Vietoris-Rips and Čech Complexes of Certain Finite Metric Spaces

by

Naga Chandra Padmini Nukala

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Approved by

Ziqin Feng, Chair, Associate Professor of Mathematics
Hannah Alpert, Assistant Professor of Mathematics
Joseph Briggs, Assistant Professor of Mathematics
Michel Smith, Professor of Mathematics
Murali Dhanasekaran, Professor of Harrison College of Pharmacy
Dean Flowers, Dean of the Graduate School and Professor of Mechanical Engineering

Abstract

We examine the homotopy types of Vietoris-Rips complexes on different collections of subsets of $[m] = \{1, 2, \ldots, m\}$ equipped with the symmetric difference metric. More specifically, we prove that the Vietoris-Rips complexes $\mathcal{VR}(\mathcal{F}_{\mathrm{n}}^{\mathrm{m}}; 2)$ and $\mathcal{VR}(\mathcal{F}_{\mathrm{n}}^{\mathrm{m}} \cup \mathcal{F}_{\mathrm{n+1}}^{\mathrm{m}}; 2)$ are either contractible or homotopy equivalent to a wedge sum of S^2's. We also show that the complexes $\mathcal{VR}(\mathcal{F}_{\mathrm{n}}^{\mathrm{m}} \cup \mathcal{F}_{\mathrm{n+2}}^{\mathrm{m}}; 2)$ and $\mathcal{VR}(\mathcal{F}_{\mathrm{n}\ \preceq A}^{\mathrm{m}}; 2)$ are homotopy equivalent to a wedge sum of S^3's. We found a more intuitive proof of the result of Adamamszek and Adams in [2] about Vietoris-Rips complexes of hypercube graphs with scale 2.

We also define Čech complexes on the finite union of finite metric spaces at scales $2, 3$ equipped with symmetric difference metric. We prove that the Čech complexes $\mathcal{N}(\mathcal{F}_{\mathrm{n}}^{\mathrm{m}} \cup \mathcal{F}_{\mathrm{n+1}}^{\mathrm{m}} \cup \mathcal{F}_{\mathrm{n+2}}^{\mathrm{m}}; 2)$ and $\mathcal{N}(\mathcal{F}_{\mathrm{n}}^{\mathrm{m}} \cup \mathcal{F}_{\mathrm{n+1}}^{\mathrm{m}} \cup \mathcal{F}_{\mathrm{n+2}}^{\mathrm{m}}; 3)$ are either contractible or homotopy equivalent to a wedge sum of S^2's and that the complex $\mathcal{N}(\mathcal{F}_{\mathrm{n}}^{\mathrm{m}} \cup \mathcal{F}_{\mathrm{n+1}}^{\mathrm{m}}; 2)$ is homotopy equivalent to a wedge sum of S^1's. We also establish that the complex $\mathcal{N}(\mathcal{F}_0^m \cup \mathcal{F}_1^m \cup \mathcal{F}_2^m \cup \mathcal{F}_3^m; 3)$ for $n = 0$ is homotopy equivalent to a wedge sum of S^4's. We provide (inductive) formulae for all these homotopy types.

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Chapter 1

Introduction

Topological Data Analysis (TDA) is an emerging field that leverages techniques from topology to study the shape and structure of data. Rather than working with the raw data directly, TDA aims to extract the shape and structure of the data. The key idea behind TDA is to use topological features, such as connected components, loops, voids, and higher-dimensional structures, to gain insights into the underlying data space. A natural way to highlight some topological structure out of data is to "connect" data points that are close to each other in order to exhibit a global continuous shape underlying the data ([19]). Connecting pairs of nearby data points by edges leads to the standard notion of neighboring graph from which the connectivity of the data can be analyzed ([19]).

The key idea is just not to build a graph but to build a higher-dimensional graph equivalent by connecting more than two nearby data points- resulting in simplicial complexes. Simplicial complexes are mathematical objects that are both topological and combinatorial, a property making them particularly useful for TDA. They can be seen as higher dimensional generalization of graphs. There are different ways to build a simplicial complex on a metric space including Vietoris-Rips Complexes, $\check{C}$ech complexes. These complexes when built on a data set provide insights into the underlying topology of point clouds. Additionally, these kinds of complexes have been intensively used in computational topology as a simplicial model for the sensor networks ([28, 33, 34]), as a tool for image processing ([35]) and in approximating object shapes in digital images ([10]).

The reason for using Vietoris–Rips and Čech complexes in data analysis is their ability to capture the topological characteristics of an unknown sample space underlying the data. Along with the development of topological data analysis [16, 27], determining the homotopy types of Vietoris-Rips complex and $\check{C}$ech complex of finite metric spaces has become crucial in applied topology. In fact, the idea behind persistent homology is to compute the (co)homology of a Vietoris-Rips complex filtration built on data, which is typically a finite metric space in high dimensions ([13]). The $\check{C}$ech complexes are also used to approximate the shape of a dataset ([16]). The two main methods from TDA, persistent homology and mapper, both use Čech complexes in their construction ([35]). Nevertheless, very less is known about the homology of Vietoris-Rips complexes and $\check{C}$ech complexes of manifolds or simple graphs at large scale parameters.

1.1 Vietoris-Rips Complexes

In recent times, a lot of study was conducted on the homotopy types of Vietoris-Rips complexes of different metric spaces, like, circles ([1]) and ellipses ([5]), geodesic spaces([38, 39]), planar point sets, hypercube graphs ([2, 9, 18, 22, 23, 36]), and about the homotopy connectivity of VR complexes of spheres ([7]). The homotopy type of $\mathcal{VR}(\mathcal{F}; r)$ for $r \geq 0$ is closely related to the study of the independence complex of Kneser graphs in [12] and the Vietoris-Rips complexes of hypercube graphs in [2]. Note that the independence complex of a Kneser graph ($\mathrm{KG}_{n,k}$) is identical to the Vietoris-Rips complex $\mathcal{VR}(\mathcal{F}_n^{2n+k}; 2n-1)$, where $\mathcal{F}_n^m$ denotes the collection of all n-subsets of $[m]$. Our result on the homotopy types of $\mathcal{VR}(\mathcal{F}_n^m; 2)$ (Corollary 3.1) is a generalization of Barmak's result [12], which is, the independence complex $\mathrm{KG}_{2,k} = I(\mathrm{KG}_{2,k})$ is homotopy equivalent to $\bigvee_{\binom{k+3}{3}} S^2$. When $m = 2n$, the complex $\mathcal{VR}(\mathcal{F}_n^m; m-2)$ has $\binom{m}{n}$ vertices and is the boundary of a cross-polytope, so it is homotopy equivalent to $S^{\frac{1}{2}\binom{m}{n}-1}$.

The interest in the shape of Vietoris–Rips complexes of hypercube graphs initially emerged from the work of mathematical biologists who applied topology to study medial recombination and genetic trees ([15, 18]). The hypercube graph is a graph whose vertices are all binary strings of length m, denoted by Q_m, and whose edges are given by pairs of such strings with Hamming distance 1. The Hamming distance between any two binary strings with the same length is defined as the number of positions in which their entries differ. Therefore, Q_m is a metric space equipped with the Hamming distance metric. Note that each binary string of length m can also be represented as a subset of $[m] = 1, 2, \ldots, m$, so another representation of Q_m is the $\mathcal{P}[m]$ equipped with the symmetric difference metric. We can consider the hypercube graph (Q_m) as a metric space equipped with the Hamming distance, and then the hypercube graph can be identified as the complex $\mathcal{VR}(Q_m; 1)$. Adamaszek and Adams investigated the Vietoris-Rips complexes $\mathcal{VR}(Q_m; r)$ at scales $r = 0, 1, 2$ in their recent work [2]. The complex $\mathcal{VR}(Q_m; 0)$ is homotopy equivalent to a wedge sum of $(2^m - 1)$-many S^0's, and $\mathcal{VR}(Q_m; 1)$ is homotopy equivalent to a wedge sum of $((m-2)2^{m-1} + 1)$-many S^1's. Their main result is that the complex $\mathcal{VR}(Q_m; 2)$ is homotopy equivalent to a wedge sum of c_m copies of S^3's, where c_m is given by $c_m = \sum_{0 \leq j < i < m} (j+1)(2^{m-2} - 2^{i-1})$.

Each binary string of length m can also be considered as the characteristic function of a subset of $[m]$. Hence, there is a natural isometric map between the metric spaces Q_m and $\mathcal{P}([m])$, where $\mathcal{P}([m])$ is the collection of all subsets of $[m]$ equipped with the symmetric difference metric d. Notice that $\mathcal{P}([m])$ contains the empty set $\emptyset$ as an element. Hence, the result about the homotopy type of $\mathcal{VR}(Q_m; 2)$ by Adamaszek and Adams is a special case of Theorem 3.4 which gives a deeper understanding on how its homotopy type is formed. Adamaszek and Adams in [2] used Polymake [25] and Ripser++ [40] to compute the reduced homology groups of $\mathcal{VR}(\mathcal{P}[m]; 3)$ for $m = 5, 6, \ldots, 9$, with coefficients $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$. They found that these homology groups are nontrivial only in dimensions 4 and 7, indicating that the complex $\mathcal{VR}(\mathcal{P}[m]; 3)$ is a wedge sum of copies of S^4's and S^7's. This suggests that the homotopy type of the complex $\mathcal{VR}(\mathcal{P}[m]; 3)$ is more complicated than that of the complexes $\mathcal{VR}(\mathcal{P}[m]; r)$ with $r = 0, 1, 2$. Shukla [36] subsequently proved that for $m \geq 5$, the reduced homology group $\tilde{H}_i(\mathcal{VR}(\mathcal{P}([m]); 3))$ is nontrivial if and only if $i \in \{4, 7\}$. Feng [22] then, was able to prove that for $m \geq 5$, the complex $\mathcal{VR}(\mathcal{Q}_m; 3)$ is homotopy equivalent to a wedge sum of $2^{n-4} \cdot \binom{n}{4}$ many S^7's and $\sum_{i=4}^{n-1} \cdot 2^{i-4} \cdot \binom{i}{4}$ many S^4's.

Here, we examine the homotopy type of Vietoris-Rips complexes of finite metric spaces at scale 2. In chapter-3, we study the homotopy types of the Vietoris-Rips complexes, $\mathcal{VR}(\mathcal{F}_n^m; 2)$ (Section 3.2), $\mathcal{VR}(\mathcal{F}_{\preceq A}^m; 2)$ (Section 3.3), and $\mathcal{VR}(\mathcal{F}_n^m \cup \mathcal{F}_{n'}^m; 2)$ (Section 3.4), where $\mathcal{F}_n^m$, $\mathcal{F}_{\preceq A}^m$, and $\mathcal{F}_n^m \cup \mathcal{F}_{n'}^m$ are all collections of subsets of $[m]$. We show that:

i) the complexes $\mathcal{VR}(\mathcal{F}_n^m; 2)$ and $\mathcal{VR}(\mathcal{F}_n^m \cup \mathcal{F}_{n+1}^m; 2)$ are either contractible or homotopy equivalent to a wedge sum of S^2's;

ii) the complexes $\mathcal{VR}(\mathcal{F}_{\preceq A}^m; 2)$ and $\mathcal{VR}(\mathcal{F}_n^m \cup \mathcal{F}_{n+2}^m; 2)$ are either contractible or homotopy equivalent to a wedge sum of S^3's.

