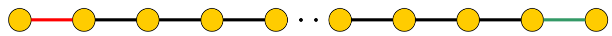

Figure 3.11: P_{k+3} k-rainbow avoiding coloring of P_{k+4}

Thus, $AR_k(P_{k+3}, P_{k+4}) \geq 3$. Now let us assume we can use 4 colors. Notice there are only two copies of P_{k+3} in P_{k+4}. One uses the left-most edge and the $k+1$ edges adjacent to the right, and the other uses the right-most edge and the $k+1$ edges adjacent to the right. Should four colors be used on the edges of P_{k+4}, then at most $|E(P_{k+4})| - 4 + 1 = k + 3 - 4 + 1 = k$ edges can be colored the same and thus no subgraph can avoid a k-rainbow. Thus, $AR_k(P_{k+3}, P_{k+4}) = 3$.

■

Lemma 26 $AR_k(P_{k+3}, P_{k+5}) = 3$

Proof: Assume for contradiction that there is some coloring of P_{k+5} using 4 colors that avoids a k-rainbow copy of P_{k+3}. Notice P_{k+3} can only have one edge not colored that same as all the others.

Notice, $\frac{k+4}{4} < \lfloor \frac{k}{4} \rfloor + 1$. So, by the pigeonhole principle, there must be some path of length $k+3$ with 3 colors present, meaning only k edges can be colored the same and a k-rainbow copy of P_{k+3} exists, a contradiction.

Now, let us show there is some coloring of a P_{k+5} using 3 colors where no k-rainbow copy of P_{k+3} exists. Let us use the coloring of P_{k+4} shown, such that every edge is colored c except the two at the ends, colored c_1 and c_2.

Figure 3.12: P_{k+3} k-rainbow avoiding coloring of P_{k+5}

Thus, $AR_k(P_{k+3}, P_{k+5}) = 3$. ■

3.5 Complete r-Partite Graphs

Lemma 27 $AR_1(K_3, K_{r,s,t}) \geq \max\{rs, rt, st\} + 1$.

Proof: Without loss of generality, let $\max\{rs, rt, st\} = rs$. Let us call the sets of independent vertices $R, S,$ and T respectively, such that $|R| = r$, $|S| = s$, and $|T| = t$.

Color every edge uv with color c if either

1. $u \in R$ and $v \in T$ or

2. $u \in T$ and $v \in S$.

Now let us color all edges uv such that $u \in R$ and $v \in S$; there are rs of these edges. Color these edges with rs colors. Then, since a K_3 can only be formed by using an edge from each group and two groups have all edges colored c, a rainbow copy of K_3 is avoided and $rs + 1$ colors were used.

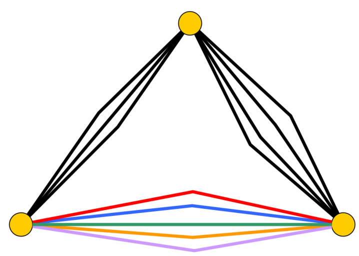

Figure 3.13: K_3 k-rainbow avoiding coloring of $K_{r,s,t}$

Thus for all cases, $AR_k(K_3, K_{r,s,t}) \geq \max\{rs, rt, st\} + 1$. ■

Lemma 28 $AR(K_3, K_{r,s,t}) \geq \max\{rs, rt, st\} + \min\{r, s, t\}$.

Proof: Use the following coloring. Without loss of generality, let $r \geq s \geq t$. Thus, $rs = \max\{rs, rt, st\}$. Color every edge with one vertex in the independent set of size r and the other vertex of size s with rs colors. Label the vertices in the independent set of size t as $v_1, v_2, ..., v_t$. color every edge adjacent to v_i using some new color c_i, $1 \leq i \leq t$. See coloring in Figure 4.14.

