A Hereditarily Indecomposable Inverse Limit of Finite Path Graphs

by

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Abstract

In this dissertation, we explore the properties and significance of inverse limit spaces where the factor spaces are path graphs. We define the graph topology for finite graphs and discuss the properties of a continuous map between graphs as well as properties of a traditional inverse limit of graphs. Most importantly, that a traditional inverse limit of finite path graphs is non-Hausdorff. We introduce a generalized inverse limit, where the first space is a metric arc and all other spaces are finite path graphs. By example, a technique is shown for constructing a generalized inverse limit, where the first space is a metric arc and the others are finite path graphs, that is homeomorphic to a traditional inverse limit of Hausdorff arcs.

Using crooked chains, we construct and analyze a non-Hausdorff hereditarily indecomposable continuum. This continuum has some interesting properties. These properties and the continuum's relationship with the Pseudo-arc is discussed. Ongoing work and open problems are stated.

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Chapter 1

Introduction

1.1 Motivation

In [7] Smith and Varagona describe a method for representing the Bucket Handle continuum as a generalized inverse limit where the first factor space is a metric arc and the remaining factor spaces are finite path graphs. It is noted that a point in the inverse limit is completely determined by whether each coordinate is on the left or right side of the roof top map. Then it is reasonable to consider the left and right intervals as single points of the space, or better yet, as edges of a graph. The ends of such intervals would then be vertices of the graph. By doing so, we obtain the generalized inverse limit described above. Smith and Varagona then extend this idea to n-type Bucket Handle continua.

In [2] we generalize this technique even further, by considering any traditional inverse limit space with continuous and onto bonding maps that are piecewise-linear and constructing a homeomorphic generalized inverse limit space. However, these bonding maps must meet certain conditions that are somewhat restricting. This dissertation explores the limitations of the technique used in [2] as well as the limitations of the inverse limits of graphs in general. A possible solution using infinite graphs is presented.

According to Smith's theorem proved in [6], if a traditional inverse limit of Hausdorff arcs is hereditarily indecomposable, then the space is metric. In Chapter 3, we construct an inverse limit of non-Hausdorff arcs that is hereditarily indecomposable but not metric. An example showing that without Hausdorffness, the conclusion of the theorem does not hold, and so Smith's theorem cannot be improved.

1.2 Definitions and Notation

A topological space X is a *continuum* if it is compact and connected. A topological space H is a *subcontinuum* of X if $H \subset X$ and H is closed compact and connected in X. A continuum is *decomposable* if it is the union of two proper subcontinua. A continuum is *indecomposable* if it is not decomposable and is *hereditarily indecomposable* if every subcontinuum is indecomposable. A *chain* is a collection of sets $C = \{c_1, c_2, ..., c_n\}$ such that $c_i \cap c_j \neq \emptyset$ if and only if $|i - j| \leq 1$. A chain is an *i-chain* if $\text{diam}(c_j) < i$ all j. A metric continuum X is *chainable* if for every open cover M of X, there is a chain with open links that refines M and covers X. For information on chainable hereditarily indecomposable continua see [4] and [5].

Suppose $\{X_i\}_{i=1}^{\infty}$ is a sequence of topological spaces, called *factor spaces*, and $\{f_i\}_{i=1}^{\infty}$, $f_i : X_{i+1} \to X_i$, is a sequence of continuous onto functions, called *bonding maps*. Then, $X = \varprojlim_T \{X_i, f_i\}_{i=1}^{\infty}$ denotes the *traditional inverse limit* space where $(x_i)_{i=1}^{\infty} \in X$ if $f_i(x_{i+1}) = x_i$ for all i. For an open set $U_i \subset X_i$, define $\overleftarrow{U_i} = \{(x_i)_{i=1}^{\infty} \in X : x_i \in U_i\}$. Then, $\mathcal{B} = \{\overleftarrow{U_i} : U_i \text{ open in } X_i\}$ forms a basis for the topology on X. Now, suppose $\{g_i\}_{i=1}^{\infty}$, $g_i : X_{i+1} \to 2^{X_i}$, is a sequence of upper semi-continuous functions. Then, $X = \varprojlim_G \{X_i, g_i\}_{i=1}^{\infty}$ denotes the *generalized inverse limit* space where $(x_i)_{i=1}^{\infty} \in X$ if $x_i \in g_i(x_{i+1})$ for all i. The basic open sets of X are as in the product topology, of the form $\prod_{i=1}^{n} U_i \times \prod_{i=n+1}^{\infty} X_i$ where U_i is an open set of X_i for all i.

Consider a finite graph X. In this thesis, we will almost always consider finite graphs. A simple graph is an undirected graph that has no loops and no multiple edges. From now on, we assume every graph is finite and simple unless otherwise stated. Then X is a space where each vertex is an element of X and each edge is an element of X. Let $\mathcal{V}(X)$ and $\mathcal{E}(X)$ denote the set of vertices and edges of X respectively, so that $X = \mathcal{V}(X) \cup \mathcal{E}(X)$. A vertex v and edge e are *incident* if v is one of the ends of e. Two vertices are *adjacent* if they share an edge. The *degree* of a vertex v is the number of edges incident to v, denoted $d(v)$. A graph C is a component of a graph G if it is a maximal connected subgraph of G. If C_1 and C_2 are components of a graph G with $C_1 \cap C_2 = \emptyset$, then we say C_1 and C_2 are incident if there is some $x_1 \in C_1$ and $x_2 \in C_2$ that are incident. A *path graph* is a graph X with vertex set $\mathcal{V}(X) = \{v_1, v_2, ..., v_n\}$ where

$d(v_1) = d(v_n) = 1$, $d(v_i) = 2$ all $2 \leq i \leq n-1$, and v_i is adjacent to v_{i+1} for all $1 \leq i \leq n-1$. The vertices v_1 and v_n are the *ends* of the path.

Now we define the *graph topology* on the graph X. The basic open set around an edge $e \in \mathcal{E}(X)$ is $B(e) = \{e\}$. The basic open set around a vertex $v \in \mathcal{V}(X)$ is $B(v) = \{v\} \cup \{e \in \mathcal{E}(X) : e \text{ is incident to } v\}$. Then the graph topology on X is generated by the basis $\mathcal{B}(X) = \{B(v) : v \in \mathcal{V}(X)\} \cup \{B(e) : e \in \mathcal{E}(X)\}$. A topological space X is T_0 if for any $x, y \in X, x \neq y$, there exists an open set U so that $x \in U, y \notin U$ or $y \in U, x \notin U$. We note that a graph X with the graph topology is T_0 but is not T_1 or Hausdorff, as a vertex cannot be separated from an incident edge.

Suppose $f : X \to Y$ is a map from a graph X to a graph Y. We say f has *level pieces* if there is some $v \in \mathcal{V}(X)$ and $e \in \mathcal{E}(X)$ that are incident and $f(v) = f(e)$.

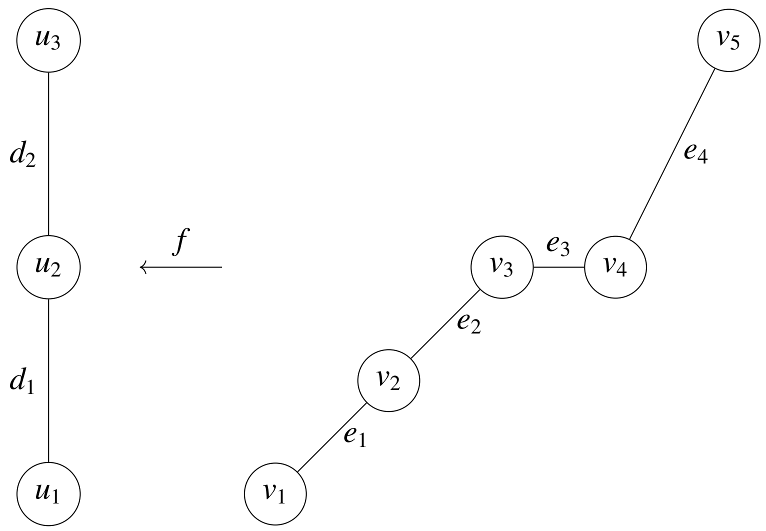


Figure 1.1: A Map with Level Pieces

In Figure 1.1, we see two different types of level pieces. The vertex v_3 and the edge e_3 both map to the vertex u_2. Notice that this portion of the map f is visually a level. The vertex v_2 and the edge e_1 both map to the edge d_1. Then, by definition, this is a level piece even though it does not visually look level.

We note that two possible interpretations of the word graph are used in this thesis: one the mathematical object made up of vertices and edges and the other a representation of a map $f(x)$ in the plane with points $(x, f(x))$. From now on, a graph will refer to the former and we will use $gr(f)$ to refer to the latter.

Chapter 2

Inverse Limits of Path Graphs

2.1 Properties of Continuous Maps

Let X and Y be path graphs with the graph topology and consider a continuous and onto function $f : X \to Y$. First, we investigate the pre-image of elements of Y. If $e \in \mathcal{E}(Y)$, then the pre-image $f^{-1}(e)$ must be a union of components where every component is a path that starts and ends with an edge. Some or all of these components could be a single edge, as a single edge is open in X. Consequently, if $v \in \mathcal{V}(Y)$, then the pre-image $f^{-1}(v)$ must be a union of components where every component is a path that starts and ends with a vertex. Some or all of these components could be a single vertex, as a single vertex is closed in X. Now, we consider two properties that will be referred to as *Incidence Properties*.

1. if $v \in \mathcal{V}(X), e \in \mathcal{E}(X)$ and v and e are incident, then $f(v) = f(e)$ or $f(v)$ and $f(e)$ are incident, where $f(v)$ is a vertex of Y and $f(e)$ is an edge of Y

2. assume X is a path graph, if $u \in \mathcal{V}(Y), d, d' \in \mathcal{E}(Y)$ and d, d' are both incident to u, then there are components $C_2 \in f^{-1}(u), C_1 \in f^{-1}(d), C_3 \in f^{-1}(d')$ such that C_2 is incident to C_1 and C_3 is incident to C_1

Proof of 1. Suppose $v \in \mathcal{V}(X), e \in \mathcal{E}(X)$ and v is incident to e. Then $f(v) = u$, some $u \in Y$ and $f(e) = d$ some $d \in Y$. By way of contradiction, assume $f(v)$ is not incident to $f(e)$ in Y and $f(v) \neq f(e)$. Now we show that in each case we reach a contradiction.

Case 1: u and d are both vertices

By assumption, $u \neq d$ so that u and d are distinct vertices in $\mathcal{V}(Y)$. Consider the basic open set $B(u)$; the set containing the vertex u and its incident edges. Because f is continuous, the pre-image $f^{-1}(B(u))$ must be open. Clearly $v \in f^{-1}(B(u))$, as we assumed $f(v) = u$. Then, all edges incident to v must also be contained in $f^{-1}(B(u))$. But v is incident to the edge e, and $f(e) = d$, some vertex that is not in $B(u)$. Then $e \notin f^{-1}(B(u))$, contradicting that f is continuous.

Case 2: u and d are both edges

By assumption, $u \neq d$ so that u and d are distinct edges in $\mathcal{E}(Y)$. Consider the basic open set $B(u) = \{u\}$. Because f is continuous, the pre-image $f^{-1}(B(u)) = f^{-1}(\{u\})$ must be open. Clearly $v \in f^{-1}(\{u\})$, as we assume $f(v) = u$. Then, all edges incident to v must also be contained in $f^{-1}(B(u))$. But v is incident to the edge e, and $f(e) = d$, some edge that is not in $B(u) = \{u\}$. Then $e \notin f^{-1}(B(u))$, contradicting that f is continuous.

Case 3: u is a vertex and d is an edge

By assumption, u and d are not incident in Y. Consider the basic open set $B(u)$; the set containing the vertex u and its incident edges. Because f is continuous, the pre-image $f^{-1}(B(u))$ must be open. Clearly $v \in f^{-1}(B(u))$, as we assumed $f(v) = u$. Then, all edges incident to v must also be contained in $f^{-1}(B(u))$. But v is incident to e, and $f(e) = d$. By assumption, u and d are not incident, so $d \notin B(u)$. Then, $e \notin f^{-1}(B(u))$, contradicting that f is continuous.

Case 4: u is an edge and d is a vertex

Consider the basic open set $B(u) = \{u\}$. Because f is continuous, the pre-image $f^{-1}(B(u))$ must be open. Clearly $v \in f^{-1}(B(u))$ as we assumed $f(v) = u$. Then, all edges incident to v must also be contained in $f^{-1}(B(u))$. But v is incident to e and $f(e) = d \neq u$, so $d \notin f^{-1}(B(u))$, contradicting that f is continuous.

Then, we must have that $f(v) = f(e)$ or $f(v)$ and $f(e)$ are incident in Y. Now, we may show that if $f(v) = u$ and $f(e) = d$ are incident in Y, then u must be a vertex and d must be an edge. By way of contradiction, assume u is an edge. Then $f^{-1}(B(u))$ contains v but not e, a contradiction. So, u must be a vertex. If d is incident to u, it must be an edge. $\square$

Proof of 2. Let $\mathcal{V}(X) = \{u_1, u_2, ..., u_n\}$ so that $u = u_j$, $d = d_{j-1}$, and $d' = d_j$ for some j. Because u_j is incident to d_{j-1}, d_j in Y, by *Incidence Property 1.*, all components of $f^{-1}(u_j)$ are either incident to a component of $f^{-1}(d_{j-1})$ or $f^{-1}(d_j)$.