Furthermore, we identify inductive formulas for determining the homotopy types of these complexes.

1.2 Čech Complexes

Compared to Vietoris-Rips complexes, $\check{C}$ech complexes are more complicated to compute than Vietoris-Rips complexes ([27]) but $\check{C}$ech complexes offer more topological guarantee than the Rips complexes [41]. Few works on the Čech complexes include the studies of hypercube graphs([8]), circular arcs ([3]), metric gluings ([4]) and also on approximating Čech complexes in low and high dimensions ([30]). More elaborately, the Čech complexes of cycle graphs are explored in [3] in which they proved that the nerve complex and the clique complex of any finite collection of arcs in the circle are homotopy equivalent to either a point, an odd-dimensional sphere, or a wedge sum of spheres of the same even dimension. Furthermore, in [4], they showed that the Čech (resp. Vietoris–Rips) complex of the metric wedge sum is homotopy equivalent to the wedge sum of the Čech (resp. Vietoris–Rips) complexes. And in [30], they presented a linear-sized approximation of the filtration of Čech complex of S, where S is a fixed dimensional point set.

In all these studies, a Čech complex on X has been defined in multiple ways. Here, we define the Čech complex as the nerve of all balls with centers in X. And in order to come up with our main results, we construct the Čech complexes on graphs. For G, a finite connected graph, unlike the Vietoris-Rips complex of G, it is often easier to produce a list of the finite maximal simplices in a Čech complex of G. Using this definition, in their recent work, Adams, Shukla, and Singh studied about the Čech complexes of the metric space Q_m at scales 2 and 3 ([8]). They proved that the Čech complex $\mathcal{N}(Q_m; 2)$ for $m \geq 2$ is homotopy equivalent to a wedge sum of 2-dimensional spheres. They also showed that the $\mathcal{N}(Q_m; 3)$ is homotopy equivalent to a simplicial complex of dimension at most 4 and that for $n \geq 4$, the reduced homology is nonzero only in dimensions $3, 4$.

Using the lemma from [8], we define the list of the finite maximal simplices in a Čech complex built on the finite metric spaces. Based of on those simplices we have investigated the homotopy types of these complexes. Following is an overview of our study of $\check{C}$ech complexes of certain finite metric spaces at scales $2, 3$.

In chapter - 4 we study the homotopy types of Čech complexes $\mathcal{N}(\mathcal{F}_{\mathrm{n}}^{\mathrm{m}} \cup \mathcal{F}_{\mathrm{n+1}}^{\mathrm{m}}; 2)$ (Section-4.1), $\mathcal{N}(\mathcal{F}_{\mathrm{n}}^{\mathrm{m}} \cup \mathcal{F}_{\mathrm{n+1}}^{\mathrm{m}} \cup \mathcal{F}_{\mathrm{n+2}}^{\mathrm{m}}; 2)$ (Section-4.2). We show that:

(i) the complexes $\mathcal{N}(\mathcal{F}_{\mathrm{n}}^{\mathrm{m}} \cup \mathcal{F}_{\mathrm{n+1}}^{\mathrm{m}}; 2)$ are either contractible or homotopy equivalent to S^1's.

(ii) the complexes $\mathcal{N}(\mathcal{F}_{\mathrm{n}}^{\mathrm{m}} \cup \mathcal{F}_{\mathrm{n+1}}^{\mathrm{m}} \cup \mathcal{F}_{\mathrm{n+2}}^{\mathrm{m}}; 2)$ are either contractible or homotopy equivalent to S^2's.

Additionally, we establish inductive formulas to determine the homotopy types of these complexes. Just as in the case of Vietoris-Rips complexes, the study of the homotopy types of $\mathcal{N}(\bigcup_{i=1}^{k} f_i^m; r)$ for $r \geq 2$, is strongly linked to the study of the Čech complexes of hypercube graphs in [8]. We show that the Čech complex of a hypercube graph Q_m with scale 2 is equivalent to the Čech complex of a union of $m - r$ finite metric spaces with scale 2.

Using the inductive formulae for the homotopy type of $\mathcal{N}(\mathcal{F}_{\mathrm{n}}^{\mathrm{m}} \cup \mathcal{F}_{\mathrm{n+1}}^{\mathrm{m}} \cup \mathcal{F}_{\mathrm{n+2}}^{\mathrm{m}}; 2)$ and $\mathcal{N}(\mathcal{F}_{\mathrm{n}}^{\mathrm{m}} \cup \mathcal{F}_{\mathrm{n+1}}^{\mathrm{m}}; 2)$, we derived a precise formula for the number of S^2's that $\mathcal{N}(Q_m, 2)$ is homotopy equivalent to.

In Chapter-5, we show that the complexes $\mathcal{N}(\mathcal{F}_{\mathrm{n}}^{\mathrm{m}} \cup \mathcal{F}_{\mathrm{n+1}}^{\mathrm{m}} \cup \mathcal{F}_{\mathrm{n+2}}^{\mathrm{m}}; 3)$ (Section- 5.1) are homotopy equivalent to a wedge sum of S^2's. Table-1.1 below gives the homotopy types in few instances.

$\mathcal{N}(\mathcal{F}_{\mathrm{n}}^{\mathrm{m}} \cup \mathcal{F}_{\mathrm{n+1}}^{\mathrm{m}} \cup \mathcal{F}_{\mathrm{n+2}}^{\mathrm{m}}; 3)$	$m = 4$	$m = 5$	$m = 6$	$m = 7$	$m = 8$
$n = 1$	S^2	$\vee_4 S^2$	$\vee_{10} S^2$	$\vee_{20} S^2$	$\vee_{35} S^2$
$n = 2$	$*$	$\vee_4 S^2$	$\vee_{19} S^2$	$\vee_{55} S^2$	$\vee_{125} S^2$
$n = 3$	$*$	$*$	$\vee_{10} S^2$	$\vee_{55} S^2$	$\vee_{210} S^2$
$n = 4$	$*$	$*$	$*$	$\vee_{20} S^2$	$\vee_{216} S^2$

Table 1.1: Homotopy types of $\mathcal{N}(\mathcal{F}_{\mathrm{n}}^{\mathrm{m}} \cup \mathcal{F}_{\mathrm{n+1}}^{\mathrm{m}} \cup \mathcal{F}_{\mathrm{n+2}}^{\mathrm{m}}; 3)$

We also show that the complexes $\mathcal{N}(\mathcal{F}_0^m \cup \mathcal{F}_1^m \cup \mathcal{F}_2^m \cup \mathcal{F}_3^m; 3)$(Section- 5.2) are homotopy equivalent to a wedge sum of S^4's. For $0 \leq n \leq 4$ and $3 \leq m \leq 7$, computer computations using Polymake [25] give us the following reduced homology groups:

(n,m)	$i=3$	$i=4$	$i \neq 3,4$
$(1,4)$	0	$\mathbb{Z}^{12}$	0
$(1,5)$	$\mathbb{Z}$	$\mathbb{Z}^{60}$	0
$(2,6)$	$\mathbb{Z}^5$	$\mathbb{Z}^{180}$	0
$(2,7)$	$\mathbb{Z}^{29}$	$\mathbb{Z}^{630}$	0
$(3,7)$	$\mathbb{Z}^{15}$	$\mathbb{Z}^{420}$	0

Table 1.2: $\tilde{H}_i(\mathcal{N}(\mathcal{F}_{\mathrm{n}}^{\mathrm{m}} \cup \mathcal{F}_{\mathrm{n+1}}^{\mathrm{m}} \cup \mathcal{F}_{\mathrm{n+2}}^{\mathrm{m}} \cup \mathcal{F}_{\mathrm{n+3}}^{\mathrm{m}}; 3))$

$\mathcal{N}(\mathcal{F}_{\mathrm{n}}^{\mathrm{m}} \cup \mathcal{F}_{\mathrm{n+1}}^{\mathrm{m}} \cup \mathcal{F}_{\mathrm{n+2}}^{\mathrm{m}} \cup \mathcal{F}_{\mathrm{n+3}}^{\mathrm{m}}; 3)$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$
$n=0$	$\vee_3 S^4$	$\vee_{12} S^4$	$\vee_{30} S^4$	$\vee_{60} S^4$	$\vee_{105} S^4$
$n=1$	$*$	$\beta_4 = 12$	$\beta_3 = 1$; $\beta_4 = 60$	$\beta_3 = 5$; $\beta_4 = 180$	$\beta_3 = 15$; $\beta_4 = 420$
$n=2$	$*$	$*$	$\beta_4 = 30$	$\beta_3 = 5$; $\beta_4 = 180$	$\beta_3 = 29$; $\beta_4 = 630$
$n=3$	$*$	$*$	$*$	$\beta_4 = 60$	$\beta_3 = 15$; $\beta_4 = 420$

Table 1.3: Homotopy types or Betti numbers of $\mathcal{N}(\mathcal{F}_{\mathrm{n}}^{\mathrm{m}} \cup \mathcal{F}_{\mathrm{n+1}}^{\mathrm{m}} \cup \mathcal{F}_{\mathrm{n+2}}^{\mathrm{m}} \cup \mathcal{F}_{\mathrm{n+3}}^{\mathrm{m}}; 3)$

For $n \geq 1$, based on proof techniques used to prove other results and the above computations, we conjectured that $\mathcal{N}(\mathcal{F}_{\mathrm{n}}^{\mathrm{m}} \cup \mathcal{F}_{\mathrm{n+1}}^{\mathrm{m}} \cup \mathcal{F}_{\mathrm{n+2}}^{\mathrm{m}} \cup \mathcal{F}_{\mathrm{n+3}}^{\mathrm{m}}; 3)$ is homotopy equivalent to a wedge sum of S^3's and S^4's. We have an explicit formula for the number of S^3's and S^4's.

Chapter 2

Notations and Preliminary Results

We now introduce few notations and lemmas that will be used consistently throughout the work.

Topological Spaces and Wedge Sums: Consider two topological spaces X, Y. The wedge sum $X \vee Y$ is the space obtained by gluing the spaces X and Y at a point. And a function $f : X \to Y$ is said to be null-homotopic when f is homotopic to a constant map. $X \simeq Y$ denotes that the spaces X and Y are homotopy equivalent. $\sum X$ is the suspension of the space X. S^n denotes a n-dimensional sphere and for any S^n, $\sum S^n$ is homeomorphic to S^{n+1}.

A cross-polytope with $2n$ vertices is a regular, convex polytope that exists in a n-dimensional Euclidean space. It is homeomorphic to the unit ball in $\mathbb{R}^n$ whose boundary is homeomorphic to S^{n-1}.