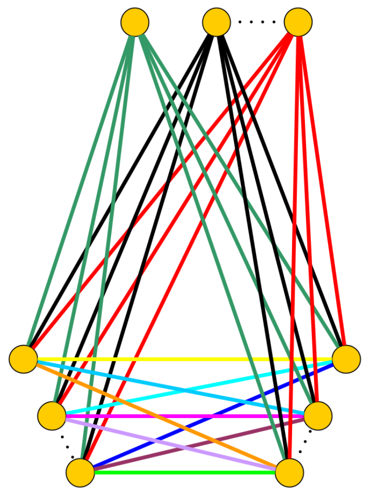

Figure 3.14: K_3 k-rainbow avoiding coloring of $K_{r,s,t}$

Then, no rainbow copy of K_3 exists since every copy of K_3 must include a vertex in the independent set of size t and each vertex in the independent set of size t has only one color present, meaning two edges in every K_3 must be colored the same and a rainbow K_3 is avoided. Thus $AR_k(K_3, K_{r,s,t}) \geq \max\{rs, rt, st\} + \min\{r, s, t\}$. ■

Lemma 29 $AR(K_3, K_{r,s,t}) \leq \max\{rs, rt, st\} + \min\{r, s, t\}$.

PROOF: Assume for contradiction that there is some coloring of $K_{r,s,t}$ using $\max\{rs, rt, st\} + \min\{r, s, t\} + 1$ colors that permits no rainbow copy of K_3.

Without loss of generality, let $r \geq s \geq t$. Thus $rs = \max\{rs, rt, st\}$ and $t = \min\{r, s, t\}$. So, we need to show that when we use $rs + t + 1$ colors, that there is a copy of K_3 with 3 different colored edges.

Notice, $E(K_{r,s,t}) = rs + rt + st$. Additionally the number of triangles in $K_{r,s,t} = rst$.

Since we are in $K_{r,s,t}$ for a triangle to form each vertex must be in a different independent set, which we will call R, S, T, respectively.

Notice to avoid a rainbow copy of K_3, some vertex in the triangle must have only one color present. Thus, rst triangles must be colored such that at some vertex there is only one color present using $rs + t + 1$ colors. ■

Chapter 4

Future Work

This section details some remaining questions and potential extensions of the work done in this dissertation.

The first question arises from the definition of AR_k-bounded graphs.

Question 1: Given a fixed AR_k-bounded graph G and a fixed k, is $AR_k(G,n)$ eventually monotone in n for all $n \geq N$? That is, eventually, is $AR_k(G,n+1) \geq AR_k(G,n)$?

Additionally, we are interested in the connections between n and k when the k-anti-rainbow bound is known. Question 2 gives an example of a question on this relationship.

Question 2: For a given graph G, what is the smallest value of k for a fixed n such that $AR_k(G,n) = 1$?

Another future direction for this work could be the extension into multi-graphs with K and G permitting multiple edges and loops. This would introduce several new restrictions but also many more possibilities.

In Chapter 2 we provided an upper bound for AR_k-bounded graphs in edge colorings of complete graphs and several conditions that can decrease that bound. There may exist other conditions that decrease this bound. As seen in the result by the tree exempted graphs from Theorem 2.3, many of the actual AR_k numbers are much lower than the bound we found. Much more work can be done to decrease this bound and determine $AR_k(G,n)$ for graphs this work has determined are AR_k-bounded.

Additionally, we established lower bounds for AR_k-unbounded graphs in Theorem 2.1. For certain graphs these bounds are tight although for most graphs more colors may be used.

More work can be done to increase these limit for graphs of undetermined conditions.

Finally, there are many unknown bounds of $AR_k(G, H)$ when H is not a complete graph. This work has found some values of $AR_k(G, H)$ for certain graphs. However, since the definition and idea of AR_k-bounded does not apply in a natural way to families of graphs besides the complete graph, it would be useful to find some extension similar to AR_k-bounded that arises from the bounds seen on various families, such as trees, paths, and r-partite graphs.

Bibliography

[1] Alon, N.: On a conjecture of Erdős, Simonovits, and Sós concerning anti-Ramsey theorems. J. Graph Theory 7(1), 91–94 (1983)

[2] Babai, László, and Joel Spencer. "Paul Erdős (1913–1996)." Notices of the AMS, American Mathematical Society, Jan. 1998, www.ams.org/notices/199801/comm-erdos.pdf.