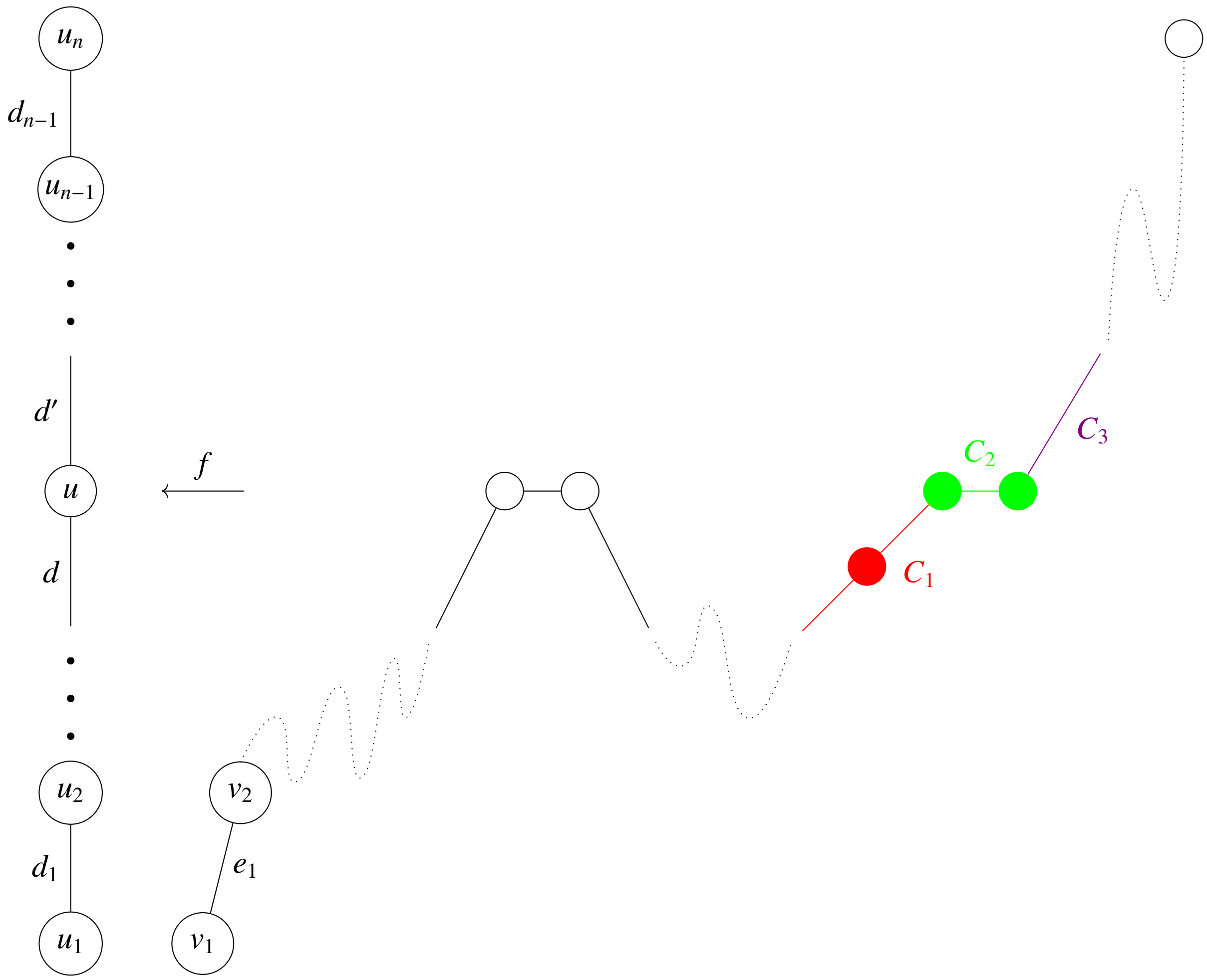


Figure 2.1: Incidence Property 2

Suppose, by way of contradiction, that there is no component of $f^{-1}(u_j)$ that is incident to both a component of $f^{-1}(d_{j-1})$ and a component of $f^{-1}(d_j)$.

Notice that the path graph X is made up of the components of the pre-images of elements of Y. Meaning, $X = \cup_{y\in Y} f^{-1}(y)$. Let $S_1, S_2, ..., S_t$ be these components of X, ordered naturally in X. Then, by definition, $f(S_i) = y$, some $y \in Y$, meaning every element of $S_i \subset X$ maps to the same element of Y. Let S_i be the first component so that $f(S_i) = u_j$. By assumption, S_j is not incident to both a component of $f^{-1}(d_{j-1})$ and a component of $f^{-1}(d_j)$. Without loss of generality, assume S_i is only incident to a component of $f^{-1}(d_j)$. Then, $f(S_{i-1}) = d_j = f(S_{i+1})$.

Now, we will consider the components S_k with $k > i$ and show that no component maps to d_{j-1}. Because f is continuous and by *Incidence Property 1.*, we have the following possibilities:

(1) if $f(S_k) = u_j$, then $f(S_{k+1}) = d_j$

(2) if $f(S_k) = u_m$, then $f(S_{k+1}) = d_m$ or d_{m+1}, $m > 0$

(3) if $f(S_k) = d_m$, then $f(S_{k+1}) = u_{m-1}$ or u_m, $m \geq 1$

Notice that the only way a component S_k can map to d_{j-1} is if $f(S_{k-1}) = u_j$, but this is precisely (1) above. So, no component $S_k, k > j$ maps to d_{j-1} in Y.

The same proof shows that no component $S_k, k < j$ maps to d_{j-1}, contradicting that f is continuous. In Figure 2.1 above, we see an example of Incidence Property 2. □

2.2 Representing Inverse Limits of Metric Arcs as a Generalized Inverse Limit of Path Graphs

A finite graph with the graph topology is a very simple space, not only being finite but also non-Hausdorff. However, when we consider an inverse limit of such spaces, we are able to create something much more complex. It is natural to ask the broad question: for what traditional inverse limits of metric arcs are we able to choose finite graphs and bonding maps so that the inverse limits are homeomorphic? In an attempt to answer this question, we first notice that the traditional inverse limit of path graphs is non-Hausdorff.

Proof. If $X = \varprojlim_T \{X_i, f_i\}_{i=1}^{\infty}$ is a traditional inverse limit with each X_i a path graph with the graph topology, then we can construct two points $v = (v_i)_{i=1}^{\infty}$ and $e = (e_i)_{i=1}^{\infty}$ in X that cannot be separated. In the first factor space, choose any incident vertex and edge, v and e. Let $v_1 = v$ and $e_1 = e$. By Incidence Property 2., there is some $u \in f_1^{-1}(v)$ and some $d \in f_1^{-1}(e)$ such that u and

d are incident in X_2. Let $v_2 = u$ and $e_2 = d$. Continue this process so that v_i and e_i are incident in every X_i and therefore every open set containing v_i also contains e_i. Then v and e are points in X that cannot be separated. □

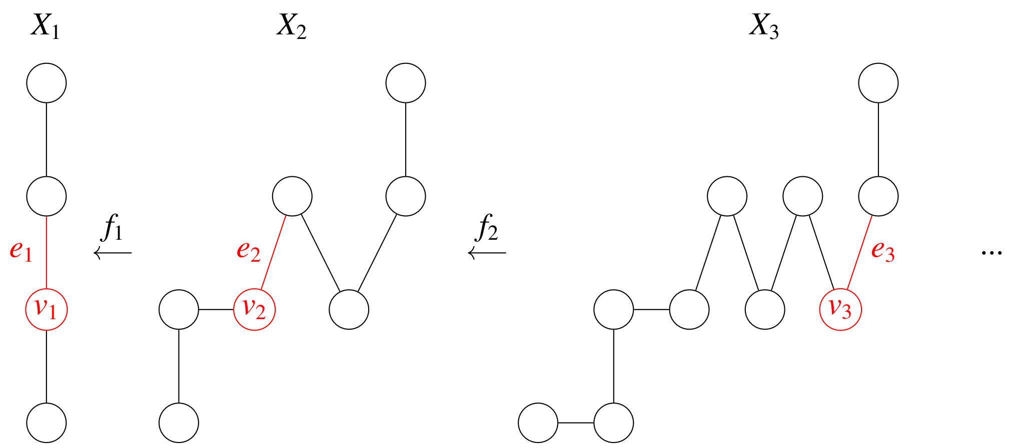


Figure 2.2: Non-Hausdorffness of Traditional Inverse Limits of Path Graphs

Then no traditional inverse limit of metric arcs can be represented as a traditional inverse limit of graphs, as the inverse limits of graphs is non-Hausdorff and therefore not metric. Instead, we will use a generalized inverse limit where the first space is a metric arc and the remaining spaces are path graphs. There are two methods for choosing such graphs and bonding maps. Both will be shown using the example below.

At the end of this section, we give a generalization of Method 2. This generalization requires that the bonding maps have no level pieces, otherwise the inverse limit will not be Hausdorff. So, we will first prove that if $Y = \varprojlim_G \{Y_i, g_i\}_{i=0}^{\infty}$ is a generalized inverse limit space where Y_0 is a metric arc and $Y_i, i > 0$, is a path graph, then if any g_i has level pieces, the inverse limit space Y is non-Hausdorff.

Proof. Assume there is some g_j that has at least one level piece. We will construct two points, $y = (y_i)_{i=0}^{\infty}$ and $y' = (y'_i)_{i=0}^{\infty}$, in Y that cannot be separated. Because g_j has a level piece, there is some $v \in \mathcal{V}(Y_{j+1})$ and $e \in \mathcal{E}(Y_{j+1})$ incident in Y_{j+1} such that $g_j(v) = g_j(e)$.

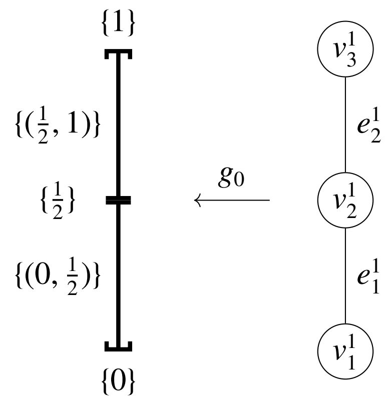
Let $y_{j+1} = v$ and $y'_{j+1} = e$. Then, $y_j = g_j(v) = g_j(e) = y'_j$, so that $y_i = y'_i$ for all $i \leq j$. By Incidence Property 2., there is some $u \in g_{j+1}^{-1}(v)$ and some $d \in g_{j+1}^{-1}(e)$ so that u and d are incident in Y_{j+2}. Let $y_{j+2} = u$ and $y'_{j+2} = d$. Continue this process so that y_i and y'_i are incident for all $i > j$. This is the same process used in Figure 2.2. Then for all i, every open set in Y_i

containing y_i must also contain y_i' so that every open set in Y containing y must also contain y'. Then y and y' are points in Y that cannot be separated. $\square$

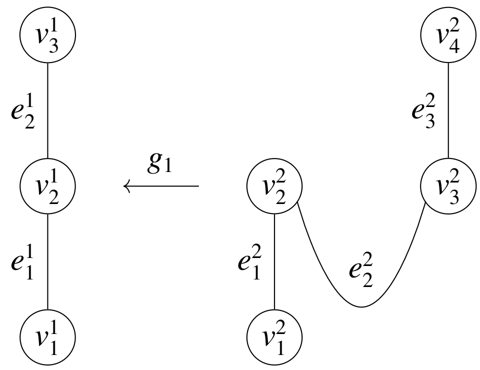
Naturally, we wonder if the other direction of the implication is true. If Y is non-Hausdorff, must it be true that some g_j has level pieces? We provide a counter-example below.

Counter-example. Let $Y = \varprojlim_G \{Y_i, g_i\}_{i=0}^{\infty}$ where $Y_0 = [0,1]$ with the usual metric topology. Let Y_1 be a path graph with 3 vertices where $\mathcal{V}(Y_1) = \{v_1^1, v_2^1, v_3^1\}$ and Y_i be a path graph with 4 vertices for $i > 1$ where $\mathcal{V}(Y_i) = \{v_1^i, v_2^i, v_3^i, v_4^i\}$. We will define the maps g_0 and g_1 below. All other maps, g_i with $i > 1$, will be like the identity, so that $g_i(v_j^{i+1}) = \{v_j^i\}$ and $g_i(e_j^{i+1}) = \{e_j^i\}$.

Define $g_0 : Y_1 \to 2^{[0,1]}$ as follows:



Define $g_1 : Y_2 \to 2^{Y_1}$ as follows:

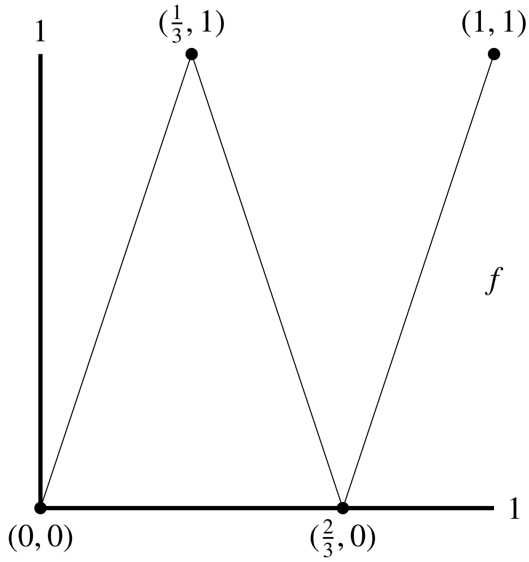


Note that for each g_i, there are no incident elements that map to the same element, so every g_i has no level piece. Consider the point $y = (y_i)_{i=0}^{\infty}$ where $y_0 = \frac{1}{2}$, $y_1 = v_2^1$, and $y_i = v_2^i$ for all $i > 1$. Consider a second point $y' = (y_i')_{i=0}^{\infty}$ where $y_0' = \frac{1}{2}$, $y_1' = v_2^1$, and $y_i = v_3^i$ for all $i > 1$. Then, y and y' cannot be separated in the first factor space, as $y_1 = \frac{1}{2} = y_1'$. Similarly, they cannot be separated in the second factor space, as $y_1 = v_2^1 = y_1'$. When $i > 1$, $y_i = v_2^i$ and $y_i' = v_3^i$, but $B^i(v_2^i) \cap B^i(v_3^i) = \{e_2^i\}$, so that any two open sets that contain y and y' must intersect.