Any two metric spaces (X, d_X) and (Y, d_Y) are said to be isometric if there exists a map $g : X \to Y$ which is bijective and distance-preserving, that is; $d_X(x_1, x_2) = d_Y(g(x_1), g_Y(x_2))$ for any $x_1, x_2 \in X$.

Graph: Let $G = (V(G), E(G))$ be a finite simple connected graph and suppose we equip the vertex set $V(G)$ with the distance metric, d. $v \sim w$ denotes that the two distinct vertices v, w are adjacent. For $v \in V(G)$, the *open neighborhood* of v is $N_G(v) = \{w \in V(G) | v \sim w\}$. For $v \in V(G), r \geq 0$, the *closed* r*-neighborhood* of v is defined to be $N_{G,r}[v] = \{w \in V(G) | d(w, v) \leq r\}$. We drop the subscript G, when the graph G is clear from the context. Note that v is not an element of $N_G(v)$, but v is an element of $N_{G,r}[v]$. For radius $r = 1, N_1[v] = N(v) \cup \{v\}$.

Let $n, k \geq 1$ be two natural numbers. $KG_{n,k}$ denotes a Kneser graph with a total of $2n+k$ elements and each vertex corresponding to a n- element subset of the $2n + k$ elements. Any two vertices v, v' are adjacent in the Kneser graph if and only if $\{v\} \cap \{v'\} = \emptyset$.

Simplicial Complexes: An *abstract simplicial complex* is a collection K of finite non-empty sets, such that if A is an element of K, so is every element of A. The element A of K is called a *simplex* of K; its *dimension* is one less than the number of its elements. Each non-empty subset of A is called a *face* of A. The *dimension* of K is the largest dimension of all its simplices. The 0-dimension simplices of K are called its *vertices.* A sub-collection of K which itself is a complex is called a *sub-complex* of K. S is a *full subcomplex* of K if it contains all the simplicies in K spanned by the vertices in S.

For a simplicial complex K, $K^{(n)}$ denotes the n-skeleton of the complex K which is a sub-complex of K. We say a simplex σ to be maximal (facet) if it is not the face of any other simplex in K. All the simplicial complexes that we work with are finite.

Let σ be a simplex. Then σ can be extended to a simplicial complex, the collection of all subsets of σ. In this dissertation, we also use σ to represent the simplicial complex generated by itself.

For a vertex v in simplicial complex K, the *star* of v in K is defined as the subcomplex $st_K(v) = \{\tau \in K| \ v \cup \tau \in K\}$. The *link* of v is $lk_K(v) = \{\tau \in K|v \notin \tau \text{ and } \tau \cup \{v\} \in K\}$, and $K \setminus v$ is the induced simplicial complex with simplices that are obtained by removing the vertex v. Thus, the $st_K(v)$ is always a cone with the vertex v, namely $v * lk_K(v)$, and therefore contractible.

Following are two results that help us in studying about the homotopy types of a complex. Following lemma which helps in investigating the homotopy type of a complex by splitting into two subcomplexes has been proved in [26].

Lemma 2.1. *If the simplicial complex $K = K_1 \cup K_2$ satisfies the conditions that the inclusion maps $\iota_1 : K_1 \cap K_2 \to K_1$ and $\iota_2 : K_1 \cap K_2 \to K_2$ are null-homotopies, then the complex $K \simeq K_1 \vee K_2 \vee \sum(K_1 \cap K_2)$.*

The following lemma ([2]) is an easy corollary of the above mentioned result.

Lemma 2.2. *Let K be a simplicial complex and v a vertex in K. If the inclusion map ι : $lk_K(v) \to K$ is null-homotopic, then K is homotopic to $(K \setminus v) \vee \sum lk_K(v)$.*

Proof. For any vertex $v \in K$, $st_K(v)$ is the collection of all simplices that contain v, hence, we can write the complex K as $K = K \setminus v \cup st_K(v)$. Using the facts that $K \setminus v \cap st_K(v) = lk_K(v)$ and $st_K(v)$ is contractible, we get the inclusion map $\iota_2 : lk_K(v) \hookrightarrow st_K(v)$ to be null-homotopic. And in the hypothesis, we assumed the inclusion map $\iota_1 : lk_K(v) \hookrightarrow K \setminus v$ to be null-homotopic. Thus, by lemma-2.1, $K \simeq K \setminus v \vee \sum lk_K(v)$. $\square$

The *star cluster* of any simplex σ in K is defined as the subcomplex $SC_K(\sigma) = \bigcup\limits_{v \in \sigma} st_K(v)$. Given a simplicial complex K and its vertex set V, if $\sigma \in K$ for any finite set $\sigma \in V$ with $\{v, w\} \in K$ for every $v, w \in \sigma$, then K is called a *clique.* The *clique complex* of a graph G is an abstract simplicial complex with all its simplices as the cliques of the graph G. Let $Cl(G)$ denote the clique complex of G. Note that $Cl(G)$ is the maximal simplicial complex with 1-skeleton equal to G. Given a graph G with vertex set V and edge set E, then the *complementary graph* $\bar{G}$ is the graph with vertex set V and $\{u, v\}$ an edge in $\bar{G}$ if and only if $\{u, v\} \notin E$. For a graph G, the *independence complex* $I(G)$ is defined to be the clique complex of complementary graph $\bar{G}$, that is; $I(G) = Cl(\bar{G})$. The independence complex of the Kneser Graph $KG_{n,k}$ is the clique of the $\overline{KG}_{n,k}$, that is; it is the simplicial complex with vertex set of $2n + k$ elements with each vertex corresponding to a n-element subset of the $2n + k$ elements. $\{v, w\} \in I(KG_{n,k})$ if and only if $\{v\} \cap \{w\} \neq \emptyset$.

Lemma 2.3. *If σ is a k-simplex and K_σ is the simplicial complex generated by σ, then $K_\sigma^{(n)}$ is homotopy equivalent to a wedge sum of $\binom{k}{n+1}$-many of S^n for any $n < k$.*

Proof. Assume $\sigma = \{v_0, v_1, \ldots, v_k\}$ and K_σ is the simplicial complex generated by σ. Denote $K = K_\sigma^{(n)}$. Notice that there are $\binom{k}{n+1}$-many n-simplices which don't contain v_0. List such n-simplices as $\tau_1, \tau_2, \ldots, \tau_{\binom{k}{n+1}}$. Then $K = \mathrm{st}_K(v_0) \cup \bigcup\{K_{\tau_i} : i = 1, 2, \ldots, \binom{k}{n+1}\}$ where K_{τ_i} is the simplicial complex generated by τ_i for each i. Next we show that $K \simeq \bigvee\limits_{\binom{k}{n+1}} S^n$ by induction.

First notice that $\mathrm{st}_K(v_0) \cap K_{\tau_1}$ is the boundary of τ_1, and hence is homotopy equivalent to S^{n-1}. Since both $\mathrm{st}_K(v_0)$ and K_{τ_1} are contractible, the inclusion maps from their intersection to each of them are null-homotopic. By Lemma 2.2, $\mathrm{st}_K(v_0) \cup K_{\tau_1} \simeq \sum S^{n-1} \simeq S^n$. Then inductively, $K \simeq \bigvee_{\binom{k}{n+1}} S^n$. $\square$

Nerve: Let X be a topological space, and let $\mathcal{U} = \{U_\alpha\}_{i \in I}$ be a covering of X. Then, the nerve of $\mathcal{U}$, denoted by $\mathcal{N}(\mathcal{U})$, is the abstract simplicial complex with vertex set I and a k-simplex $\{i_0, i_1, \ldots, i_k\}$ if and only if $\bigcap_{i=0}^{k} U_i \neq \emptyset$.

Symmetric Difference metric and Finite Metric Spaces: Let $\mathcal{F}$ be a collection of subsets of $[m]$ for some $m \in \mathbb{N}$, where $[m] = \{1, 2, \ldots, m\}$. We define a metric d on $\mathcal{F}$ such that, for any A and B in $\mathcal{F}$, $d(A, B) = |A \Delta B|$, where $A \Delta B$ denotes the symmetric difference of A and B, i.e., $(A \setminus B) \cup (B \setminus A)$. Hence, $(\mathcal{F}, d)$ is a finite metric space. Define the space $\mathcal{F}_{\mathrm{n}}^{\mathrm{m}} = \{A : A \subseteq [m] \wedge |A| = n\}$ equipped with a symmetric difference metric.

Let's define a total ordering $\prec$ on $\mathcal{P}[m]$. For any $A \subset [m]$ with $|A| = n$, we represent $A = \{i_1, i_2, i_3, \ldots, i_n\}$ as $i_1 i_2 i_3 \ldots i_n$ where $i_1 < i_2 < i_3 < \ldots < i_n$. For any $A, B \subset [m]$ with $A = i_1 i_2 \ldots i_n,$ and $B = j_1 j_2 \ldots j_n$ we say $A \prec B$ when one of the following holds:

1. $|A| < |B|$;
2. if $|A| = |B|$, then there exists $1 \leq k \leq n$, such that $i_k < j_k$ and $i_l = j_l$ for any $l < k$.

Clearly, this is a total order on $\mathcal{P}([m])$ and for any subcollection $\mathcal{F}$ of $\mathcal{P}([m])$, $(\mathcal{F}, \prec)$ is also a total order. For any $A \subset [m]$, we denote $\mathcal{F}_{\prec A}^{m} = \{B : B \prec A \wedge B \subset [m]\}$ and $\mathcal{F}_{\preceq A}^{m} = \mathcal{F}_{\prec A}^{m} \cup \{A\}$. Notice that the set $[m]$ is the maximal elements in $\mathcal{P}([m])$; hence if $A = [m]$, then, $\mathcal{F}_{\preceq A}^{m} = \mathcal{P}([m])$.

From next section on, each vertex is a subset of $[m]$ and we will use A, B, C, D to represent these subsets of $[m]$. To specify these vertices as subsets of $[m]$, we interchangeably use A or $\{A\}$. We will use notations like $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{P}$ to denote a set of vertices. The finite metric space $\mathcal{F}_0^m$ has only one element which is the empty set. In the upcoming chapters, we will either use $\emptyset$ or 0 to denote the element of $\mathcal{F}_0^m$.

Vietoris-Rips Complex: The *Vietoris-Rips complex of X at scale r*, $\mathcal{VR}(X; r)$, of a metric space (X, d) with $r \geq 0$, is the simplicial complex with vertex set X and a finite subset σ as a simplex if and only if $diam(\sigma) \leq r$.

Recent work has focused on studying the Vietoris-Rips complexes of circles, geodesics, hypercube graphs, etc. In the next chapter, we examine the homotopy type of the Vietoris-Rips complex $\mathcal{VR}(\mathcal{F}, r)$ of a specific class of finite metric spaces with scale 2.

***Čech* Simplicial complex of graph *G*:** For a finite, simple, connected graph G and $r \geq 0$, the $\check{C}$ech simplicial complex at scale r is the nerve of all closed balls of radius $\dfrac{r}{2}$ centered at vertices of G.