[3] Balachandran, N., & Khare, N. (2009). Graphs with restricted valency and matching number. Discrete Mathematics, 309(12), 4176-4180.

[4] Chen, H., Li, X., Tu, J.: Complete solution for the rainbow number of matchings. ar-Xiv:math.CO/0611490

[5] Erdős, P., Simonovits, M., Sós, V.T.: Anti-Ramsey theorems. In: Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), vol. II, pp. 633–643. Colloq. Math. Soc. János Bolyai, vol. 10. North-Holland, Amsterdam (1975)

[6] Fujita, S., Magnant, C., & Ozeki, K. (2010). Rainbow generalizations of Ramsey theory: a survey. Graphs and Combinatorics, 26, 1-30.

[7] Gouge, A., Hoffman, D., Johnson, P., Nunley, L., & Paben, L. (2010). Edge-colorings of Kn which forbid rainbow cycles. Utilitas Mathematica, 83.

[8] Gyárfás, A. and Simonyi, G. (2004), Edge colorings of complete graphs without tricolored triangles. J. Graph Theory, 46: 211-216. https://doi.org/10.1002/jgt.20001

[9] Jiang,T. (2002), Edge-colorings with no large polychromatic stars, Graphs Combin.18(2), 303–308.

[10] Jiang, T., & West, D. B. (2002). Edge-colorings of complete graphs that avoid polychromatic trees. Electronic Notes in Discrete Mathematics, 11, 376-385.

[11] Katz, M., & Reimann, J. (2018). An introduction to Ramsey theory (Vol. 87). American Mathematical Soc..

[12] Manoussakis, Y., Spyratos, M., Tuza, Z., & Voigt, M. (1996). Minimal colorings for properly colored subgraphs. Graphs and Combinatorics, 12, 345-360.

[13] Montellano-Ballesteros, J. J. (2006). On totally multicolored stars. Journal of Graph Theory, 51(3), 225-243.

[14] Montellano-Ballesteros, J. J., & Neumann-Lara, V. (2005). An Anti-Ramsey Theorem on Cycles. Graphs & Combinatorics, 21(3).

[15] Ramsey, F.P. (1930), On a Problem of Formal Logic. Proceedings of the London Mathematical Society, s2-30: 264-286. https://doi.org/10.1112/plms/s2-30.1.264

[16] Simonovits, M., Sós, V.T.: On restricted colourings of Kn. Combinatorica 4(1), 101–110 (1984)

Appendices

Appendix A

Dean Hoffman's Proof for $k = 1$

This is Dean Hoffman's unpublished proof on the graphs that AR_1-bounded on complete graphs.

Theorem A.1 *If a graph is one of the following four, it is AR_1-bounded. Otherwise, it is AR_1-unbounded.*

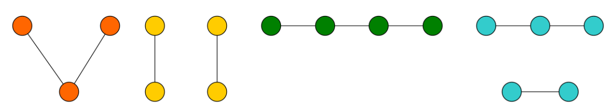

Figure A.1: Four AR-bounded graphs

1. Let G be a sequence of graphs. $G = (G_n | n \in \mathbb{P})$.

2. Def: $G = (G_n | n \in \mathbb{P})$ is **good** if each G_n is a graph on n vertices, and ϵ_n edges where $\{\epsilon_n | n \in \mathbb{P}\}$ is unbounded.

3. Def: H is a **G-graph** if for all $n \in \mathbb{P}$, every graph isomorphic to subgraph of both H and G_n has at most $\epsilon(H) - 2$.

4. Need to find a sequence of graphs so H is G-graph to prove H is unbounded.

Theorem A.2 *If H is a G-graph, then H is not AR-bounded.*

Proof: We will show the lower bound gets arbitrarily large for every $n \in \mathbb{P}$.

We will show, for every $n \in \mathbb{P}$, $n \geq \epsilon$, that $AR(H, n) \geq \epsilon_n + 1$ (which is an unbounded sequence and implies AR is unbounded).