So, the absence of level pieces is necessary for Hausdorffness, but does not guarantee Hausdorffness. Now we see the example illustrating the two methods mentioned above.

Example. Let $X = \varprojlim_T \{X_i, f_i\}_{i=1}^{\infty}$ be a traditional inverse limit space where each $X_i = [0, 1]$ with the usual topology and each $f_i = f$, shown below.

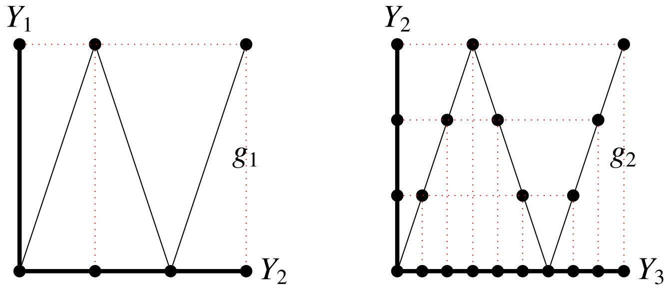
$$f(x) = \begin{cases} 3x \text{ if } 0 \leq x \leq \frac{1}{3}, \\ -3x + 2 \text{ if } \frac{1}{3} \leq x \leq \frac{2}{3}, \\ 3x - 2 \text{ if } \frac{2}{3} \leq x \leq 1 \end{cases}$$



We offer two methods for constructing a homeomorphic generalized inverse limit of graphs.

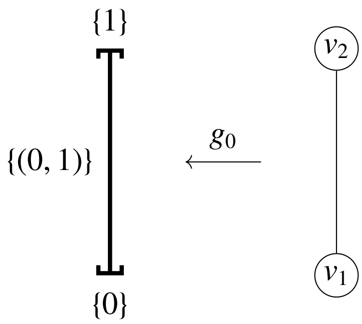
Method 1. Consider the generalized inverse limit space $Y = \varprojlim_G \{Y_i, g_i\}_{i=0}^{\infty}$ where $Y_0 = [0, 1]$ with the usual metric topology, each $Y_i, i \geq 1$ a graph with the graph topology, g_0 an upper semi-continuous map, and each $g_i, i \geq 1$ a continuous map. Our goal is to choose appropriate

Y_i's and g_i's so that X is homeomorphic to Y. We do so by using edges and vertices to mimic the shape of f, like so:



So that $|\mathcal{V}(Y_1)| = 2$ and $|V(Y_i)| = 4 + 3(|\mathcal{V}(Y_{i-1})| - 2)$ for $i \geq 2$. Now, we only need to define the map $g_0 : Y_1 \to [0,1]$. Let $\mathcal{V}(Y_1) = \{v_1, v_2\}$ and $\mathcal{E}(Y_1) = \{e_1\}$. Then, define the map g_0:

$$g_0(v_1) = \{0\}\,;\; g_0(e_1) = \{x : x \in (0,1)\}\,;\; g_0(v_2) = \{1\}$$



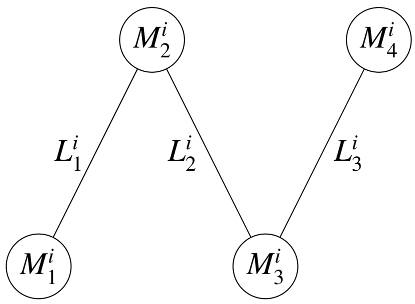
In [2], it is shown that X and Y are homeomorphic.

Method 2. We again consider a generalized inverse limit space $Y = \varprojlim_G \{Y_i, g_i\}_{i=1}^{\infty}$, where $Y_1 = [0,1]$, but the remaining factor spaces and bonding maps will be defined differently than in Method 1. We look at $gr(f_i) \subset X_i \times X_{i-1}$. In this example, every $gr(f_i)$ can be written as the union of $M_1^i = (0,0), L_1^i = \{(x, 3x) : 0 < x < \frac{1}{3}\}, M_2^i = (\frac{1}{3}, 1), L_2^i = \{(x, -3x+2) : \frac{1}{3} < x < \frac{2}{3}\}, M_3^i = (\frac{2}{3}, 0), L_3^i = \{(x, 3x-2) : \frac{2}{3} < x < 1\}, M_4^i = (1,1)$.

Then, the path graph $Y_i, i > 1$ is the graph $gr(f_i)$ where $\mathcal{V}(Y_i) = \{M_1^i, M_2^i, M_3^i, M_4^i\}$ and $\mathcal{E}(Y_i) = \{L_1^i, L_2^i, L_3^i\}$.

We define the map $g_1 : Y_2 \to 2^{[0,1]}$ as follows:

$$g_1(M_1^2) = g_1(M_3^2) = \{0\}\,;\; g_1(M_2^2) = g_1(M_4^2) = \{1\}$$



$$g_1(L_1^2) = g_1(L_2^2) = g_1(L_3^2) = \{(0,1)\}$$

And we define the maps $g_i : Y_{i+1} \to Y_i, i > 1$ as follows:

$$g_i(M_1^{i+1}) = g_i(M_3^{i+1}) = \{M_1^i\}\,;\ g_i(M_2^{i+1}) = g_i(M_4^{i+1}) = \{M_4^i\}$$

$$g_i(L_1^{i+1}) = g_i(L_2^{i+1}) = g_i(L_3^{i+1}) = \{L_1^i, M_2^i, L_2^i, M_3^i, L_3^i\}$$

In [2], it is shown than X and Y are homeomorphic and that, in general, if $X = \varprojlim_G \{X_i, f_i\}_{i=1}^{\infty}$ where each X_i is a metric arc and each f_i is a piecewise-linear map, we can find a homeomorphic Y as in the example if each Y_i meets the following conditions:

H1. If $(x, y) \in \mathcal{V}(Y_{i+1})$, then $(y, z) \in \mathcal{V}(Y_i)$, some z.

H2. If $(x_1, y) \in \mathcal{V}(Y_i)$ and $(x_2, y) \in \mathcal{V}(Y_i)$, then $B^i((x_1, y)) \cap B^i((x_2, y)) = \emptyset$.

We note that we may add vertices to the Y_i's to meet condition H1. and that H2. implies that each map g_i has no level pieces. Then with the map $g_1 : Y_2 \to 2^{Y_0}$:

$$g_1(p) = \begin{cases} \{y\} \text{ if } p = (x, y) \in \mathcal{V}(Y_2), \\ D_n^2 \text{ if } p = L_n^2 \in \mathcal{E}(Y_2), n \leq \epsilon_1. \end{cases}$$

And g_i, for $i > 1$, $g_i : Y_{i+1} \to 2^{Y_i}$:

$$g_i(p) = \begin{cases} \{(y,z) | (y,z) \in \mathcal{V}(Y_i)\} \text{if } p = (x,y) \in \mathcal{V}(Y_{i+1}) \\ \\ \{L^i_k | k \leq \epsilon^i \text{ and } D^{i+1}_n \subseteq R^i_k\} \cup \{(x,y) : (x,y) \in \mathcal{V}(Y_i) \text{ and } x \in D^{i+1}_n\} \\ \quad \text{if } p = L^{i+1}_n, n \leq \epsilon^{i+1} \end{cases}$$

where D^i_n and R^i_n represent the domain and range of L^i_n respectively and ϵ^i is the size of $\mathcal{V}(Y_i)$. The inverse limit space $Y = \varprojlim_G \{Y_i, g_i\}_{i=1}^{\infty}$ is homeomorphic to $X = \varprojlim_T \{X_i, f_i\}_{i=1}^{\infty}$.

2.3 Infinite Graphs

If the bonding maps meet conditions H1. and H2., then we can use Method 1 and Method 2. If they do not, it is unclear if the traditional inverse limit can be represented as a generalized inverse limit of graphs. In this section, we offer a possible solution when all bonding maps are the same but do not meet the conditions.

Consider an arbitrary graph G equipped with the graph topology which contains a ray R. Let G_∞ be the graph made up of the singleton $\{v_\infty\}$ and no edges so that $G \cup G_\infty$ is the one-point compactification of G. Then, we say that v_∞ is the *ray end* of the ray R.

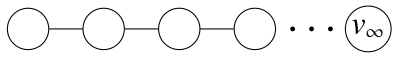


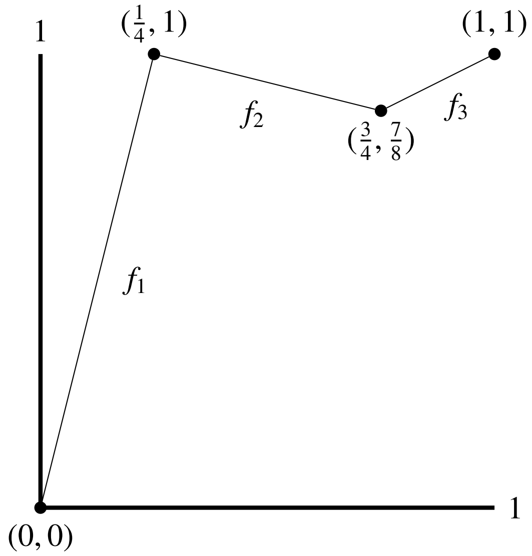
Figure 2.3: Ray and Ray End v_∞

We note that the rays we will be considering have a natural ordering, so that we may modify the graph topology for such an infinite graph as follows: an open set containing the point v_∞ is as in the order topology. There may be multiple rays, which do not intersect, that have the same ray end, v_∞. Suppose $R_1, R_2, ..., R_n$ are such rays with $R_i = \{v^i_0, e^i_1, v^i_1, e^i_2, ...\}$. Then an open set containing v_∞ is of the form $\{e^1_j, v^1_{j+1}, ...\} \cup ... \cup \{e^n_k, v^n_{k+1}, ...\} \cup \{v_\infty\}$.

Example. Let $X = \varprojlim_T \{X_i, f\}_{i=1}^{\infty}$ be a traditional inverse limit space where each $X_i = [0,1]$

with the usual topology and each $f_i = f$, shown below.

$$f(x) = \begin{cases} f_1(x) = 4x \text{ if } 0 \leq x \leq \frac{1}{4}, \\ f_2(x) = -\frac{1}{4}x + \frac{17}{16} \text{ if } \frac{1}{4} \leq x \leq \frac{3}{4}, \\ f_3(x) = \frac{1}{2}x + \frac{1}{2} \text{ if } \frac{3}{4} \leq x \leq 1 \end{cases}$$



Now, we attempt to define a generalized inverse limit $Y = \varprojlim_G \{Y_i, g_i\}_{i=1}^{\infty}$ that is homeomorphic to X using the methods described in the previous section. Then, $Y_1 = [0, 1]$ and the graphs Y_i must at least have the following vertices and edges:

$$\mathcal{V}(Y_i) = \left\{(0, 0), \left(\frac{1}{4}, 1\right), \left(\frac{3}{4}, \frac{7}{8}\right), (1, 1)\right\}$$

$$\mathcal{E}(Y_i) = \left\{\left\{(x, f_1(x)) : 0 < x < \frac{1}{4}\right\}, \left\{(x, f_2(x)) : \frac{1}{4} < x < \frac{3}{4}\right\}, \left\{(x, f_3(x)) : \frac{3}{4} < x < 1\right\}\right\}$$

Then, the vertex $(\frac{3}{4}, \frac{7}{8})$ is in every Y_i, but there is no vertex (x, y) with $x = \frac{7}{8}$, failing condition H1. However, it is permissible to add vertices so that the conditions are met. To do so, we must add the vertex $(\frac{7}{8}, \frac{15}{16})$ to each Y_i. But now this new vertex fails our conditions. We notice that $f_3^i(x) = \frac{1}{2^i}x + \frac{2^i - 1}{2^i}$, so that $f_3^i < 1$ for all $\frac{3}{4} < x < 1$ and is increasing, so that an attempt to meet our conditions will result in adding infinitely many vertices (x, y) with y approaching 1.