In [8], for a finite, simple, connected graph G and integer $r \geq 0$, the $\check{C}$ech complex of graph $\mathcal{N}(G; r)$ was equivalently defined as follows: let $V(G)$ be the vertex set, the maximal simplices are defined as follows: if r is even then the simplices are generated by the closed neighborhoods $N_{\frac{r}{2}}[v]$ for any vertex v and if r is odd then the simplices are generated by the sets $N_{\frac{r-1}{2}}[v] \cup N_{\frac{r-1}{2}}[w]$ for all the adjacent vertices v, w in the graph G. From here on, we will use the above definition to generate the maximal simplices of $\check{C}$ech complex.

Note that for $n \geq 0, m \geq 1$, the symmetric distance between any two arbitrary vertices in the space $\mathcal{F}_{\mathrm{n}}^{\mathrm{m}}$ is at least 2. Therefore, we cannot define a Čech simplicial complex with scales 2 or 3 on the space $\mathcal{F}_{\mathrm{n}}^{\mathrm{m}}$ since the maximal simplices are generated by closed 1-neighborhoods, resulting in a disconnected graph. Instead, we construct a graph by taking a finite union of finite metric spaces, $\bigcup\limits_{i=n}^{n+k} \mathcal{F}_i^m$ for $k \geq 1$, where the vertices are the elements of the union and any two vertices are adjacent if their symmetric distance is 1. We then define a Čech complex on this graph. We extend the total ordering $\prec$ defined above on the union spaces as follows. For any two elements $A, B \in \bigcup\limits_{i=n}^{n+k} \mathcal{F}_i^m$, with $A \in \mathcal{F}_i^m, B \in \mathcal{F}_j^m$ and $i \neq j$ then $A \prec B$ when $i < j$. If both A, B are from $\mathcal{F}_i^m$, then $A \prec B$ as defined above.

Following is the isometric lemma that we use in almost every proof to find the homotopy type of a desired complex.

Lemma 2.4. *X, Y are topological spaces which are isometric, then the following holds true for any $r \geq 0$:*

1. *$\mathcal{VR}(X; r)$ is homeomorphic to $\mathcal{VR}(Y; r)$*

2. *$\mathcal{N}(X; r)$ is homeomorphic to $\mathcal{N}(Y; r)$*

For convenience, we set $\sum_{i=a}^{b} f(i) = 0$ when $b < a$, where f is a function on the set of natural numbers.

In the following proofs, we adopt an inductive process to study these complexes and our approach is potentially applicable to the investigation of these complexes at larger scales. Firstly, we take a sub-complex of the main complex and then inductively add the remaining vertices. We examine how the addition of each vertex alters the homotopy type of the entire complex.

Chapter 3

Vietoris-Rips Complex with Scale 2

We start with introducing notations for certain collections of subsets of $[m]$. For $n \leq m$, let $\mathcal{F}^m_{\leq n}$ be the collection of all subsets of $[m]$ with cardinality $\leq n$. It is easy to see that the complex $\mathcal{VR}(\mathcal{F}^m_{\leq r}; r)$ is contractible since it is a cone with the cone vertex being the empty set $\emptyset$. We now proceed to define a total ordering $\prec$ on $\mathcal{P}([m])$ to facilitate the conduction of induction process. For each $A \subseteq [m]$ with $|A| = n$, we represent $A = \{i_1, i_2, \ldots, i_n\}$ as $i_1 i_2 \cdots i_n$ with $i_1 < i_2 < \cdots < i_n$. For any $A, B \subseteq [m]$, we say $A \prec B$ if one of the followings holds:

i) $|A| < |B|$;

ii) there is a $k \in \mathbb{N}$ such that $i_k < j_k$ and $i_\ell = j_\ell$ for any $\ell < k$, when $n = |A| = |B|$, $A = i_1 i_2 \cdots i_n$ and $B = j_1 j_2 \cdots j_n$.

Clearly this is a total order on $\mathcal{P}([m])$ and for any subcollection $\mathcal{F}$ of $\mathcal{P}([m])$, $(\mathcal{F}, \prec)$ is also a total order. For any $A \subset [m]$, we denote $\mathcal{F}^m_{\prec A} = \{B : B \prec A \text{ and } B \subset [m]\}$ and $\mathcal{F}^m_{\preceq A} = \mathcal{F}^m_{\prec A} \cup \{A\}$. Notice that the set $[m]$ is the maximal elements in $\mathcal{P}([m])$; hence if $A = [m]$, $\mathcal{F}^m_{\preceq A} = \mathcal{P}([m])$.

We start with some easy observations of the homotopy types of such complexes. For any collection $\mathcal{F}$ of subsets of $[m]$, $\mathcal{VR}(\mathcal{F}; 0)$ is a complex with $|\mathcal{F}|$-many disjoint vertices. Also, for any $1 \leq n \leq m-1$, $\mathcal{VR}(\mathcal{F}^m_n; 1)$ is the space of $\binom{m}{n}$ disjoint vertices since $d(A, B) \geq 2$ for any two different subsets A, B with cardinality n. And for each $i = 0, 1, \ldots, m$, the metric space $\mathcal{F}^m_i$ is isometric to $\mathcal{F}^m_{m-i}$ since the complementary mapping with $\phi(A) = [m] \setminus A$ preserves the symmetric distance from $\mathcal{F}^m_i$ to $\mathcal{F}^m_{m-i}$. Therefore, $\mathcal{VR}(\mathcal{F}^m_i; r)$ is homotopy equivalent to $\mathcal{VR}(\mathcal{F}^m_{m-i}; r)$ for each $r \geq 0$. We see that the complexes $\mathcal{VR}(\mathcal{F}^m_1; 2)$ and $\mathcal{VR}(\mathcal{F}^m_{m-1}; 2)$ are contractible because each pair of their vertices has distance 2. Hence, the complex $\mathcal{VR}(\mathcal{F}^m_n; 2)$ is contractible when $n = 0, 1, m-1$, or m. Similarly, the complex $\mathcal{VR}(\mathcal{F}^m_n \cup \mathcal{F}^m_{n+1}; 2)$ is contractible if $n = 0$ or $m - 1$

In this chapter, we investigate the homotopy type of the Vietoris-Rips complex $\mathcal{VR}(\mathcal{F}; 2)$ of a specific class of finite metric spaces with scale 2. We study the homotopy types of the Vietoris-Rips complexes, $\mathcal{VR}(\mathcal{F}^m_n; 2), \mathcal{VR}(\mathcal{F}^m_{\preceq A}; 2)$, and $\mathcal{VR}(\mathcal{F}^m_n \cup \mathcal{F}^m_{n'}; 2)$ where $\mathcal{F}^m_n$, $\mathcal{F}^m_{\preceq A}$, and $\mathcal{F}^m_n \cup \mathcal{F}^m_{n'}$ are all collections of subsets of $[m]$. We show that:

i) the complexes $\mathcal{VR}(\mathcal{F}^m_n; 2)$ and $\mathcal{VR}(\mathcal{F}^m_n \cup \mathcal{F}^m_{n+1}; 2)$ are either contractible or homotopy equivalent to a wedge sum of S^2's;

ii) the complexes $\mathcal{VR}(\mathcal{F}^m_{\preceq A}; 2)$ and $\mathcal{VR}(\mathcal{F}^m_n \cup \mathcal{F}^m_{n+2}; 2)$ are either contractible or homotopy equivalent to a wedge sum of S^3's.

In the Kneser graph $\mathrm{KG}_{n,k}$, any two vertices v, v' are adjacent if and only if $\{v\} \cap \{v'\} = \emptyset$. In other words, any two vertices in $\mathrm{KG}_{n,k}$ are not disjoint if and only if their symmetric difference distance is at most $2n-1$. Therefore, the independence complex of $\mathrm{KG}_{n,k}$ is identical to the Vietoris-Rips complex $\mathcal{VR}(\mathcal{F}^{2n+k}_n; 2n-1)$, where $\mathcal{F}^m_n$ denotes the collection of all n-subsets of $[m]$.

Barmak proved in [12] (Theorem 4.11) that the independence complex of $\mathrm{KG}_{2,k}$, $\mathrm{I}(\mathrm{KG}_{2,k})$, is homotopy equivalent to $\bigvee_{\binom{k+3}{3}} S^2$. For any $m \geq 4$, note that $\mathcal{VR}(\mathcal{F}^m_2; 2) = \mathcal{VR}(\mathcal{F}^m_2; 3) = \mathrm{I}(\mathrm{KG}_{2,m-4})$; thus, the complex $\mathcal{VR}(\mathcal{F}^m_2; 2)$ is homotopy equivalent to a wedge sum of $\binom{m-1}{3}$ copies of S^2. Our result on the homotopy types of $\mathcal{VR}(\mathcal{F}^m_n; 2)$ is a generalization of Barmak's result. When $m = 2n$, the complex $\mathcal{VR}(\mathcal{F}^m_n; m-2)$ has $\binom{m}{n}$ vertices and is the boundary of a cross-polytope, so it is homotopy equivalent to $S^{\frac{1}{2}\binom{m}{n}-1}$.

3.1 Star Cluster of a Subcomplex

The star cluster of a simplex is a powerful tool used in different papers to prove various results of the independence complex of graphs. Extending this idea, we now define the *star cluster of a subcomplex* L in a simplicial complex K :

$$SC_K(L) = \bigcup_{v \in L^0} st_K(v)$$

In a clique complex K, Barmak proved that the star cluster of any simplex $\mathrm{SC}_K(\sigma)$ is contractible, and therefore, homotopy equivalent to σ. But in general, given a subcomplex L of K. $\mathrm{SC}_K(L)$ is not homotopy equivalent to L as can be seen in the excample below.

Example 3.1. *Let K be the $\mathcal{VR}(\mathcal{P}([2]); 1)$ and L a full subcomplex whose vertices are $\{\emptyset, \{1\}, \{2\}\}$. Then, the $SC_K(L) = K$ which is homotopy equivalent to S^1 while L itself is contractible.*

We are now going to add few more conditions and extend the result that Barmak proved regarding a star cluster.

Lemma 3.1. *Let K be a clique complex and L a clique subcomplex of K. Suppose $(st_K(v) \cap st_K(w)) \setminus L \neq \emptyset$ for any $(v, w) \in L^0$, then the following hold true:*

1. *L is a full subcomplex of K.*
2. *for any collection of vertices, $v_1, v_2, \ldots, v_\ell$ in L, the complex $L' = L \cup \bigcup_{i=1}^{\ell} st_K(v_i))$ is homotopy equivalent to L.*

In particular, (2) implies that $SC_K(L)$ is homotopy equivalent to L.

Proof. First, let us prove (1) by proof of contradiction.