We need to find an edge coloring of K_n with $\epsilon_n + 1$ colors, having no rainbow copy of H.

Color a copy of G_n in K_n so it is rainbow.

Notice $\epsilon_n < \binom{n}{2}$.

Color all other edges of K_n with a new color, c.

Claim: There is no rainbow copy of H in this edge-colored K_n.

Proof: Let H' be a copy of H in K_n. Let K be the subgraph with edges $H' \cap G_n$.

So, $\epsilon(K) \leq \epsilon - 2$ but H' has ϵ edges. So at least two edges of H' are c. ■ ■

Theorem A.3 *The following sequences are **good**.*

1. *Let G_n be a graph consisting of a matching with $\epsilon_n = \lfloor \frac{n}{2} \rfloor$.*

2. *Let G_n be the star $K_{1,n-1}$.*

Proof: **Case 1:** G_n is the matching graph.

Notice: ϵ_n edges and ϵ_n is unbounded.

If $\epsilon \geq n$, H is G-good iff $\alpha'(H) \leq \epsilon(H) - 2$ (where $\alpha'(H)$ is the max matching number of H).

Can H contain a cycle C?

Assume so. Then at least two edges of H are not in any one matching. Thus, if H contains a cycle, it is unbounded.

Likewise, if H has a vertex of degree 3 or more, then H contains at least 2 edges not in any matching. Thus, if H contains a vertex of degree 3 or more, it is AR-unbounded.

Thus, H must be a forest with no vertices of degree 3 or more.

Thus, every component of H must be a path if H is AR-bounded.

Case 2: G_n is the star $K_{1,n-1}$.

Notice: $\epsilon_n = n - 1$ and ϵ_n is unbounded.

H is a G-graph iff $\epsilon(H) \geq \Delta(H) + 2$ (where $\Delta(H)$ is the max degree of H).

If H is AR bounded, then $\epsilon(H) \leq 3$ (since max degree of a path is 2).

Notice the graphs remaining are precisely the graphs listed as the four possible (assuming $\epsilon \geq 2$). ■

Theorem A.4 *P_3 is AR-bounded.*

Table A.1: AR_1 values for P_3

n	AR(G,n)
1	0
≥ 2	1

PROOF: $n = 2$ has only one edge. K_n is connected so there exists vertex v such that 2 edges at v are 2 different colors, a contradiction. ■

Theorem A.5 *$2K_2$ is AR-bounded.*

Table A.2: AR_1 values for $2K_2$

n	AR(G,n)
1	0
2	1
3	3
4	3
≥ 5	1

PROOF: If $|V(G)| > n$, then $AR(G, n) = \binom{n}{2}$ so $n = 1, 2, 3$ proved.

If $n = 4$, see the coloring below.

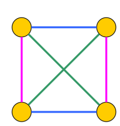

Figure A.2: Coloring of K_4 that avoids rainbow copies of $2K_2$

If $n \geq 5$, color K_n with two colors. Then there is a rainbow P_2 by Theorem A.4.

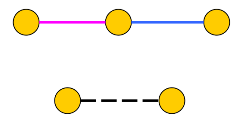

Figure A.3: Coloring of $P_3 \cup P_2$ that cannot avoid a rainbow copy of $2K_2$

If the dashed line is pink, there is a rainbow G. And if the dashed line is blue, there is a rainbow G. Thus, there can be only one color. ■

Theorem A.6 *P_4 is AR_1-bounded.*

Table A.3: AR_1 values for P_4

n	AR(G,n)
1	0
2	1
3	3
4	3
≥ 5	2

PROOF: For $n = 1, 2, 3$, $|V(G)| \geq n \implies AR(G,n) = \binom{n}{2}$.

If $n = 4$, opposite edges of K_4 cannot be different colors or there is a rainbow. Thus, the three sets of opposite edges provide maximum of three colors. See image:

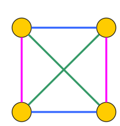

Figure A.4: Coloring of K_4 that avoids rainbow copies of P_4

If $n \geq 5$, color K_n with three colors.