Instead, we will define new Y_i's equipped with the graph topology, allowing for rays in our graphs. We define the vertices of Y_i as follows:

$$M_1^i = (0,0)\,;\; M_2^i = \left(\frac{1}{4}, 1\right)$$

$$N_1^i = \left(\frac{3}{4}, f_3\left(\frac{3}{4}\right)\right)\,;\; N_2^i = \left(f_3\left(\frac{3}{4}\right), f_3^2\left(\frac{3}{4}\right)\right) = \left(\frac{7}{8}, \frac{15}{16}\right)$$

$$...$$

$$N_j^i = \left(f_3^j\left(\frac{3}{4}\right), f_3^{j+1}\left(\frac{3}{4}\right)\right)$$

$$...$$

$$N_\infty^i = (1,1)$$

where the N^i's represent the vertices approaching $(1,1)$, so that the set of vertices $\left\{N_j^i\right\}_{j=1}^{\infty}$ is a ray in Y_i with end N_∞^i. Now we may define the edges of Y_i as follows:

$$L_1^i = \left\{(x, f(x)) : 0 < x < \frac{1}{4}\right\}\,;\; L_2^i = \left\{(x, f(x)) : \frac{1}{4} < x < \frac{3}{4}\right\}$$

$$T_1^i = \left\{(x, f(x)) : \frac{3}{4} < x < f_3\left(\frac{3}{4}\right)\right\}\,;\; T_2^i = \left\{(x, f(x)) : f_3\left(\frac{3}{4}\right) < x < f_3^2\left(\frac{3}{4}\right)\right\}$$

$$...$$

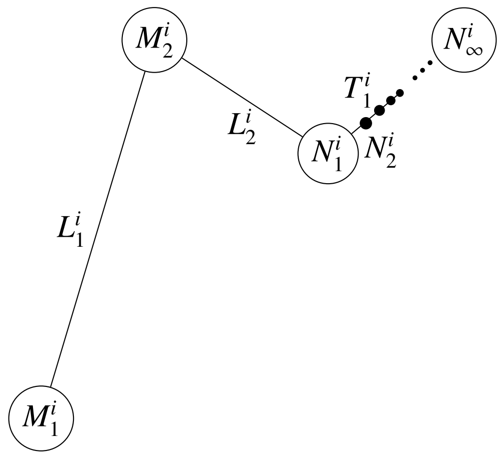
$$T_j^i = \left\{(x, f(x)) : f_3^{j-1}\left(\frac{3}{4}\right) < x < f_3^j\left(\frac{3}{4}\right)\right\}$$

$$...$$

Then $g_1 : Y_2 \to 2^{[0,1]}$ is the map:

$$g_1(M_1^2) = g_1((0,0)) = \{0\}\,;\; g_1(M_2^2) = g_1((\frac{1}{4}, 1)) = \{1\}\,;\; g_1(N_j^2) = g_1((x,y)) = \{y\}$$

$$g_1(L_j^2) = D_j^2(L_j^2)\,;\; g_1(T_j^2) = D_j^2(T_j^2)$$



And $g_i : Y_{i+1} \to 2^{Y_i}, i > 1$ is the map:

$$g_i(M^{i+1}_1) = \{M^i_1\}\,;\; g_i(M^{i+1}_2) = g_i(N^{i+1}_\infty) = \{N^i_\infty\}\,;\; g_i(N^{i+1}_j) = \{N^i_{j+1}\}$$

$$g_i(L^{i+1}_1) = Y_i - \{M^i_1, N^i_\infty\}\,;\; g_i(L^{i+1}_2) = \{N^i_2, T^i_2, N^i_3, T^i_3, ...\}\,;\; g_i(T^{i+1}_j) = \{T^i_{j+1}\}$$

We first show that Y is Hausdorff.

Proof. Suppose $p = (p_i)^\infty_{i=1}, q = (q_i)^\infty_{i=1} \in Y$ and $p \neq q$.

Case 1. $p_1 \neq q_1$

If $p_1 \neq q_1$, then because $Y_1 = [0, 1]$ is metric, we may find disjoint open sets U and V separating p and q.

If $p_1 = q_1$, we may assume $p_j \neq q_j$ for some smallest $j > 1$.

Case 2. $p_j \in \mathcal{E}(Y_j), q_j \in \mathcal{E}(Y_j)$

Let $U = \overleftarrow{\{p_j\}}$ and $V = \overleftarrow{\{q_j\}}$ so that U and V separate p and q.

Case 3. $p_j \in \mathcal{E}(Y_j), q_j \in \mathcal{V}(Y_j)$

Let $U = \overleftarrow{\{p_j\}}$ and $V = \overleftarrow{B(q_j)}$. Clearly $p \in U$ and $q \in V$, then we only need to check that $U \cap V = \emptyset$. By way of contradiction, assume $U \cap V \neq \emptyset$. Then, p_j must be incident to q_j. But by assumption, $p_{j-1} = q_{j-1} \Rightarrow g_{j-1}(p_j) = g_{j-1}(q_j)$, contradicting that each g_i has no level pieces. The case when $p_j \in \mathcal{V}(Y_j), q_j \in \mathcal{E}(Y_j)$ is the same as Case 3.

Case 4. $p_j \in \mathcal{V}(Y_j), q_j \in \mathcal{V}(Y_j)$

Let $U = \overleftarrow{B(p_j)}$ and $V = \overleftarrow{B(q_j)}$. Clearly $p \in U$ and $q \in V$, then we only need to check that $U \cap V = \emptyset$. By assumption, $p_j \neq q_j$ and $p_{j-1} = q_{j-1} \Rightarrow g_{j-1}(p_j) = g_{j-1}(q_j)$. But, there are only two possible vertices that map to the same element. We may assume $p_j = M_1^j$ and $q_j = N_\infty^j$, otherwise they would just be switched, so $U \cap V = \emptyset$. $\square$

Now we show that X and Y are homeomorphic.

Proof. Define a homeomorphism $h : X \to Y$ as follows:

$$y_1 = x_1$$

$$\text{for } i \geq 1, \text{ if } (x_i, f(x_i)) = M_j^{i+1} \text{ then } y_{i+1} = M_j^{i+1}, j \in \{0, 1\}$$

$$\text{if } (x_i, f(x_i)) \in L_j^{i+1} \text{ then } y_{i+1} = L_j^{i+1}, j \in \{1, 2\}$$

$$\text{if } (x_i, f(x_i)) = N_j^{i+1} \text{ then } y_{i+1} = N_j^{i+1}, j \geq 0$$

$$\text{if } (x_i, f(x_i)) \in T_j^{i+1} \text{ then } y_{i+1} = T_j^{i+1}, j \geq 1$$

Proof that h is one-to-one. Let $p = (p_i)_{i=1}^\infty, q = (q_i)_{i=1}^\infty \in X$ and assume $p \neq q$. Then, for some first j, $p_j \neq q_j$ so that $f(p_j) \neq f(q_j)$. Then $(p_j, f(p_j)) \neq (q_j, f(q_j))$ so that $h(p) \neq h(q)$. $\square$

Proof that h is continuous. Let U be an open set in Y, so that $U = \overleftarrow{U_i}$ some U_i open in Y_i. Define $V_i = \{x : (x, f(x)) \in \mathcal{V}(U_i)\} \cup \{x : a < x < b \in \mathcal{E}(U_i)\}$. Then, $h^{-1}(U) = \overleftarrow{V_i} = V$. The set V_i is open in X_i, so the set V is open in X. $\square$

Proof that h^{-1} is continuous. We will show that h is a closed map. Let $H \subset X$ be a closed set. Because X is compact, the closed subset H is compact and because h is continuous, $h(H) = K$ is compact and therefore closed in the Hausdorff space Y. $\square$

Certainly we could generalize this technique as in the previous section whenever the bonding maps f_i are all a piecewise defined function f that has no level pieces. By allowing infinite graphs, we increase the number of spaces that can be represented as a generalized inverse limit of graphs where the first space is a metric arc, as this technique does not require condition H1.

Chapter 3

A Hereditarily Indecomposable Inverse Limit of Path Graphs

3.1 Construction

Though in [7] and [2] we are able to give some generalizations of these inverse limits of graphs, we still question what limitations they have. In an attempt to test these limitations, we consider a complicated space, the Pseudo-arc, and want to determine if such a space can be represented using graphs.

A *Pseudo-arc* is a hereditarily indecomposable chainable metric continuum. Before giving a construction of the Pseudo-arc, we must define crooked chains. Suppose we have a chain $C = \{c_1, c_2, ..., c_n\}$ with $|C| \geq 5$. Then the chain $D = \{d_1, d_2, ..., d_k\}$ is *crooked* in C if it is a proper refinement of C and if $C' = \{c_m, ...c_l\}$, $|C'| \geq 5$, is a subchain of C and $D' = \{d_p, ..., d_q\}$ is a subchain of D such that:

- if $d_p \subset c_m$ and $d_q \subset c_l$ then there is some s, t with $p < s < t < q$ such that $d_s \subset c_{l-1}$ and $d_t \subset c_{m+1}$
- if $d_p \subset c_l$ and $d_q \subset c_m$ then there is some s, t with $q < t < s < p$ such that $d_t \subset c_{l-1}$ and $d_s \subset c_{m+1}$

Suppose X is a metric continuum and there is a sequence of chains $\{C^i\}_{i=1}^{\infty}$ such that each C^{i+1} is a proper refinement of C^i, each C^i is a $\frac{1}{i}$-chain, and each C^{i+1} is crooked in C^i. Notice that the size of the links of the chains decreases as i increases. Then, $\cap_{i=1}^{\infty} \cup C^i$ is a hereditarily indecomposable chainable metric continuum, and therefore a Pseudo-arc. We will reference [8] for one construction of such a sequence of chains.

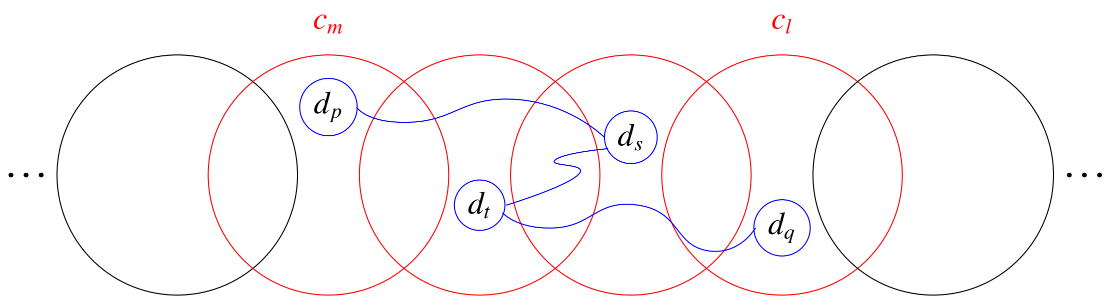


Figure 3.1: A Crooked Chain

The given definition of crookedness relies on the metric of the space X. Because our spaces with the graph topology are not metric, we give another definition of crookedness using maps and graphs. Suppose X and Y are path graphs and C and D are chains covering X and Y respectively, $|C| \geq 5$. Then D is *mapped crookedly* in C if there exists a continuous map $f : Y \to X$ such that if $\{c_m, ..., c_l\}$ is a subchain of C and $\{d_p, ..., d_q\}$ is a subchain of D such that:

- if $f(d_p) \subseteq c_m$ and $f(d_q) \subseteq c_l$ then there is some s, t with $p < s < t < q$ such that $f(d_s) \subseteq c_{l-1}$ and $f(d_t) \subseteq c_{m+1}$
- if $f(d_p) \subseteq c_l$ and $f(d_q) \subseteq c_m$ then there is some s, t with $q < t < s < p$ such that $f(d_t) \subseteq c_{l-1}$ and $f(d_s) \subseteq c_{m+1}$

From now on, when proving crookedness we will assume we are in the case where $f(d_p) \subseteq c_m$ and $f(d_q) \subseteq c_l$, as the other case would be very similar.

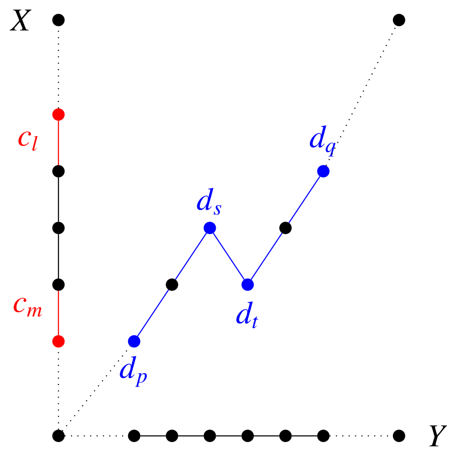


Figure 3.2: Mapping Crookedly

This new definition of crookedness gives us, in a sense, a decreasing size of links as in the construction of the Pseudo-arc. If a link d of D maps into a link c of C, so $f(d) \subseteq c$, then this takes the role of subsets in the usual construction of the Pseudo-arc. If multiple links $d_1, d_2, ..., d_n$ of D map into a link c of C, so $f(d_i) \subseteq c$ all $1 \leq i \leq n$, then we can think of the links $d_1, d_2, ..., d_n$ as being "smaller" than the link c.

We now give a detailed construction of an inverse limit of path graphs that is non-metric and hereditarily indecomposable. Our goal is to mimic the shapes in the construction of the Pseudo-arc as in [8].

Theorem 1. *If X is a path graph with $|\mathcal{V}(X)| = n \geq 5$ and $C = \{B(v) : v \in \mathcal{V}(X)\}$ is a chain covering X, then there is some path graph Y and continuous map g_n so that $D = \{B(u) : u \in \mathcal{V}(Y)\}$ maps crookedly in C.*

Note that the chains C and D are chosen for simplicity, but certainly the theorem could be re-stated and proved with different chains covering X and Y.

Proof. We will give a proof by induction, but first we must construct two maps g_3 and g_4.

Construction of g_3. The map g_3 will be used to map crookedly into a graph with vertex set of size 3. Suppose $\mathcal{V}(X) = \{v_1, v_2, v_3\}$, so that $C = \{B(v_1), B(v_2), B(v_3)\}$. Then, define $\mathcal{V}(Y) = \{u_1, u_2, ...u_{k_3}\}$, $k_3 = 10$, so Y is a path graph with 10 vertices. Then, g_3 is the map satisfying:

$$g_3(u_{4i-3}) = g_3(u_{4i-3}u_{4i-2}) = g_3(u_{4i-2}) = v_i \in \mathcal{V}(X),\ i \in \{1, 2, 3\}$$

$$g_3(u_{4i-2}u_{4i-1}) = g_3(u_{4i-1}) = g_3(u_{4i-1}u_{4i}) = g_3(u_{4i}) = g_3(u_{4i}u_{4i+2}) = v_iv_{i+1} \in \mathcal{E}(X),\ i \in \{1, 2\}$$

We construct g_3 so that the pre-image of a vertex contains two vertices and the pre-image of an edge also contains two vertices. By doing so, the links of the chain D are smaller than the links of the chain C, in the sense that multiple links of D are mapped into a single link of C.

Certainly g_3 is continuous, as the pre-image of an edge is a component of Y with an edge at both ends and the pre-image of a vertex is a component of Y with a vertex at both ends. It is vacuously true that D maps crookedly in C, as $|C| < 5$.