1. Assume L is not a full subcomplex of K. Then, for some simplex $\sigma = \{w_0, w_1, \ldots, w_k\}$ in K, all the vertices w_j for each $j = 0, 1, \ldots, k$ are in L but for some arbitrary pair $w_j, w_{j'}$ of vertices from σ with $j \neq j'$ the edge $\{w_j, w_j'\} \notin L$. As the 1-simplex $\{w_j, w_j'\} \in K$, it is in both $\mathrm{st}_K(w_j)$ and $\mathrm{st}_K(w_{j'})$, implying, $(\mathrm{st}_K(w_j) \cap \mathrm{st}_K(w_j')) \setminus L \neq \emptyset$. Hence, by our hypothesis, $\{w_j, w_j'\} \in L$, which is a contradiction. Therefore, each pair of vertices in σ form an edge in L and L is a clique complex, implying $\sigma \in L$. Thus, L is a full subcomplex of K.

   We next prove (2) by induction.

2. Consider a set of vertices $v_1, v_2, \ldots, v_{k-1}$ in L and define

$$L_{k-1} = L \cup \bigcup_{i=1}^{k-1} \{\mathrm{st}_K(v_i)\}$$

For $k = 1, L_0 = L$ and the result follows. Suppose the statement holds true for the $k-1$ set of vertices, that is, $L_{k-1} \simeq L$. Consider a new vertex $v_k \in L$ and say, $L_k = L_{k-1} \cup \mathrm{st}_K(v_k)$. We will show that $L_k \simeq L$.

Firstly, we claim that $L_{k-1} \cap \mathrm{st}_K(v_k) = \mathrm{st}_{L_{k-1}}(v_k)$. One side inclusion $\mathrm{st}_{L_{k-1}}(v_k) \subseteq L_{k-1} \cap \mathrm{st}_K(v_k)$ is clear from definition. To prove that $L_{k-1} \cap \mathrm{st}_K(v_k) \subseteq \mathrm{st}_{L_{k-1}}(v_k)$, consider a simplex $\tau \in L_{k-1} \cap \mathrm{st}_K(v_k)$. Then $\tau \in L_{k-1}$ and $\tau \in \mathrm{st}_K(v_k)$.$\tau \in L_{k-1}$ implies that $\tau \in L$ or $\tau \in \bigcup\{\mathrm{st}_K(v_i) : i = 1, 2, \ldots, k-1\}$. We'll prove that $\tau \in \mathrm{st}_{L_{k-1}}(v_k)$ in either of the two cases:

Case (a): Suppose $\tau \in L$, that is, all the vertices of τ are in L. As $\tau \in \mathrm{st}_K(v_k), \tau \cup \{v_k\}$ is a simplex in K whose vertices are in L. Then by (1), $\tau \cup \{v_k\} \in L \subseteq L_{k-1}$, implying, $\tau \in \mathrm{st}_{L_{k-1}}(v_k)$.

Case (b): Suppose $\tau \in \bigcup\{\mathrm{st}_K(v_i) : i = 1, 2, \ldots, k-1\}$, then there exists at least one k_0 with $1 \leq k_0 \leq k-1$ such that $\tau \in \mathrm{st}_K(v_{k_0})$. Then $\tau \cup \{v_{k_0}\}$ is a simplex in K. We also have that $\tau \in \mathrm{st}_K(v_k)$, implying, $\tau \cup \{v_k\}$ is also a simplex in K. Note that $\tau \in (\mathrm{st}_K(v_{k_0}) \cup \mathrm{st}_K(v_k)) \setminus L$. By our assumption $\{v_{k_0}, v_k\}$ is an edge in L. Since K is a clique complex, $\tau \cup \{v_{k_0}, v_k\}$ is a simplex in K; and this simplex is in $\mathrm{st}_K(v_{k_0}) \subseteq L_{k-1}$. Hence the simplex $\tau \cup \{v_{k_0}, v_k\} \in \mathrm{st}_{L_0}(v_k)$ which implies that $\tau \in \mathrm{st}_{L_0}(v_k)$.

Thus, we now have that $L_{k-1} \cap \mathrm{st}_K(v_k) = \mathrm{st}_{L_{k-1}}(v_k)$. Note that both $\mathrm{st}_K(v_k)$ and $\mathrm{st}_{L_{k-1}}(v_k)$ are contractible, and so is $\sum(\mathrm{st}_{L_{k-1}}(v_k))$. And the inclusion maps are null-homotopies. Hence, by Lemma 2.1 and the inductive assumption,

$$L_k = L_{k-1} \cup \mathrm{st}_K(v_k) \simeq L_{k-1} \vee \mathrm{st}_K(v_k) \vee \sum(\mathrm{st}_{L_{k-1}}(v_k)) \simeq L_0 \simeq L.$$

$\square$

Next is a result that splits a complex K into a union of two sub-complexes using the concept of star cluster. Based on this split, we can then apply Lemma 2.1 to investigate the homotopy type of the complex K.

Lemma 3.2. *Let K be a simplicial complex and K_1, K_2 be subcomplexes of K such that*

i) $K^{(0)} = K_1^{(0)} \cup K_2^{(0)}$;

ii) K_2 *is a full subcomplex of* K.

Then $K = SC_K(K_1) \cup K_2$.

Proof. Let σ be a simplex of K. If one of σ's vertices, namely v, is in K_1, then $\sigma \in \mathrm{st}_K(v) \subseteq \mathrm{SC}_K(K_1)$; otherwise, $\sigma \in K_2$ by the assumption. $\square$

3.2 Vietoris-Rips Complex $\mathcal{VR}(\mathcal{F}_n^m; 2)$

From hereon, each vertex of a complex is a subset of $[m]$ and we'll use A, B, C, or D to represent them. For any subset C of $[m]$, denote $N[C] = \{A \in \mathcal{P}([m]) : C \subset A \text{ and } |A \setminus C| = 1\}$ and $L[C] = \{A \in \mathcal{P}([m]) : A \subset C \text{ and } |C \setminus A| = 1\}$.

Fix $n, m \in \mathbb{N}$ with $n < m$. For any $\{i_1, i_2, \ldots, i_n, i_{n+1}\} \in [m]$ with $i_1 < i_2 < \ldots < i_n < i_{n+1}$, we get that

$$N[i_1, i_2, \ldots, i_{n-1}] = \{A \in \mathcal{F}_n^m : \{i_1, i_2, \ldots, i_{n-1}\} \subset A\}, \text{ and}$$

$$L[i_1, i_2, \ldots, i_{n+1}] = \{i_1 i_2 \ldots \hat{i_j} \ldots i_{n+1} : j \in \{1, ..., n+1\}\},$$

here, $i_1 i_2 \ldots \hat{i_j} \ldots i_{n+1}$ is defined to be $\{i_1, i_2, \ldots, i_n, i_{n+1}\} \setminus \{i_j\}$ for each j.

Lemma 3.3. *Assume that $m \geq n+2$ with $n \geq 2$ and $\{i_1, i_2, \ldots, i_{n+1}\} \subseteq [m]$. Then $N[i_1, i_2, .., i_{n-1}]$ and $L[i_1, i_2, \ldots, i_{n+1}]$ are maximal simplices in the complex $\mathcal{VR}(\mathcal{F}_n^m; 2)$.*

Proof. One can easily verify that $N[i_1, i_2, \ldots, i_{n-1}]$ is an $(m-n)$-simplex and $L[i_1, i_2, \ldots, i_{n+1}]$ is an n-simplex in $\mathcal{VR}(\mathcal{F}_n^m; 2)$.

(i) $N[i_1, i_2, \ldots, i_{n-1}]$ is a maximal simplex in $\mathcal{VR}(\mathcal{F}_n^m; 2)$:

Let A be an n-subset of $[m]$ such that $A \notin N[i_1, i_2, \ldots, i_{n-1}]$. Without loss of generality, we assume that $i_1 \notin A$, then we pick $i, j \in A \setminus \{i_1, i_2, \ldots, i_{n-1}\}$ and $k \in [m] \setminus \{i, j, i_1, i_2, \ldots, i_{n-1}\}$. Let $B = \{i_1, i_2, \ldots, i_{n-1}, k\}$ which is clearly in $N[i_1, i_2, \ldots, i_{n-1}]$. Clearly, $d(A, B) \geq 4$ since $\{i, j, k, i_1\} \subseteq A\Delta B$. Hence $N[i_1, i_2, .., i_{n-1}]$ is a maximal simplex in $\mathcal{VR}(\mathcal{F}_n^m; 2)$.

(ii) $L[i_1, i_2, \ldots, i_{n+1}]$ is a maximal simplex in $\mathcal{VR}(\mathcal{F}_n^m; 2)$:

Let A be an n-subset of $[m]$ such that $A \notin L[i_1, i_2, \ldots, i_{n+1}]$. Then there is an $i \in [m] \setminus \{i_1, i_2, \ldots, i_{n+1}\}$ such that $i \in A$. Suppose $A \cap \{i_1, i_2, \ldots, i_{n+1}\} \neq \emptyset$. Then we can pick $B \in L[i_1, i_2, \ldots, i_{n+1}]$ such that $(\{i_1, i_2, \ldots, i_{n+1}\} \setminus A) \subset B$. Then $|A \setminus B| \geq 1$ since $i \in A \setminus B$. Also, $|B \setminus A| \geq 2$ since at least 2 elements in $\{i_1, i_2, \ldots, i_{n+1}\}$ are not in A. So $d(A, B) \geq 3$. If $A \cap \{i_1, i_2, \ldots, i_{n+1}\} = \emptyset$, $d(A, B) \geq 2n \geq 4$ for any $B \in L[i_1, i_2, \ldots, i_{n+1}]$. Hence $L[i_1, i_2, \ldots, i_{n+1}]$ is a maximal simplex in $\mathcal{VR}(\mathcal{F}_n^m; 2)$.

□

For convenience in this paper, we will use $N[i_1, i_2, \ldots, i_{n-1}]$ or $L[i_1, i_2, \ldots, i_{n+1}]$ to represent both a simplex and the subcomplex generated by the simplex in $\mathcal{VR}(\mathcal{F}_n^m; 2)$ or any other complexes containing them.

For a complex K, let $M(K)$ be the collection of maximal simplices in K. Clearly $K = \bigcup M(K)$. Hence, it is important to understand the collection of maximal simplices in a complex. Next, we show that these two are the only maximal simplices in $\mathcal{VR}(\mathcal{F}_n^m; 2)$.

Lemma 3.4. *Fix $n, m \in \mathbb{N}$ with $1 < n < m$. Let K be the complex $\mathcal{VR}(\mathcal{F}_n^m; 2)$.*

i) Any maximal simplex σ in K is either $N[i_1, i_2, .., i_{n-1}]$ or $L[i_1, i_2, .., i_{n+1}]$ for $i_1, i_2, i_3, ..., i_{n+1} \in [m]$ with $i_1 < i_2 < ... < i_n < i_{n+1}$.

ii) For any $k \geq 2$ and $\{A_1, A_2, \ldots, A_{k+1}\}$ being a k-simplex in K such that $|\bigcap_{\ell=1}^{k+1} A_\ell| < n - 1$, the only maximal simplex containing $\{A_1, A_2, \ldots, A_{k+1}\}$ as a face is $L[A_1 \cup A_2]$.

Proof. To prove i), we pick a maximal simplex σ in the complex K.