Can all colors at a vertex be the same?

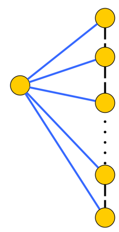

Figure A.5: Example of all colors at one vertex being the same color

Notice c_2 and c_3 must be present in the graph. If they meet at a vertex, there is a rainbow P_4.

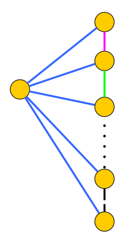

Figure A.6: Coloring of a vertex with monochromatic incident edges that permits a rainbow copy of P_4 Version 1

If they do not meet at some vertex, what color is the dashed line? Regardless of choice, we will get a rainbow P_4.

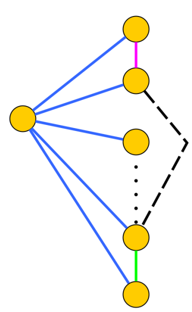

Figure A.7: Coloring of a vertex with monochromatic incident edges that permits a rainbow copy of P_4 Version 2

Thus, no vertex can have all the same color.

Can all three colors appear at one vertex?

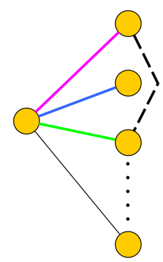

Figure A.8: Coloring of a vertex with three colors incident to a single vertex

So we see we must color the dashed line blue.

So, we see a K_3 coloring such as the one below. We see, we cannot color the dotted line without getting a rainbow P_4.

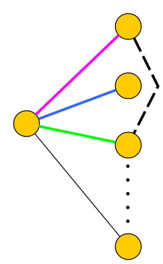

Figure A.9: Coloring of a vertex with three colors incident to a single vertex that permits a rainbow P_4

Thus, we cannot have all three colors on one vertex.

Can two colors appear at a vertex?

Group vertices according to which two colors appear.

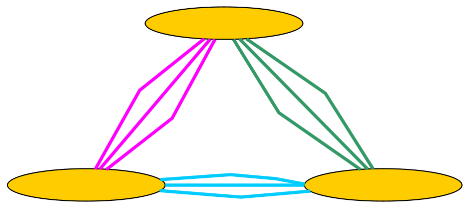

Figure A.10: Grouping of vertices by the two colors incident

A set cannot be empty, otherwise there is a rainbow P_3, see below.

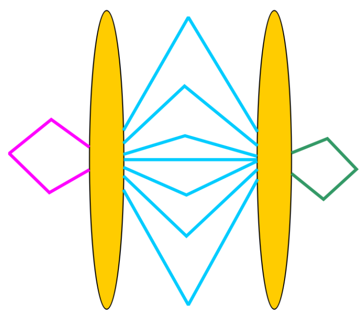

Figure A.11: Groups of two sets of vertices with two colors incident

Thus no sets can be empty. So, by the pigeonhole principle, at least one set must have at least two vertices. Thus, there is a rainbow P_3.

Thus, we cannot color K_n with three colors. ■

Theorem A.7 *$P_3 \cup P_2$ is AR_1-bounded.*

Table A.4: AR_1 values for $P_3 \cup P_2$

n	AR(G,n)
1	0
2	1
3	3
4	6
≥ 5	2

PROOF: For $n = 1, 2, 3, 4$, $|V(G)| \geq n \implies AR(G, n) = \binom{n}{2}$.

For $n \geq 5$, assume we can use 3 colors.

By Theorem 3.6, there is a rainbow P_4.

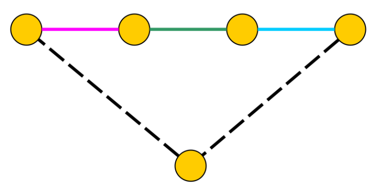

Figure A.12: Extension of rainbow P_4 that permits a rainbow $P_3 \cup P_2$

There is no way to color the dashed lines to avoid a rainbow G. Thus, we cannot use 3 colors. ■