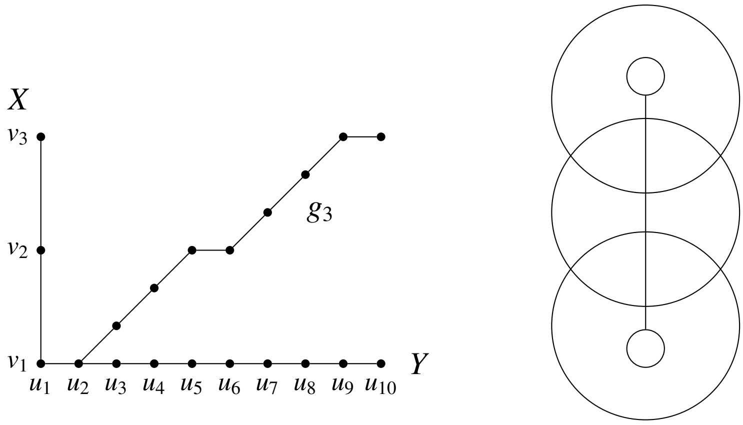


Figure 3.3: Mapping Crookedly for Chain of Size 3

Construction of g_4. The map g_4 will be used to map crookedly into a graph with a vertex set of size 4. Suppose $\mathcal{V}(X) = \{v_1, v_2, v_3, v_4\}$, so that $C = \{B(v_1), B(v_2), B(v_3), B(v_4)\}$. Then, define $\mathcal{V}(Y) = \{u_1, u_2, ..., u_{k_4}\}$, $k_4 = 22$, so Y is a path graph with 22 vertices.Then, g_4 is any map satisfying:

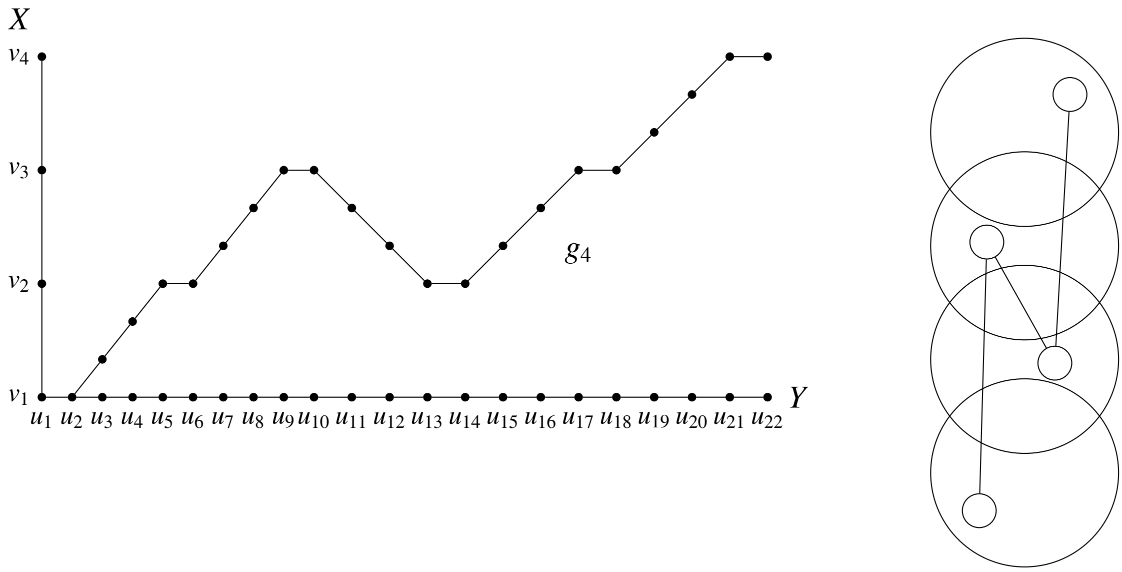


Figure 3.4: Mapping Crookedly for Chain of Size 4

$$g_4(u_{4i-3}) = g_4(u_{4i-3}u_{4i-2}) = g_4(u_{4i-2}) = v_i \in \mathcal{V}(X),\ i \in \{1, 2, 3\}$$

$$g_4(u_{4i-2}u_{4i-1}) = g_4(u_{4i-1}) = g_4(u_{4i-1}u_{4i}) = g_4(u_{4i}) = g_4(u_{4i}u_{4i+2}) = v_iv_{i+1} \in \mathcal{E}(X),\ i \in \{1, 2\}$$

$$g_4(u_{4i+5}) = g_4(u_{4i+5}u_{4i+6}) = g_4(u_{4i+6}) = v_i \in \mathcal{V}(X),\ i \in \{2, 3, 4\}$$

$$g_4(u_{4i+6}u_{4i+7}) = g_4(u_{4i+7}) = g_4(u_{4i+7}u_{4i+8}) = g_4(u_{4i+8}) = g_4(u_{4i+8}u_{4i+9}) = v_iv_{i+1} \in \mathcal{E}(X),\ i \in \{2, 3\}$$

As in the previous construction, g_4 is continuous and D is mapped crookedly in C. We proceed to prove the theorem using induction.

Basis Step. Assume $|\mathcal{V}(X)| = 5$. Let $\mathcal{V}(X) = \{v_1, v_2, v_3, v_4, v_5\}$, so that $C = \{B(v_1), B(v_2), B(v_3), B(v_4), B(v_5)\}$. Let Y be the path graph with $|\mathcal{V}(Y)| = (k_4 + k_3 + k_4) - 4 = (22 + 10 + 22) - 4 = 50 = k_5$. Let $Y_1 = Y(\{u_1, ..., u_{22}\}), Y_2 = Y(\{u_{21}, ..., u_{30}\}, Y_3 = Y(\{u_{29}, ..., u_{50})\}$. Define the map g_5 as follows:

$$g_4 : Y_1 \to X(\{v_1, v_2, v_3, v_4\})$$

$$g_3 : Y_2 \to X(\{v_4, v_3, v_2\})$$

$$g_4 : Y_3 \to X(\{v_2, v_3, v_4, v_5\})$$

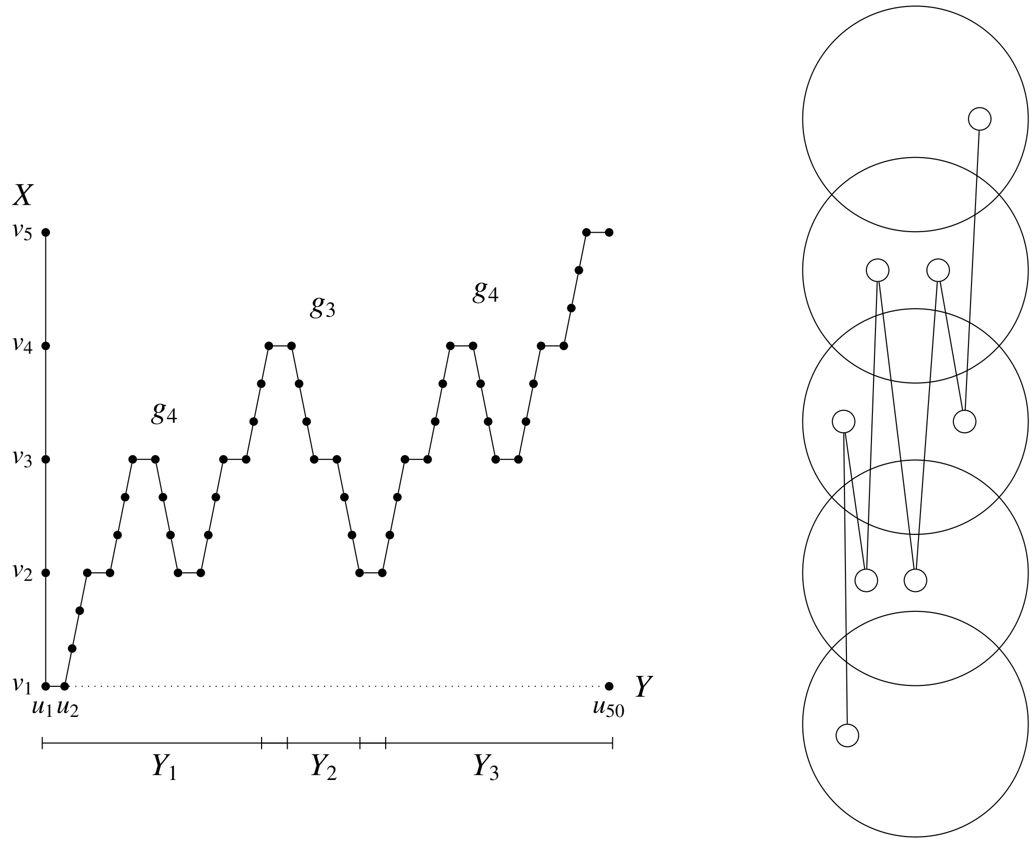


Figure 3.5: Basis Step: Mapping Crookedly for Chain of Size 5

Now we must show that D is mapped crookedly in C. Notice that $C' = \{B(v_1), B(v_2), B(v_3), B(v_4), B(v_5)\}$ is the only subchain of C of size at least 5, so it is the only one we need to consider. Let $D' = \{B(u_p), B(u_{p+1}), ..., B(u_q)\}$ be a subchain of D such that $g_5(B(u_p)) \subset B(v_1)$

and $g_5(B(u_q)) \subset B(v_5)$. Then, u_p is some vertex with $p \in \{1, ..., 4\}$ and u_q is some vertex with $q \in \{47, ..., 50\}$

Choose $u_s = u_{22}$ and $u_t = u_{29}$. Then, $p \leq 4 < 22 = s < t = 29 < 47 \leq q$. The vertex $u_{22} \in Y_1$, so $g_5(B(u_{22})) = g_4(B(u_{22})) = v_3v_4 \subset B(v_4)$. The vertex $u_{29} \in Y_3$, so $g_5(u_{29}) = g_4(u_{29}) = v_2v_3 \subset B(v_2)$. Then, D maps crookedly in C.

Now, we may assume such a Y and g_{n-1} exists for any path graph X with $|\mathcal{V}(X)| = n-1$ and $C = \{B(v_1), B(v_2), ..., B(v_{n-1})\}$ so that Y maps crookedly in X.

Induction Step. Assume $|\mathcal{V}(X)| = n$. Let $\mathcal{V}(X) = \{v_1, v_2, ..., v_n\}$. Let Y be the path graph $|\mathcal{V}(Y)| = (k_{n-1} + k_{n-2} + k_{n-1}) - 4 = k_n$. Let $Y_1 = Y(\{u_1, ..., u_{k_{n-1}}\})$, $Y_2 = Y(\{u_{k_{n-1}-1}, ..., u_{k_{n-1}+k_{n-2}-2}\})$, $Y_3 = Y(\{u_{k_{n-1}+k_{n-2}-3}, ..., u_{2k_{n-1}+k_{n-2}-3}\})$, so that $|V(Y_1)| = k_{n-1}$, $|V(Y_2)| = k_{n-2}$, and $|V(Y_3)| = k_{n-1}$. Define the map g_n as follows:

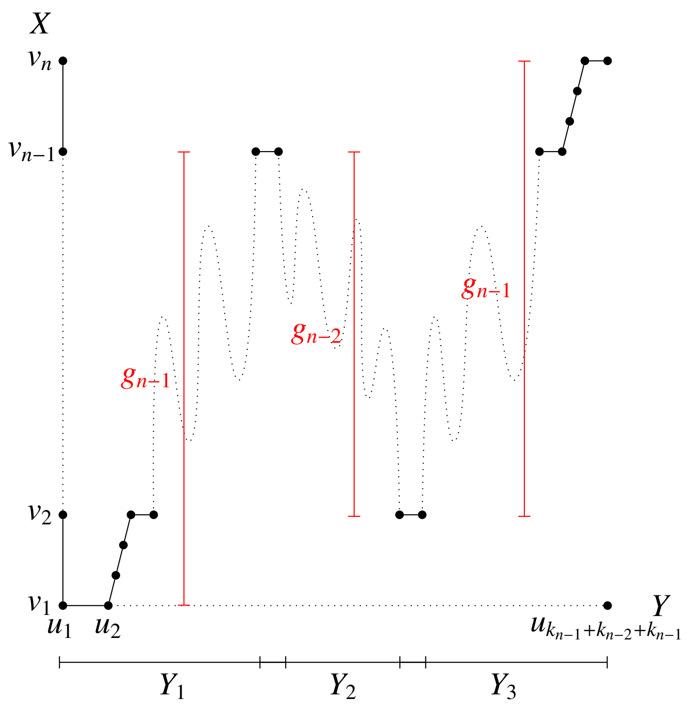


Figure 3.6: Induction Step: Mapping Crookedly for Chain of Size n

- $g_{n-1} : Y_1 \to X(\{v_1, v_2, ..., v_{n-1}\})$
- $g_{n-2} : Y_2 \to X(\{v_{n-1}, v_{n-2}, ..., v_2\})$
- $g_{n-1} : Y_3 \to X(\{v_2, v_3, ..., v_n\})$

Now we must show that D is mapped crookedly in C. Suppose $C' = \{B(v_m), B(v_{m+1}), ..., B(v_l)\}$ is a subchain of C, $|C'| \geq 5$, and $D' = \{B(u_p), B(u_{p+1}), ..., B(u_q)\}$ with $g_n(B(u_p)) \subset B(v_m)$ and $g_n(B(u_q)) \subset B(v_l)$ is a subchain of D. Because X is a path graph and C' must be connected, the subchain C' will be the chain C with some number of links removed from both ends of C.

Case 1. C' is C with at least two links removed from both ends

Then, D' is fully contained in Y_1, Y_2, or Y_3. By induction, we can choose u_s and u_t that satisfy the conditions for D to map crookedly in C.