Note that the vertices of K are subsets of $[m]$. Hence, σ is a collection of subsets of $[m]$ and $\bigcap \sigma \subset [m]$. If $|\bigcap \sigma| = n-1$, then clearly σ is one of the simplices in the form $N[i_1, i_2, .., i_{n-1}]$.

We claim that the size of the set $\bigcap \sigma$ can't be greater than 0 and less than $n-1$. For the purpose of contradiction, we suppose that $0 < |\bigcap \sigma| < n-1$. Let $|\bigcap \sigma| = k$ with $0 < k < n-1$ and list $\bigcap \sigma$ as $\{i_1, i_2, \dots, i_k\}$. Pick $A \in \sigma$ such that $A \setminus \bigcap \sigma = \{j_1, j_2, \dots, j_{n-k}\}$. For each $\ell = 1, 2, \dots, n-k$, pick $B_\ell \in \sigma$ such that $j_\ell \notin B_\ell$. Also $|B_\ell \setminus A| = 1$ because $d(B_\ell, A) = 2$ for each ℓ. Since $k < n-1$, $n-k \geq 2$. Then we let i_0 be the number in $B_1 \setminus A$ and j_0 be the number in $B_2 \setminus A$. If $i_0 \neq j_0$, then the $B_1 \Delta B_2 = \{j_1, i_0, j_2, j_0\}$ which is a contradiction. So $i_0 = j_0$. Therefore, by induction, $B_\ell \setminus A = \{i_0\}$ for each $\ell = 1, 2, \dots, n-k$. Then if $C = \{i_0, i_2, \dots, i_k, j_1, \dots j_{n-k}\}$, then $C \Delta A = \{i_0, i_1\}$ and $C \Delta B_\ell = \{i_1, j_\ell\}$ for each $\ell = 1, 2, \dots, n-k$, i.e., $d(C, A) = 2$ and $d(C, B_\ell) = 2$ for each $\ell = 1, 2, \dots, n-k$. If C is in σ, then $i_1 \notin \bigcap \sigma$; and if C is not in σ, then σ is not a maximal simplex. These contradictions show that it is impossible that $0 < |\bigcap \sigma| < n-1$.

Now we suppose that $\bigcap \sigma = \emptyset$. Pick $A \in \sigma$ and represent A as $i_1 i_2 \dots i_n$. For each $\ell = 1, 2, \dots, n$, there exists $B_\ell \in \sigma$ such that $i_\ell \notin B_\ell$. Using the argument above, we can show that $B_\ell \setminus A = B_{\ell'} \setminus A$ for each $\ell, \ell' = 1, 2, \dots, n$. Denote $B_1 \setminus A = \{i_{n+1}\}$. Then clearly $\sigma = L[i_1, i_2, \dots, i_{n+1}]$.

To prove ii), we start with a k-simplex $\{A_1, A_2, \dots, A_{k+1}\}$ in K such that $|\bigcap_{\ell=1}^{k+1} A_\ell| < n-1$ and $k \geq 2$. Then if σ is a maximal simplex in K such that $\{A_1, A_2, \dots, A_{k+1}\} \subseteq \sigma$, then $\bigcap \sigma = \emptyset$ by the argument above, and hence σ is in the form $L[i_1, i_2, \dots, i_{n+1}]$. Clearly $A_1 \cup A_2 \subseteq \{i_1, i_2, \dots, i_{n+1}\}$ which means $A_1 \cup A_2 = \{i_1, i_2, \dots, i_{n+1}\}$ because $|A_1 \cup A_2| = n+1$. It is clear that no other maximal simplex contains this simplex. $\square$

We need one more result before the discussion of the homotopy types of the complex $\mathcal{VR}(\mathcal{F}_n^m; 2)$. Assume $n \geq 1$. Fix a number $a \in [m]$, let $\mathcal{S}_a = \{A : A \in \mathcal{F}_n^m \text{ and } a \in A\}$. There is a natural isometric mapping between the metric spaces $\mathcal{F}_{n-1}^{m-1}$ and $\mathcal{S}_a$. Hence, $\mathcal{VR}(\mathcal{F}_{n-1}^{m-1}; 2)$

is homeomorphic to $\mathcal{VR}(\mathcal{S}_a; 2)$. Next, we show that the homotopy type of the star cluster of the latter in K remains the same.

Lemma 3.5. *Let n, m be in $\mathbb{N}$ such that $n < m$. Define $\mathcal{S}_1 = \{A \subset [m] : |A| = n$ and $1 \in A\}$ and let L be the complex $\mathcal{VR}(\mathcal{S}_1; 2)$. Then*

$$SC_{\mathcal{VR}(\mathcal{F}_n^m;2)}(L) \simeq L.$$

Proof. Let $K = \mathcal{VR}(\mathcal{F}_n^m; 2)$. We'll show that the condition in Lemma 3.1 is satisfied. Then the result follows.

Pick vertices A and B in L such that $\{A, B\}$ is not an edge in L, i.e., $|A\Delta B| \geq 4$. Hence there exist natural numbers i_1, i_2, j_1, and j_2 such that $\{i_1, i_2\} \subseteq A \setminus B$ and $\{j_1, j_2\} \subseteq B \setminus A$. Suppose, for contradiction, that $(\mathrm{st}_K(A) \cap \mathrm{st}_K(B)) \setminus L \neq \emptyset$. We pick a vertex $C \in (\mathrm{st}_K(A) \cap \mathrm{st}_K(B)) \setminus L$. Clearly $1 \notin C$. We claim that $A \setminus \{1\} \subset C$, otherwise there exists $i_0 \neq 1$ such that $i_0 \in A \setminus C$, whence $|A \setminus C| \geq 2$ which is a contradiction. Similarly, $B \setminus \{1\} \subset C$. Therefore, $\{i_1, i_2, j_1, j_2\} \subset C$. Notice that $(A \cap B) \setminus \{1\}$ has size $\leq n - 3$. Suppose that $|(A \cap B) \setminus \{1\}| = n - 3$. Then the vertex C has $n + 1$ elements because $(A \cap B) \setminus \{1\} \subset C$ and $\{i_1, i_2, j_1, j_2\} \subset C$. This means that C has at least $n + 1$ elements which is a contradiction. This finishes the proof. □

Now we are ready to give a complete characterization of the homotopy types of $\mathcal{VR}(\mathcal{F}_n^m; 2)$.

Theorem 3.1. *Suppose that $1 < n < m - 1$. The complex $\mathcal{VR}(\mathcal{F}_n^m; 2)$ is homotopy equivalent to a wedge sum of spheres. Specifically,*

$$\mathcal{VR}(\mathcal{F}_n^m; 2) \simeq \Big(\bigvee_{\binom{m-1}{n+1}\cdot\binom{n}{2}} S^2\Big) \vee \mathcal{VR}(\mathcal{F}_{n-1}^{m-1}; 2).$$

Proof. Notice that the complex $\mathcal{VR}(\mathcal{F}_1^{m-1}; 2)$ is contractible. Hence the result holds when $n = 2$ by Barmak's result mentioned above.

Assume that $n > 2$ and $\mathcal{VR}(\mathcal{F}_{n-1}^{m-1}; 2)$ is homotopic to a wedge sum of spheres S^2. We denote $K = \mathcal{VR}(\mathcal{F}_n^m; 2)$. As in Lemma 3.5, let $\mathcal{S}_1 = \{A \subset [m] : |A| = n$ and $1 \in A\}$ and

L be the complex $\mathcal{VR}(\mathcal{S}_1;2)$. Then, the complex L is homotopy equivalent to $\mathcal{VR}(\mathcal{F}_{n-1}^{m-1};2)$ which is a wedge sum of S^2's by the assumption. Also by Lemma 3.5, the star cluster $\mathrm{SC}_K(L)$ is homotopy equivalent to L.

Now we examine the collection of maximal simplices in K to decide which of them is not in $\mathrm{SC}_K(L)$. Notice that any maximal simplex in the form $N[i_1,i_2,\ldots,i_{n-1}]$ or $L[1,i_1,\ldots,i_n]$ contains at least one vertex containing 1 for any $i_1,i_2,\ldots,i_n\in[m]$; hence any such simplex is in $\mathrm{SC}_K(L)$. Therefore in the complement of $\mathrm{SC}_K(L)$, namely $K\setminus\mathrm{SC}_K(L)$, there is only one kind of maximal simplicies in the form $L[i_1,i_2,\ldots,i_{n+1}]$ with $i_k\neq 1$ for any $k=1,2,\ldots,n+1$; and there are $\binom{m-1}{n+1}$-many such simplices and list them as $\{\sigma_1,\sigma_2,\ldots,\sigma_{\binom{m-1}{n+1}}\}$. Here, K_{σ_ℓ} is the complex generated by σ_ℓ for each $\ell=1,2,\ldots,\binom{m-1}{n+1}$.

For each ℓ with $1\le\ell\le\binom{m-1}{n+1}$, we denote L_ℓ to be the complex whose maximal simplices are $\{\sigma_j : j=1,2,\ldots,\ell\}$. Hence the complex $L_{\binom{m-1}{n+1}}$ is the complex $\mathcal{VR}(\mathcal{S}_2;2)$ where $\mathcal{S}_2$ is the collection of n-subsets of $[m]$ not containing 1. Therefore, $K=\mathrm{SC}_K(L)\cup L_{\binom{m-1}{n+1}}$.

We claim that $\mathrm{SC}_K(L)\cup L_\ell$ is homotopic to $(\bigvee_{\ell\cdot\binom{n}{2}} S^2)\vee\mathcal{VR}(\mathcal{F}_{n-1}^{m-1};2)$ for each $\ell=1,2,\ldots,\binom{m-1}{n+1}$. This claim finishes the proof. Next, we'll prove this claim by induction. For convenience, denote $L_0=\emptyset$.

Suppose, for induction, that

$$\mathrm{SC}_K(L)\cup L_{\ell-1}\simeq(\bigvee_{(\ell-1)\cdot\binom{n}{2}} S^2)\vee\mathcal{VR}(\mathcal{F}_{n-1}^{m-1};2).$$

This holds when $\ell=1$ since $L_0=\emptyset$. Then $\mathrm{SC}_K(L)\cup L_\ell=\mathrm{SC}_K(L)\cup L_{\ell-1}\cup\{K_{\sigma_\ell}\}$. Denote σ_ℓ to be $L[i_1,i_2,\ldots,i_{n+1}]$ where $i_k\neq 1$ for each $k=1,2,\ldots,n+1$. Next we'll find the homotopy type of $(\mathrm{SC}_K(L)\cup L_{\ell-1})\cap\{K_{\sigma_\ell}\}$.