Case 2. $C' = C$

Then, $C' = \{B(v_1), B(v_2), ..., B(v_n)\}$. Because $g_n(B(u_p)) \subset B(v_1)$, the vertex u_p is some vertex with $p \in \{1, ..., 4\}$. Because $g_n(B(u_q)) \subset B(v_n)$, the vertex u_q is some vertex with $q \in \{k_n - 3, ..., k_n\}$.

Let $u_s = u_{k_{n-1}}$, the last vertex in Y_1, so that $g_n(B(u_{k_{n-1}}) = \{v_{n-2}v_{n-1}, v_{n-1}\} \subset B(v_{n-1}) = B(v_{l-1})$. Let $u_t = u_{k_{n-1}+k_{n-2}-2}$, the last vertex in Y_2, so that $g_n(B(u_{k_{n-1}+k_{n-2}-2})) = \{v_2, v_2v_3\} \subset B(v_2) = B(v_{m+1})$. And $p \leq 4 < k_{n-1} = s < t = k_{n-1} + k_{n-2} - 2 < k_{n-3} \leq q$, so that $p < s < t < q$.

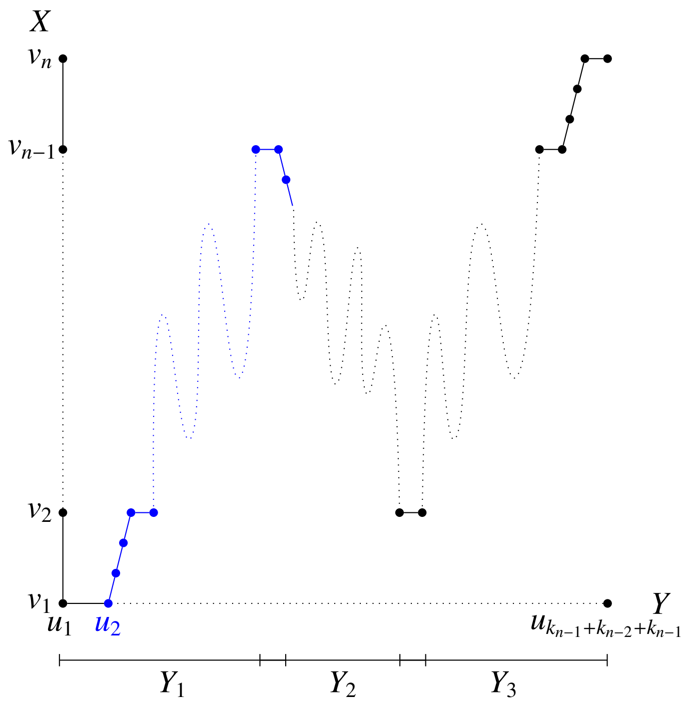
Case 3. C' is C with one link removed from one end

Assume the last link is removed, so $C' = \{B(v_1), ...B(v_{n-1})\}$. Then, either 1. or 2. is true:

1. D' is fully contained in Y_1

2. $D' \cap Y_1 \neq \emptyset, D' \cap Y_2 \neq \emptyset, |\mathcal{V}(D') \cap \mathcal{V}(Y_2)| \leq 4, D' \cap Y_3 = \emptyset$

If 1. is true, then by induction, we can choose u_s and u_t that satisfy the conditions for D to map crookedly in C. If 2. is true, then D' is almost fully contained in Y_1, but contains at most two links in Y_2 that are not in Y_1. One such possibility is shown in blue in the figure below. Then, we can choose the same u_s and u_t as in 1. If instead $C' = \{B(v_2), ..., B(v_n)\}$, then the proof is similar.



Case 4. C' is C with one link removed from both ends

Then, $C' = \{B(v_2), ..., B(v_{n-1})\}$ and one of the following must be true:

1. D' is fully contained in Y_1
2. D' is fully contained in Y_2
3. D' is fully contained in Y_3
4. $D' \cap Y_1 \neq \emptyset$, $D' \cap Y_2 \neq \emptyset$, $|\mathcal{V}(D') \cap \mathcal{V}(Y_2)| \leq 4$, $D' \cap Y_3 = \emptyset$
5. $D' \cap Y_3 \neq \emptyset$, $D' \cap Y_2 \neq \emptyset$, $|\mathcal{V}(D') \cap \mathcal{V}(Y_2)| \leq 4$, $D' \cap Y_1 = \emptyset$
6. $D' \cap Y_1 \neq \emptyset$, $D' \cap Y_2 \neq \emptyset$, $D' \cap Y_3 \neq \emptyset$

If 1., 2., or 3. is true, then by induction, we can choose u_s and u_t that satisfy the conditions for D to map crookedly in C. If 4. is true, then D' is almost fully contained in Y_1 but contains at most two links in Y_2 that are not in Y_1. As in Case 3., we may choose the same u_s and u_t as in 1. If 5. is true, then D' is almost fully contained in Y_3 but contains at most two links in Y_2 that are not in Y_3. We may choose the same u_s and u_t as in 3. Finally, if 6. is true, then we may choose the same u_s and u_t as in Case 2.

Case 5. C' is C with one link removed from one end and two links removed from the other end

Assume $C' = \{B(v_2), ..., B(v_{n-2})\}$. Then, one of the following must be true:

1. D' is fully contained in Y_1
2. D' is fully contained in Y_2
3. D' is fully contained in Y_3
4. $D' \cap Y_2 \neq \emptyset$, $D' \cap Y_3 \neq \emptyset$, $|\mathcal{V}(D') \cap \mathcal{V}(Y_3)| \leq 4$, $D' \cap Y_1 = \emptyset$
5. $D' \cap Y_3 \neq \emptyset$, $D' \cap Y_2 \neq \emptyset$, $|\mathcal{V}(D') \cap \mathcal{V}(Y_2)| \leq 4$, $D' \cap Y_1 = \emptyset$

If 1., 2., or 3. is true, then by induction, we can choose u_s and u_t that satisfy the conditions for D to map crookedly in C. If 4. is true, then D' is almost fully contained in Y_2 but contains at most two links in Y_3 that are not in Y_2. We may choose the same u_s and u_t as in 2. If 5. is true, then D' is almost fully contained in Y_3 but contains at most two links in Y_2 that are not in Y_3. We may choose the same u_s and u_t as in 3. If instead $C' = \{v_3, ..., v_{n-1}\}$, then the proof is similar. □

Then we can construct an inverse limit space $X = \varprojlim_T \{X_i, f_i\}_{i=1}^{\infty}$ where X_1 is a path graph of any size. Choose X_2 and f_1 to be constructed as Y and g_n are in Theorem 1. Continue this process to choose each X_i and f_i. In the next section, we will show that an X constructed this way is hereditarily indecomposable.

3.2 Properties of the Space

Consider the traditional inverse limit space $X = \varprojlim_T \{X_i, f_i\}_{i=1}^{\infty}$ where X_1 is a path graph and $X_2, X_2, ...$ and $f_1, f_2, ...$ are chosen using the construction in Theorem 1. Notice that the size of X_1 is not specified, so that this is a family of inverse limit spaces where X_{i+1} is mapped crookedly in X_i for all i. Call this family $\mathcal{X}$. We prove some observations about $X \in \mathcal{X}$.

Observation 1(a). If H and K are subcontinua of $X \in \mathcal{X}$ and $H \cup K$ is a continuum, then either $H \subseteq K$ or $K \subseteq H$.

Proof. Assume by way of contradiction that there is some $h = (h_i)_{i=1}^{\infty} \in H - K$ and $k = (k_i)_{i=1}^{\infty} \in K - H$. Let L_h^i be the link in X_i containing h_i and L_k^i be the link in X_i containing k_i. If h_i is an edge, then it is contained in two links of X_i. If this is the case, just choose either link to be L_h^i. Do the same for L_k^i.

Let X_j be the first space where $L_h^j \cap \pi_j(K) = \emptyset$ and $L_k^j \cap \pi_j(H) = \emptyset$. We know such a j exists, if not, then every open set containing h_i contains a point of $\pi_i(K)$. Therefore every open set containing h intersects K, so that h is a limit point of K. But $h \notin K$, contradicting that K is closed. Note that $L_h^j \cap L_k^j = \emptyset$, so that there is at least one link between them.

Let X_m be the first space after X_j where at least one of the links next to L_h^m contains a point of H that is not h. First, we must show such an m exists. Choose $h' = (h_i')_{i=1}^{\infty} \in H - \{h\}$. Let X_m be the first space where $m > j$ and $h_m \neq h_m'$. Because H is connected, $\pi_m(H)$ must also be connected. Then, there is some sequence of links $L_h^m, L_{2_h}^m, L_{3_h}^m, ..., L_{h'}^m$ that start at h_m, end at h_m', and are fully contained in $\pi_m(H)$. Then the link $L_{2_h}^m$ is the one we are looking for. Assume L_2^m occurs after L_h^m in the graph X^m. We use a similar proof to choose a link $L_{2_k}^m$ that contains a point of $\pi_m(K)$ and assume it occurs before L_k^m in the graph X^m. Let $C' = \{L_h^m, L_{2_h}^m, ..., L_{2_k}^m, L_k^m\}$. Without our assumptions, the first two links might be switched and the last two links might be switched and the proof would be the same. Because there was at least one link between L_h^j and L_k^j and because of the level pieces of the maps, it must be true that $|C'| \geq 6 > 5$. We note that without the level pieces, we may have $|C'| < 5$, and would not be able to get crookedness.

Now, consider the pre-images of the four links $L_h^m, L_{h_2}^m, L_{k_2}^m, L_k^m$. According to Incidence Property 2, we are able to find a sequence of components of these pre-images that are incident and start in L_h^{m+1} and end in L_k^{m+1}. Let $D' = \{L_h^{m+1}, L_2^{m+1}, L_3^{m+1}, ..., L_k^{m+1}\}$, this sequence of incident components. By our construction, D maps crookedly in C, so that there is some links L_s^{m+1} and L_t^{m+1} in D' such that $f_m(L_s^{m+1}) \subseteq L_{k_2}^m$ and $f_m(L_t^{m+1}) \subseteq L_{h_2}^m$.

But then we can write $H = (H \cap \cup_{i=1}^{s} \overleftarrow{L_i^{m+1}}) \cup (H \cap \cup_{i=s}^{|\mathcal{V}(X_{m+1})|} \overleftarrow{L_i^{m+1}})$, so that H is disconnected, contradicting that H is a continuum. □

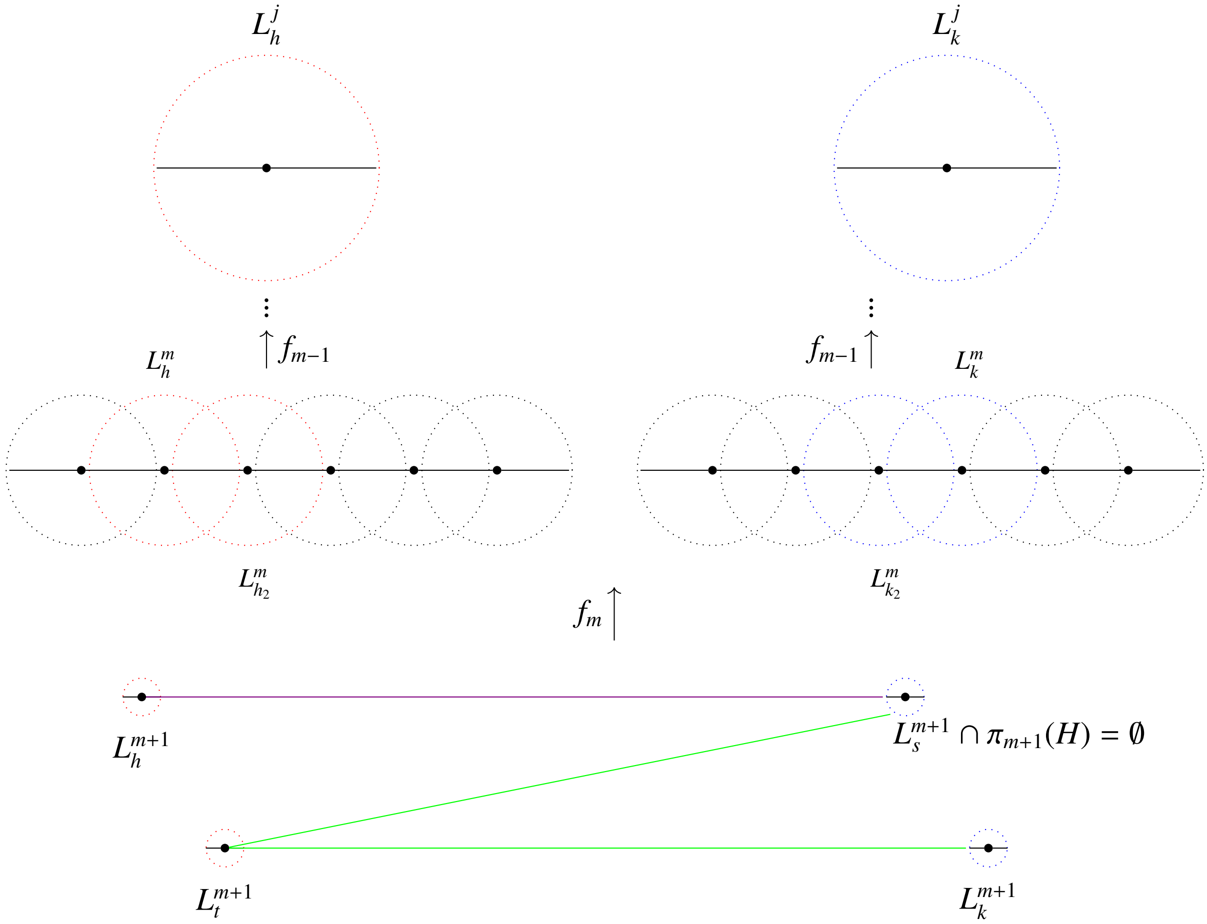


Figure 3.7: Indecomposability of $X \in \mathcal{X}$

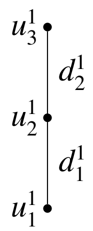
Observation 1(b). Any $X \in \mathcal{X}$ is indecomposable and hereditarily indecomposable.

Proof. Follows from Observation 1(a). □

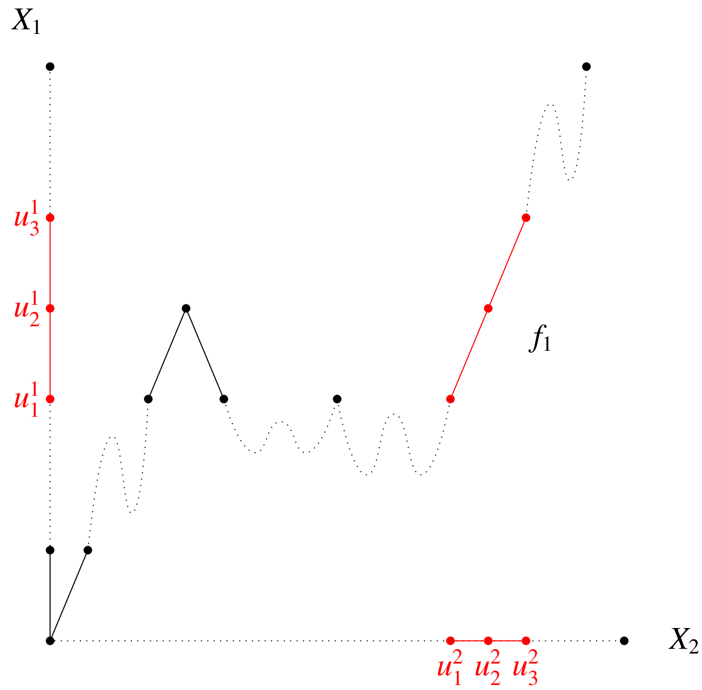
Then X is hereditarily indecomposable but is not Hausdorff because it is a traditional inverse limit of path graphs, as shown in Section 2.2, so it is not metric and therefore not a Pseudo-arc. If we attempt to find a homeomorphic generalized inverse limit where the first space is a metric arc using one of the given methods, we still will not have Hausdorffness because of the level pieces of the maps, as shown in Section 2.2. We then wonder, if use the same construction described in the Theorem but without the level pieces, will we get a Pseudo-arc?

Observation 2(a). If X is an inverse limit of path graphs and each f_i has no level pieces, then X is not hereditarily indecomposable.

Proof. We will choose a set $\{u_1, d_1, u_2, d_2, u_3\} \subset X$ that is decomposable. Choose any three vertices in X_1 that are adjacent and the two edges between them, $u_1^1, d_1^1, u_2^1, d_2^1, u_3^1$.



We claim that there are three vertices in X_2 that are adjacent along with the edges between them, $u_1^2, d_1^2, u_2^2, d_2^2, u_3^2$, so that $f_1(u_j^2) = u_j^1$ and $f_1(d_j^2) = d_j^1$. We prove this claim by noticing the bonding map f_1 is continuous, onto, and has no level pieces and the path graph X_2 is connected, so that the map f_1, assuming it starts at the first vertex, must eventually reach the last vertex, as shown in the figure below.



Using the same proof, we can find three vertices in X_3 that are adjacent along with the edges between them, $u_1^3, d_1^3, u_2^3, d_2^3, u_3^3$, so that $f_2(u_j^3) = u_j^2$ and $f_2(d_j^3) = d_j^2$. We continue this process and let $u_j = (u_j^i)_{i=1}^{\infty}$ and $d_j = (d_j^i)_{i=1}^{\infty}$. Then the set $\{u_1, d_1, u_2, d_2, u_3\} = H \cup K$, where $H = \{u_1, d_2, u_2\}$ and $K = \{u_2, d_2, u_3\}$.

□

Observation 2(b). If we remove the level pieces in the construction of X, then X will not be hereditarily indecomposable.

Proof. The proof follows from Observation 2(a). □

So, if we try to remove the level pieces in our construction, we lose the property of hereditary indecomposability.

However, we notice that there is only one type of set causing X to be non-Hausdorff, a pair $\{(v_i)_{i=1}^{\infty}, (e_i)_{i=1}^{\infty}\}$ that cannot be separated. We define a *two-point incidence set* to be a set $\{(v_i)_{i=1}^{\infty}, (e_i)_{i=1}^{\infty}\}$ such that v_i is incident to e_i for all i or $v_i = e_i$ for all $i < k$ and v_i is incident to e_i for all $i \geq k$.

Suppose $\{v = (v_i)_{i=1}^{\infty}, e = (e_i)_{i=1}^{\infty}\}$ is a two-point incidence set and v_i is incident to e_i for $i > k$. We note that one point in the set, say v, is such that v_i is a vertex for $i > k$ and similarly e_i is an edge for $i > k$. This follows from the fact that a component in the pre-image of v_k has a vertex at each end and a component in the pre-image of e_k has an edge at each end, so that the only way v_{k+1} can be incident to e_{k+1} is if v_{k+1} is a vertex and e_{k+1} is an edge.

These two-point incidence sets can also be given one of two classifications: a *left two-point incidence set* or a *right two-point incidence set*. A left two-point incidence set is an incidence set $\{(v_i)_{i=1}^{\infty}, (e_i)_{i=1}^{\infty}\}$ where e_i precedes v_i for all $i \geq k$. A right two-point incidence set is an incidence set $\{(v_i)_{i=1}^{\infty}, (e_i)_{i=1}^{\infty}\}$ where v_i precedes e_i for all $i \geq k$. The proof that any $X \in \mathcal{X}$ contains these two-point incidence sets follows from Figure 2.2 in the proof of the non-Hausdorffness of a traditional inverse limit of path graphs.

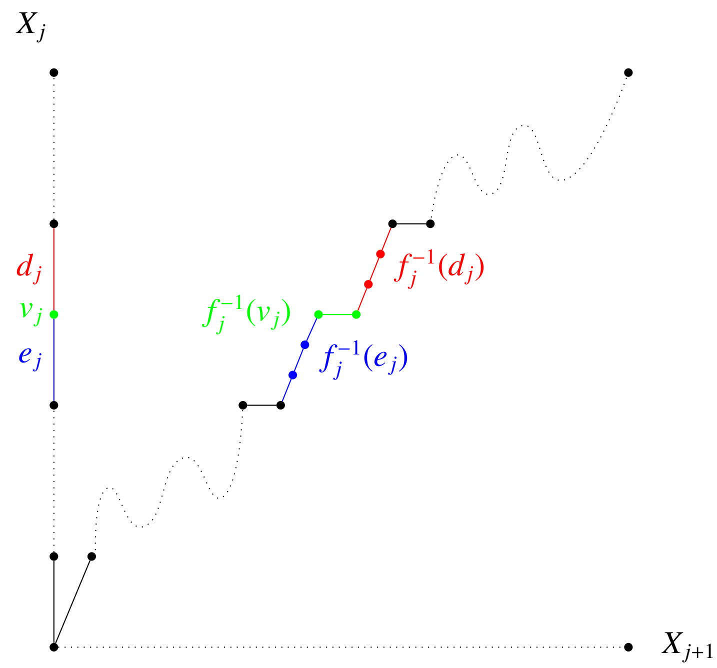
We notice some interesting properties of these two-point incidence sets in $X \in \mathcal{X}$.

Observation 3. Each two-point incidence set is unique, meaning if $\{v, e\}$ is a two-point incidence set in X then there is no other two-point incidence set containing v or e.

Proof. Let $v = (v_i)_{i=1}^{\infty}, e = (e_i)_{i=1}^{\infty}$ so that $\{v, e\}$ is a two-point incidence set of $X \in \mathcal{X}$. First, we show that v cannot be in any other two-point incidence set. Let $d = (d_i)_{i=1}^{\infty} \neq e$ and suppose, by way of contradiction, that $\{v, d\}$ is a two-point incidence set. Because $e \neq d$, for some first j, $e_j \neq d_j$.

Case 1: v_j *is incident to* e_j *and* v_j *is incident to* d_j

Because $e_j \neq d_j$, it must be true that $e_{j+1} \neq d_{j+1}$. By the definition of a two-point incidence set, v_{j+1} is incident to both e_{j+1} and d_{j+1}. But because of the shape of the maps f_i, one vertex of $f_j^{-1}(v_j)$ can only be incident to a component of $f_j^{-1}(e_j)$ or $f^{-1}(d_j)$, but not both, contradicting that v_{j+1} is incident to e_{j+1} and d_{j+1}. This is shown in the figure below. Notice that each vertex of $f_j^{-1}(v_j)$ is incident to one edge of $f_j^{-1}(e_j)$ or $f_j^{-1}(d_j)$, but not both.



Case 2: $v_j = e_j$ and v_j is incident to d_j

But $v \neq e$, so for some $k > j$, $v_k \neq e_k$. Then, v_k is incident to both e_k and d_k and $e_k \neq d_k$. Then, the proof follows as in Case 1.

Now, we show that e cannot be in any other two-point incidence set. Let $u = (u_i)_{i=1}^{\infty}$ and suppose, by way of contradiction, that $\{u, e\}$ is a two-point incidence set. Because $v \neq u$, for some first j, $v_j \neq u_j$.

Case 1: e_j is incident to v_j and e_j is incident to u_j

Because $v_j \neq u_j$, it must be true that $v_{j+1} \neq u_{j+1}$. By definition of a two-point incidence set, e_{j+1} is incident to both v_{j+1} and u_{j+1}. But because of the shape of the maps f_i, one edge of $f_j^{-1}(e_j)$ can only be incident to a component of $f_j^{-1}(v_j)$ or $f_j^{-1}(u_j)$, but not both, contradicting that e_{j+1} is incident to v_{j+1} and u_{j+1}.

Case 2: $e_j = v_j$ and e_j is incident to u_j

But $e \neq v$, so for some $k > j$, $e_k \neq v_k$. Then, e_k is incident to both v_k and u_k and $v_k \neq u_k$. Then, the proof follows as in Case 1. □

Observation 4. Each two-point incidence set is closed in X.

Proof. Let $\{v, e\}$ be a two-point incidence set and let $p = (p_i)_{i=1}^{\infty} \in X$, $p \neq v, e$. Then, by Observation 2, p does not form a two-point incidence set with v or e. Then, for some j, $p_j \neq v_j$

and p_j is not incident to v_j. Similary, for some k, $p_k \neq e_k$ and p_k is not incident to e_k. Let $l = \max\{j, k\}$, so that p_l is not incident to v_l and is not incident to e_l.

If p_l is an edge, then let $U = \overleftarrow{\{p_l\}}$ so that $p \in U, v, e \notin U$. If p_l is a vertex, then let $U = \overleftarrow{B^l(p_l)}$. Then $\{v, e\}$ contains all of its limit points, and is therefore closed. □

Let $M = \{x \in X : x$ is in some two-point incidence set$\}$.

Observation 5. The set M is dense in X.

Proof. Let $p = (p_i)_{i=1}^{\infty} \in X - M$ and U be an open set containing p. Then $U = \overleftarrow{U_j}$ some open $U_j \subset X_j$. We construct two points, v and e, that are in U and form a two-point incidence set. Let $v_i = p_i = e_i$ for all $i \leq j$. Notice that $f_j^{-1}(U_j)$ must contain an incident vertex and edge, (in fact, it contains several incident pairs). Let v_{j+1} and e_{j+1} be one such pair. Similarly, we can choose v_{j+2} and e_{j+2} to be such a pair in $f_{j+1}^{-1}(\{v_{j+1}, e_{j+1}\})$, and continue this process so that v_i is incident to e_i for all $i > j$. Then, $U \cap M \neq \emptyset$. □

3.3 The Family $\mathcal{X}$ and the Pseudo-arc

In this section, we circle back to our original question: can we use our methods to represent a Pseudo-arc as an inverse limit of path graphs? In the previous section, we have shown that our given construction does not result in a Psuedo-arc, but we wonder what relationship the family $\mathcal{X}$ has with the Pseudo-arc.

We notice that there are two ways to use any $X \in \mathcal{X}$ to create a Pseudo-arc. First, let $Y = \{\{v, e\} \subset X : v, e$ form and incidence set in $X\} \cup \{\{p\} : p \in X - M\}$. Certainly X/Y is separable and chainable as each of the factor spaces are finite. If it can be shown that X/Y is metric and hereditarily indecomposable, then X/Y is a Pseudo-arc.

Proof of Hausdorffness of X/Y. Let $p_1, p_2 \in X/Y, p_1 \neq p_2$.

Case 1: $p_1 = \{(x_i)_{i=1}^{\infty}\}, p_2 = \{(x'_i)_{i=1}^{\infty}\}$

Then, for some j, x_j and x'_j are not incident in X_j. It is possible that the basic open sets intersect in X_j, so we instead consider the basic open sets $B^{j+1}(x_{j+1})$ and $B^{j+1}(x'_{j+1})$ so that they do not intersect. Then $U = (X/Y) \cap 2^{\overleftarrow{B^{j+1}(x_{j+1})}}$ and $V = (X/Y) \cap 2^{\overleftarrow{B^{j+1}(x'_{j+1})}}$ separate p_1 and p_2. Notice that we use the power set of these open sets because points in X/Y are sets of X.