For any vertex $B\in K_{\sigma_\ell}$, $B\in L[\{1\}\cup B]\subset\mathrm{SC}_K(L)$. Hence the 0-skeleton of K_{σ_ℓ} is contained in $\mathrm{SC}_K(L)$. Let $\{B_1,B_2\}$ be a 1-simplex in K_{σ_ℓ}. Then $|B_1\cap B_2|=n-1$. Because $N[B_1\cap B_2]$ is in $\mathrm{SC}_K(L)$, the edge $\{B_1,B_2\}$ is in $\mathrm{SC}_K[L]$. So the 1-skeleton $K_{\sigma_\ell}^{(1)}$ of K_{σ_ℓ} is also contained in $\mathrm{SC}_K(L)$. Moreover, any k-simplex with $k\ge 2$ in K_{σ_ℓ} is not in $\mathrm{SC}_K(L)$; otherwise such a k-simplex would be contained in a maximal simplex which has a vertex containing 1 and hence is different from σ_ℓ. This leads to a contradiction by ii) in Lemma 3.4. For any

$\ell' = 1, 2, \ldots, \ell - 1$, the intersection of the complexes $K_{\sigma_{\ell'}}$ and K_{σ_ℓ} contains at most one vertex because of their definitions. Therefore, $(\mathrm{SC}_K(L) \cup L_{\ell-1}) \cap K_{\sigma_\ell} = K_{\sigma_\ell}^{(1)}$. Recall that σ_ℓ is an n-simplex, hence $K_{\sigma_\ell}^{(1)}$ is homotopy equivalent to a wedge sum of $\binom{n}{2}$-many copies of S^1's by Lemma 2.3.

Notice that $K_{\sigma_\ell}^{(1)}$ is null-homotopic in K_{σ_ℓ} because K_{σ_ℓ} is contractible. Also, $K_{\sigma_\ell}^{(1)}$ is null-homotopic in $\mathrm{SC}_K(L) \cup L_{\ell-1}$ because the homotopy type of former is a wedge sum of S^1's and the homotopy type of latter is a wedge sum of S^2's. Therefore by Lemma 2.1, $\mathrm{SC}_K(L) \cup L_\ell$ is homotopy equivalent to $\sum(\bigvee_{\binom{n}{2}} S^1) \vee (\mathrm{SC}_K(L) \cup L_{\ell-1})$ which is by inductive assumption $(\vee_{\ell\binom{n}{2}} S^2) \vee \mathrm{SC}_K(L)$. This finishes the proof because $\mathrm{SC}_K(L) \simeq L \simeq \mathcal{VR}(\mathcal{F}_{n-1}^{m-1}; 2)$. $\square$

By an inductive calculation, we obtain the following corollary.

Corollary 3.1. *Suppose that $1 < n < m - 1$. The complex $\mathcal{VR}(\mathcal{F}_n^m; 2)$ is homotopy equivalent to a wedge sum of $\sum\limits_{k=2}^{n} \binom{m+k-1-n}{k+1}\binom{k}{2}$-many copies of S^2's.*

3.3 Vietoris-Rips Complex $\mathcal{VR}(\mathcal{F}_{\preceq A}^m; 2)$

In this section, we'll determine the homotopy type of $\mathcal{VR}(\mathcal{F}_{\preceq A}; 2)$ for $A \in \mathcal{P}([m])$ with $|A| = n$.

As in the discussion in Section 2, $\mathcal{VR}(\mathcal{F}_{\leq r}^m; r)$ is a cone with cone vertex being the empty set, hence contractible; and similarly $\mathcal{VR}(\mathcal{F}_{\geq m-r}^m; r)$ is also contractible. Hence, for any $A \subset [m]$ with $|A| \leq 2$, the complex $\mathcal{VR}(\mathcal{F}_{\preceq A}^m; 2)$ is contractible. So in this section, we will discuss the homotopy type of $\mathcal{VR}(\mathcal{F}_{\preceq A}^m; 2)$ with $|A| \geq 3$.

The following lemma is easy to prove, but heavily used in the discussion of $\mathcal{VR}(\mathcal{F}_{\preceq A}^m; 2)$.

Lemma 3.6. *For any $A, B \in \mathcal{P}[m]$ with $|A| < |B|$, $d(A, B) \leq 2$ if and only if $A \subset B$ and $|B \setminus A| \leq 2$.*

Proof. If $A \subset B$ and $|B \setminus A| \leq 2$, then $d(A, B) = |(A \setminus B) \cup (B \setminus A)| \leq 2$.

Now we suppose $A \setminus B \neq \emptyset$, i.e. $|A \setminus B| \geq 1$. Since $|A| < |B|$, $|B \setminus A| \geq 2$, therefore $d(A, B) \geq 3$. If $A \subset B$ and $|B \setminus A| > 2$, then $d(A, B) = |B \setminus A| > 2$. This finishes the proof. $\square$

Next, we'll discuss the homotopy type of $\mathcal{VR}(\mathcal{F}_n^m \cup \mathcal{F}_{n+1}^m; 2)$ using a similar approach as in the proof of Theorem 3.1.

Theorem 3.2. *Suppose that* $1 < n < m-1$. *Then the complex* $\mathcal{VR}(\mathcal{F}_n^m \cup \mathcal{F}_{n+1}^m; 2)$ *is homotopy equivalent to a wedge sum of* $(\sum_{k=2}^{n} \binom{m+k-1-n}{k+1} \cdot \binom{k}{2} + \binom{m}{n+2} \cdot \binom{n+1}{2})$*-many copies of* S^2.

Proof. Let $K = \mathcal{VR}(\mathcal{F}_n^m \cup \mathcal{F}_{n+1}^m; 2)$ and $K_0 = \mathcal{VR}(\mathcal{F}_n^m; 2)$. By Corollary 3.1, the complex K_0 is homotopy equivalent to a wedge sum of $\sum_{k=2}^{n} \binom{m+k-1-n}{k+1} \cdot \binom{k}{2}$-many copies of S^2's.

We claim that $\mathrm{SC}_K(K_0) \simeq K_0$. We proceed to show that the condition in Lemma 3.1 is satisfied. Hence this claim holds. Take a $B \in \mathcal{F}_{n+1}^m$ such that $B \in \mathrm{st}_K(D) \cap \mathrm{st}_K(D')$ for $D, D' \in \mathcal{F}_n^m$. Then $d(B, D) = d(B, D') = 2$, hence by Lemma 3.6, D, D' are both subsets of B which implies that $d(D, D') = 2$. This finishes the proof of the claim.

By Lemma 3.4, there are two types of maximal simplicies in $\mathcal{VR}(\mathcal{F}_{n+1}^m; 2)$. If σ is a maximal simplex $\mathcal{VR}(\mathcal{F}_{n+1}^m; 2)$ which can be represented in the form $N[i_1, i_2, \ldots, i_n]$, clearly $\{i_1 i_2 \ldots i_n\} \cup N[i_1, i_2, \ldots, i_n]$ is a simplex in K; hence $N[i_1, i_2, \ldots, i_n] \in \mathrm{SC}_K(K_0)$.

Now we look at the second type of maximal simplices in $\mathcal{VR}(\mathcal{F}_{n+1}^m; 2)$. There are $\binom{m}{n+2}$-many type of maximal simplicies in $\mathcal{VR}(\mathcal{F}_{n+1}^m; 2)$ which are in the form $L[i_1, i_2, \ldots, i_{n+2}]$; and list such $(n+1)$-simplicies as $\{\sigma_1, \sigma_2, \ldots, \sigma_{\binom{m}{n+2}}\}$. Denote $L_\ell = \mathrm{SC}_K(K_0) \cup \bigcup_{j=1}^{\ell} K_{\sigma_j}$ for $\ell = 1, 2, \ldots, \binom{m}{n+2}$. Recall that the complex K_{σ_j} is the complex generated by the simplex σ_j for $j = 1, 2, \ldots, \binom{m}{n+2}$.

Assume for induction that $L_{\ell-1}$ is homotopic to

$$\bigvee_{\sum_{k=2}^{n} \binom{m+k-1-n}{k+1} \cdot \binom{k}{2} + (\ell-1) \cdot \binom{n+1}{2}} S^2.$$

This is clearly true when $\ell = 1$. We claim that $L_{\ell-1} \cap K_{\sigma_\ell} = K_{\sigma_\ell}^{(1)}$ which is homotopic to $\bigvee_{\binom{n+1}{2}} S^1$ and hence is null-homotopic in both $L_{\ell-1}$ and K_{σ_ℓ}. By Lemma 2.1, this implies that L_ℓ is homotopy equivalent to a wedge sum of $(\sum_{k=2}^{n} \binom{m+k-1-n}{k+1} \cdot \binom{k}{2} + \ell \cdot \binom{n+1}{2})$-many S^2. This finishes the proof. Next, we'll prove our claim.

By part ii) of Lemma 3.4, any 2-simplex in K_{σ_ℓ} is not in $L_{\ell-1}$. Let $\{B_1, B_2\}$ be a 1-simplex in K_{σ_ℓ}. Then $B_1 \cap B_2$ is an n-subset, i.e., a vertex in K_0; so $\{B_1, B_2, B_1 \cap B_2\}$ is a 2-simplex in K which means $\{B_1, B_2\} \in \mathrm{st}_K(B_1 \cap B_2)$. This shows that $L_{\ell-1} \cap K_{\sigma_\ell} = K_{\sigma_\ell}^{(1)}$. $\square$

To identify the homotopy types of $K = \mathcal{VR}(\mathcal{F}^m_{\preceq A}; 2)$ with $|A| \geq 3$, we'll use Lemma 2.2 by taking the vertex A so that $K = (K \setminus A) \cup \mathrm{st}_K(A)$. So the key is to understand the link of A in K, $\mathrm{lk}_K(A)$. Next lemma shows that $\mathrm{lk}_K(A)$ is a wedge sum of S^2's.

Note that when $n = 3$, $\sum_{k=2}^{n-2} \binom{k}{2}$ is set to be 0 as introduced in Section 2.

Lemma 3.7. *Suppose that $m \geq n > 2$ and $A = i_1 i_2 \dots i_n \in \mathcal{P}([m])$.*

Denote $i_0 = -1$ and define $d_\ell = i_\ell - (i_{\ell-1} + 1)$ for each $\ell = 1, 2, \dots, n$. Then

$$lk_{\mathcal{VR}(\mathcal{F}^m_{\preceq A};2)}(A) \simeq \bigvee_{\sum_{k=2}^{n-2} \binom{k}{2} + \sum_{\ell=1}^{n-2} d_\ell \cdot \binom{n-\ell}{2}} S^2.$$

Proof. Let $K = \mathcal{VR}(\mathcal{F}^m_{\preceq A}; 2)$. Note that for any B with $|B| \leq n-3$, $d(A, B) \geq 3$. Next we divide the vertices in the link of the vertex A in K, $\mathrm{lk}_K(A)$, into the following pairwise disjoint collections $\mathcal{G}_k$ for $k = 0, 1, \dots, i_{n-1}$. These collections are defined as the following:

i) $\mathcal{G}_0 = \{B \in \mathcal{P} : |B| < n \text{ and } d(B, A) = 2\}$;

ii) for $k \in \{1, 2, \dots, i_{n-1}\} \setminus \{i_1, i_2, \dots, i_{n-1}\}$, $\mathcal{G}_k$ contains all the $B's$ with $|B| = n$ such that B contains k, all i_j's with $i_j < k$, all but one of i_j's with $i_j > k$;

iii) $\mathcal{G}_{i_{n-1}}$ contains all the B's with $|B| = n$ such that $\{i_1, i_2, \dots, i_{n-1}\} \subset B$ and B contains any other number between i_{n-1} and i_n.

iv) $\mathcal{G}_{i_j} = \emptyset$ for $j = 1, 2, \dots, n-2$ for the purpose of convenience.