Case 2: $p_1 = \{(x_i)_{i=1}^{\infty}\}, p_2 = \{(v_i)_{i=1}^{\infty}, (e_i)_{i=1}^{\infty}\}$

Then, for some j_1, x_{j_1} is not incident to $v_{j_1} \in X_{j_1}$ and for some j_2, x_{j_2} is not incident to e_{j_2} in X_{j_2}. Let $j = max(j_1, j_2)$ so that x_j is not incident to v_j or e_j in X_j. However, the basic open set containing x_j might intersect the basic open set containing v_j, so instead we consider the basic open sets in X_{j+1}. Let $U = (X/Y) \cap 2^{\overleftarrow{B^{j+1}(x_{j+1})}}$ and $V = (X/Y) \cap 2^{\overleftarrow{B^{j+1}(v_{j+1})} \cup \overleftarrow{B^{j+1}(e_{j+1})}}$, so that U and V separate p_1 and p_2.

Case 3: $p_1 = \{(v_i)_{i=1}^{\infty}, (e_i)_{i=1}^{\infty}\}$ and $p_2 = \{(u_i)_{i=1}^{\infty}, (d_i)_{i=1}^{\infty}\}$

By Observation 2, the sets p_1 and p_2 are unique, so for some j_1, v_{j_1} is not incident to d_{j_1} and for some j_2, e_{j_2} is not incident to u_{j_2}. Let $j = max(j_1, j_2)$. However, the basic open set containing v_j and the basic open set containing u_j may still intersect at an edge. Instead, we consider the basic open sets in X_{j+1}. Let $U = (X/Y) \cap 2^{\overleftarrow{B^{j+1}(v_{j+1})}}$ and $V = (X/Y) \cap 2^{\overleftarrow{B^{j+1}(u_{j+1})}}$, so that U and V separate p_1 and p_2. □

To show X/Y is hereditarily indecomposable, we use the same proof as in Observation 1(a). We notice that X/Y is compact, Hausdorff, and second-countable, so it is metric. Then X/Y is a Pseudo-arc.

We could also consider the subspace $X-E$ of X, where $E = \{e :$ the edge points of the incidence sets$\}$ [1]. We can show that X/Y is homeomorphic to $X - E$ using the homeomorphism $h : X/Y \to X - E$:

$$h(\{x\}) = x, \text{ if } x \text{ is not in any two-point incidence set of } X$$

$$h(\{v, e\}) = v, \text{ if } v, e \text{ form a two-point incidence set in } X$$

Then, $X - E$ is also a Pseudo-arc. This leads us to two more observations of the spaces $X \in \mathcal{X}$.

Observation 6. The Pseudo-arc is a continuous image of any $X \in \mathcal{X}$.

Proof. We define a continuous map $f : X \to X/Y$. For $x \in X$, let $f(x) =$ the set containing x in X/Y. □

Observation 7. Any $X \in \mathcal{X}$ is a continuous image of the Pseudo-arc.

Proof. Let $X = \varprojlim_T \{X_i, f_i\}_{i=1}^{\infty} \in \mathcal{X}$. We begin by strategically defining an inverse limit space that is a Pseudo-arc. Let $P = \varprojlim_T \{P_i, g_i\}_{i=1}^{\infty}$ where $P_i = [0,1]$ with the usual metric topology for all i, and g_i is the piece-wise linear function with the same shape as f_i. Notice that if we define a chain $C^i = \{[0, \frac{1}{|\mathcal{V}(X_i)|-1}), (\frac{1}{|\mathcal{V}(X_i)|-1}, \frac{2}{|\mathcal{V}(X_i)|-1}), ..., (\frac{|\mathcal{V}(X_i)|-2}{|\mathcal{V}(X_i)|-1}, 1]\}$ covering each P_i, then $\{C^i\}_{i=1}^{\infty}$ is a sequence of chains such that each C^{i+1} is a proper refinement of C^i, each C^{i+1} maps crookedly in C^i, and each C^i is a $\frac{1}{i}$-chain, so that P is a Pseudo-arc.

Define a map $f : P \to X$ as follows: $(p_i)_{i=1}^{\infty} \to (x_i)_{i=1}^{\infty}$ where

$$\text{if } p_i = \frac{n}{|\mathcal{V}(X_i)| - 1}, n \in \{0, 1, ..., |\mathcal{V}(X_i)| - 1\} \text{ then } x_i = v_{n+1} \in \mathcal{V}(X_i)$$

$$\text{if } p_i \in \left(\frac{n}{|\mathcal{V}(X_i)| - 1}, \frac{n+1}{|\mathcal{V}(X_i)| - 1}\right), n \in \{0, 1, ..., |\mathcal{V}(X_i)| - 2\} \text{ then } x_i = v_{n+1}v_{n+2} \in \mathcal{E}(X_i)$$

Let U be an open set of X. Then $U = \overleftarrow{U_j}$ where U_j is some open set of X_j. Because U_j is open, it must be a union of components where each component has an edge at each end. Let $V = (\cup_{v_n \in \mathcal{V}(U_j)} \frac{n-1}{|V(X_j)|-1}) \cup (\cup_{v_n v_{n+1} \in \mathcal{E}(U_j)} \left(\frac{n-1}{|\mathcal{V}(X_i)|-1}, \frac{n}{|\mathcal{V}(X_i)|-1}\right))$. V is open, as it is a union of points and open intervals where an open interval is always on both sides of a point. Then $f^{-1}(U) = f^{-1}(\overleftarrow{U_j}) = \overleftarrow{V}$, which is open in P. □

Chapter 4

Conclusion and Open Problems

In this dissertation, we explore several methods for representing an inverse limit of metric arcs as a generalized inverse limit of path graphs where the first factor space is a metric arc. In Chapter 2, we present some important properties, referred to as Incidence Properties, of a continuous map from one path graph to another and use them to show that an inverse limit of path graphs is not Hausdorff. This is unfortunate, as we would like the inverse limit to be a metric space. We remedy this issue by instead considering a generalized inverse limit space.

Starting with some inverse limit of metric arcs, we present two methods to construct a homeomorphic generalized inverse limit where the first factor space is a metric arc with the usual topology and every other space is a path graph with the graph topology. For these methods to be successful, the bonding maps must meet certain conditions, H1. and H2. This limits the use of these methods. We outline one more method that allows infinite graphs, though this method also requires that the bonding maps meet certain limiting conditions.

This leads us to the main question: can a Pseudo-arc be represented as a generalized inverse limit of path graphs using methods similar to the ones in Chapter 2? In Chapter 3, we attempt to construct such an inverse limit of path graphs. This construction results in a family of hereditarily indecomposable inverse limit of path graphs. However, these spaces are not Hausdorff, and therefore not a Pseudo-arc. The other sections of Chapter 3 analyze properties of this family of spaces and their relationship with the Pseudo-arc.

In terms of future research, we first consider a continuation of the work in Chapter 3 and wonder what else the spaces in $\mathcal{X}$ have in common with the Pseudo-arc. The Pseudo-arc P is homogeneous, so for every $p, q \in P$, there is a homeomorphism $h : P \to P$ such that $h(p) = q$

[3]. However, a space $X \in \mathcal{X}$ has several different types of points. X contains points in left two-point incidence sets, right two-point incidence sets, and points that are not in any incidence set. We conjecture that every $X \in \mathcal{X}$ is non-homogeneous but that for any p and $q \in X$ that are both in left or both in right two-point incidence sets, there is a homeomorphism $h : X \to X$ so that $h(p) = q$.

To support this conjecture, we notice that if $h : X \to X$ is continuous and $\{v, e\}$ is a two-point incidence set, then $h(\{v, e\}) = \{u, d\}$ where $\{u, d\}$ is a two-point incidence set.

Proof. Note that because $\{v, e\}$ is a two-point incidence set, every open set containing v must also contain e. Let $h(v) = u$ and $h(e) = d$. Assume, by way of contradiction, that u and d do not form a two-point incidence set. Then, there is some j so that $B^j(u_j) \cap B^j(d_j) = \emptyset$. Consider the open set $U = \overleftarrow{B^j(u_j)}$. Then $v \in h^{-1}(U)$ but $e \notin h^{-1}(U)$, a contradiction. So the set $\{u, d\}$ must be a two-point incidence set. □

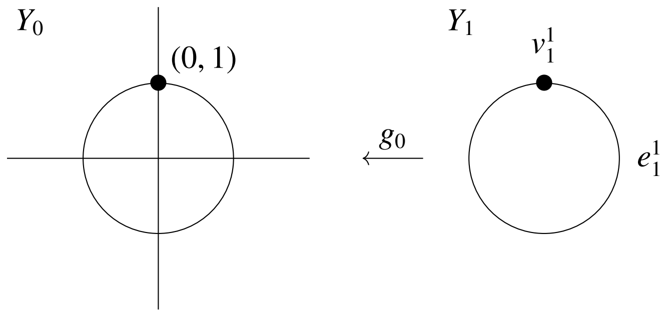
Certainly it is worth exploring what other properties the spaces $X \in \mathcal{X}$ have.

More broadly, we wonder what other spaces can be represented using graphs. In [2], a unit triod is represented using similar techniques as in Chapter 2. Below, we give an example of a generalized inverse limit of graphs where the first space is a metric space that is homeomorphic to the dyadic solenoid.

Example. Let $X = \varprojlim_T \{X_j, f_j\}_{j=1}^{\infty}$ where each X_i is the circle $\{z = \cos\theta + i\sin\theta \in \mathbb{C} : \theta \in [0, 2\pi]\}$ and each $f_i : X_{i+1} \to X_i$ is $f_i = z^2$, so that X is the dyadic solenoid. Let $Y = \varprojlim_G \{Y_j, g_j\}_{j=0}^{\infty}$ be the generalized inverse limit space with $Y_0 = \{z = \cos\theta + i\sin\theta \in \mathbb{C} : \theta \in [0, 2\pi]\}$ and Y_j be a cycle graph with $|\mathcal{V}(Y_j)| = 2^j$ so $\mathcal{V}(Y_j) = \{v_1^j, v_2^j, ..., v_{2^j}^j\}$ and $\mathcal{E}(Y_j) = \{e_1^j, e_2^j, ..., e_{2^j}^j\}$. Define the map $g_0 : Y_1 \to Y_0$ as follows:

$$g_0(v_1^1) = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)$$

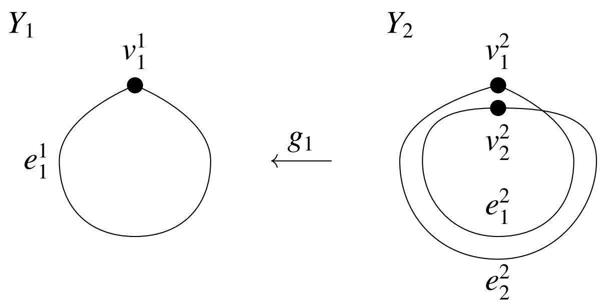
$$g_0(e_1^1) = \left\{z = \cos\theta + i\sin\theta \in \mathbb{C} : \theta \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, 2\pi\right]\right\}$$

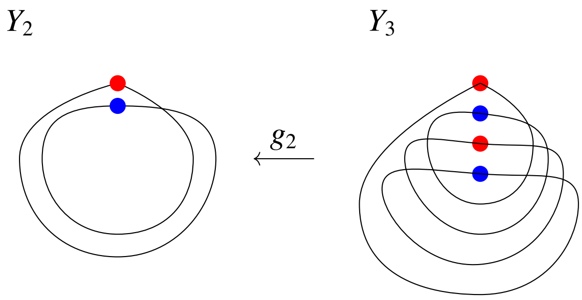


Define the map $g_j : Y_{j+1} \to Y_j$ as follows:

$$g_j(v_k^{j+1}) = g_j(v_{k+2^{j-1}}^{j+1}) = \{v_k^j\}$$

$$g_j(e_k^{j+1}) = g_j(e_{k+2^{j-1}}^{j+1}) = \{e_k^j\}$$





In the figure above, g_2 maps the red vertices in Y_3 to the red vertex in Y_2 and does the same for the blue vertices.

We can show that X and Y are homeomorphic. In general, if X is the n-adic solenoid, we can let $Y = \varprojlim_G \{Y_j, g_j\}_{j=0}^{\infty}$ be the generalized inverse limit space with $Y_0 = \{z = \cos\theta + i\sin\theta \in \mathbb{C} : \theta \in [0, 2\pi]\}$ and Y_j be a cycle graph with $|\mathcal{V}(Y_j)| = n^j$ so $\mathcal{V}(Y_j) = \{v_1^j, v_2^j, ..., v_{n^j}^j\}$ and

$\mathcal{E}(Y_j) = \{e_1^j, e_2^j, ..., e_{n^j}^j\}$. We define the map $g_j : Y_{j+1} \to Y_j$ as follows:

$$g_j(v_k^{j+1}) = g_j(v_{k+n^{j-1}}^{j+1}) = ... = g_j(v_{k+(n-1)n^{j-1}}^{j+1}) = \{v_k^j\}, 1 \leq k \leq n^{j-1}$$

$$g_j(e_k^{j+1}) = g_j(e_{k+n^{j-1}}^{j+1}) = ... = g_j(e_{k+(n-1)n^{j-1}}^{j+1}) = \{e_k^j\}, 1 \leq k \leq n^{j-1}$$

So that Y is homeomorphic to the n-adic solenoid. If instead the solenoid wraps $n_1, n_2, ...$ times, then we just replace the appropriate n with n_i.

If we consider the Pseudo-solenoid, we conjecture that as in Chapter 3 we will not be able to use our methods to construct an inverse limit of graphs homeomorphic to the Pseudo-solenoid. Finally, we consider infinite graphs with a natural ordering with the graph topology and wonder how they might expand our methods for representing spaces using inverse limits of graphs. In particular, whether we can use infinite graphs to represent the Pseudo-arc or Pseudo-solenoid.

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