By Lemma 3.6, $\mathcal{G}_0$ contains all the B's such that $B \subset A$ and $|B| = n-1$ or $n-2$. Also, it is clear that $\bigcup_{k=1}^{i_{n-1}} \mathcal{G}_k$ contains all the B's such that $B \prec A$, $d(A, B) = 2$, and $|B| = n$. Hence $\mathrm{lk}_K(A) = \mathcal{VR}(\bigcup_{k=0}^{i_{n-1}} \mathcal{G}_k; 2)$. For each $k = 0, 1, \dots, i_{n-1}$, we define $K_k = \mathcal{VR}(\mathcal{G}_k; 2)$ if $\mathcal{G}_k \neq \emptyset$ and $K_{\leq k} = \mathcal{VR}(\bigcup_{i=0}^{k} \mathcal{G}_i; 2)$. Hence $\mathrm{lk}_K(A) = K_{\leq i_{n-1}}$.

Since $\mathcal{G}_0$ is the collection of all $(n-2)$-subsets and $(n-1)$-subsets of the n-set A, the complex K_0 is homeomorphic to $\mathcal{VR}(\mathcal{F}^n_{n-2} \cup \mathcal{F}^n_{n-1}; 2)$; hence by Theorem 3.2, the complex

$K_0 = K_{\leq 0}$ is homotopy equivalent to a wedge sum of $(\sum_{k=2}^{n-2}\binom{k}{2} + \binom{n-1}{2})$-many copies of S^2's. Since $\mathcal{G}_{i_j} = \emptyset$ for $j = 1, 2, \ldots, n-2$, the complex $K_{\leq i_j}$ is same as $K_{\leq i_j - 1}$ for such j.

Now we investigate the complex K_k with $k \geq 1$ and the collection $\mathcal{G}_k \neq \emptyset$. Fix k such that $1 \leq k < i_{n-1}$ and $\mathcal{G}_k \neq \emptyset$. Then there exists an ℓ in the set $\{1, 2, \ldots, n-1\}$ such that $i_{\ell-1} < k < i_\ell$. Then, the complex K_k is the complex generated by a proper face of $L[i_1, \ldots, i_{\ell-1}, k, i_\ell, \ldots, i_n]$ which consists of all B which contains $\{i_1, \ldots, i_{\ell-1}, k\}$ and all but one of $\{i_\ell, \ldots, i_n\}$; hence it is an $(n-\ell)$-simplex. And $K_{i_{n-1}}$ is a proper face of $N[i_1, i_2, \ldots, i_{n-1}]$ which includes all B's which contains $\{i_1, i_2, \ldots, i_{n-1}\}$ and another number between i_{n-1} and i_n; hence it is a complex generated by a $(d_n - 1)$-simplex.

Next we determine the homotopy type of $K_{\leq i_{n-2}}$. If there is no k such that $k \in [i_{n-2}] \setminus \{i_1, i_2, \ldots, i_{n-2}\}$, then $d_1 = 1$ and $d_2, \ldots, d_{n-2}$ are all zeroes and the complex $K_{\leq i_{n-2}} = K_0$ which is clearly homotopy equivalent $\bigvee_{\sum_{k=2}^{n-2}\binom{k}{2} + \sum_{\ell=1}^{n-2} d_\ell \cdot \binom{n-\ell}{2}} S^2$. Now we suppose otherwise and fix k such that $1 \leq k \leq i_{n-2}$ and $i_{\ell-1} < k < i_\ell$ for some $\ell = 1, 2, \ldots, n-2$, here we define $i_0 = 0$. Suppose, for induction, that $K_{\leq (k-1)}$ is homotopy equivalent to a wedge sum of S^2's. This holds when k is the minimal natural number different from $i_1, i_2, \ldots i_{n-2}$ in which case $K_{\leq k-1}$ is homotopy equivalent to K_0. By Lemma 3.2, $K_{\leq k} = \mathrm{SC}_{K_{\leq k}}(K_{\leq (k-1)}) \cup K_k$. We'll prove the following two claims and these two claims imply that $K_{\leq k} \simeq K_{\leq (k-1)} \vee (\bigvee_{\binom{n-\ell}{2}} S^2)$ by Lemma 2.1 and the inductive assumption.

Claim i) $\mathrm{SC}_{K_{\leq k}}(K_{\leq (k-1)}) \simeq K_{\leq (k-1)}$.

Claim ii) $\mathrm{SC}_{K_{\leq k}}(K_{\leq (k-1)}) \cap K_k \simeq \bigvee_{\binom{n-\ell}{2}} S^1$.

Proof of Claim i): We'll verify that the condition in Lemma 3.1 is satisfied. Then the result follows. We'll show that $d(C_1, C_2) \leq 2$ for $C_1, C_2 \in K_{\leq (k-1)}$ whenever $st_{K_{\leq k}}(C_1) \cap st_{K_{\leq k}}(C_2) \setminus K_{\leq (k-1)} \neq \emptyset$. Pick a vertex D in K_k. Then D contains k and an $(n-1)$-subset of A, denoted by C. Then for any vertex $B \in K_{\leq (k-1)}$, $D \in \mathrm{st}_{K_{\leq k}}(B)$ if and only if B is one of the following: a) $C \subset B$ and B contains one of $1, 2, \ldots, k-1$ not in A; b) C; c) any $(n-2)$ subset of C. Any pair of such vertices have distance 2; hence they form a 1-simplex in $K_{\leq (k-1)}$. This finishes the proof of Claim i).

Proof of Claim ii): Since the complex K_k is generated by an $(n-\ell)$-simplex, $K_k^{(1)}$ is homotopy equivalent to $\bigvee_{\binom{n-\ell}{2}} S^1$ by Lemma 2.3. We'll show that $\mathrm{SC}_{K_{\leq k}}(K_{\leq(k-1)}) \cap K_k = K_k^{(1)}$. Pick any pair of vertices, B_1, B_2, in K_k. Then, $B_1 \cap B_2$ contains the number k and an $(n-2)$-subset of A, denoted by D. Note that D is a vertex in the complex $K_0 \subseteq K_{\leq k-1}$; therefore, the 1-simplex $\{B_1, B_2\} \in \mathrm{st}_{K_{\leq k}}(D)$. Hence $K_k^{(1)} \subseteq \mathrm{SC}_{K_{\leq k}}(K_{\leq(k-1)}) \cap K_k$. It is straightforward to verity that for any $B \in \mathcal{G}_i$ with $i = 1, 2, \ldots, k-1$, $\mathrm{st}_{K_{\leq k}}(B) \cap K_k$ is a complex containing only one vertex because any vertex in this complex must contains $B \cap A$ and the number k. Similarly, for any $B \in \mathcal{G}_0$ with $|B| = n-1$, there is at most 1 vertex in K_k containing B as a subset, i.e. having a distance ≤ 2 from B; and if $B \in \mathcal{G}_0$ with $|B| = n-2$, then there are at most two vertices in K_k which have distance 2 from B. Hence, $\mathrm{st}_{K \leq k}(B) \cap K_k \subseteq K_k^{(1)}$ for any vertex B in the complex $K_{\leq(k-1)}$. This finishes the proof of Claim ii).

By an inductive calculation, we have proved that the complex $K_{\leq i_{n-2}}$ is homotopy equivalent to a wedge sum of $(\sum_{k=2}^{n-2}\binom{k}{2} + \sum_{\ell=1}^{n-2} d_\ell \cdot \binom{n-\ell}{2})$-many S^2's. Next, we show that the complex $K_{\leq(i_{n-1}-1)}$ is homotopy equivalent to $K_{\leq i_{n-2}}$. If $d_{n-1} = 0$, then $K_{<i_{n-1}} = K_{\leq i_{n-2}}$; otherwise we fix k with $i_{n-2} < k < i_{n-1}$ and suppose that $K_{\leq(k-1)} \simeq K_{\leq i_{n-2}}$. The collection $\mathcal{G}_k$ contains two vertices $i_1 i_2 \ldots i_{n-2} k i_n$ and $i_1 i_2 \ldots i_{n-2} k i_{n-1}$; and the simplex $\{i_1 i_2 \ldots i_{n-2} k i_n, i_1 i_2 \ldots i_{n-2} k i_{n-1}\}$ is in $\mathrm{st}_{K_{\leq(i_{n-1}-1)}}(D)$ where $D = i_1 i_2 \ldots i_{n-2} \in K_{\leq(k-1)}$. Hence $\mathrm{SC}_{K_{\leq k}}(K_{\leq(k-1)}) = K_{\leq k}$. By a similar discussion as in the proof of claim i), we can verify that the complex $K_{\leq k}$ and its subcomplex $K_{\leq(k-1)}$ satisfy the condition in Lemma 3.1. Therefore, $\mathrm{SC}_{K_{\leq k}}(K_{\leq(k-1)}) \simeq K_{\leq(k-1)}$. Hence the complex $K_{\leq k}$ is homotopy equivalent to $K_{\leq i_{n-2}}$. Therefore by induction, the complex $K_{\leq(i_{n-1}-1)}$ is homotopy equivalent to $K_{\leq i_{n-2}}$.

In the last part, we show that the complex $K_{\leq i_{n-1}} = \mathrm{lk}_K(A)$ is also homotopy equivalent to $K_{\leq i_{n-2}}$. Again it is straightforward to verify that $K_{\leq i_{n-1}}$ and $K_{\leq(i_{n-1}-1)}$ satisfy the condition in Lemma 3.1. Hence, $\mathrm{SC}_{K_{\leq i_{n-1}}}(K_{\leq(i_{n-1}-1)}) \simeq K_{\leq(i_{n-1}-1)}$. Recall that $K_{i_{n-1}}$ is a complex generated by a proper face of the simplex $N[i_1 i_2 \ldots i_{n-1}]$. Note that $i_1 i_2 \ldots i_{n-1}$ is a vertex in $K_{\leq(i_{n-1}-1)}$; and also, $\{i_1 i_2 \ldots i_{n-1}\} \cup \mathcal{G}_{i_{n-1}}$ is a simplex in $K_{\leq i_{n-1}}$. So, $K_{i_{n-1}} \subset \mathrm{st}_{K_{\leq i_{n-1}}}(i_1 i_2 \ldots i_{n-1})$; and hence $\mathrm{SC}_{K_{\leq i_{n-1}}}(K_{\leq(i_{n-1}-1)}) = K_{\leq i_{n-1}}$. And this finishes the proof. $\square$

Motivated by the lemma above, we define a natural number r_A for each $A \subset [m]$ in the following way. For each $A = i_1 i_2 \dots i_n \subseteq [m]$ with $d_1 = i_1$ and $d_\ell = i_\ell - (i_{\ell-1} + 1)$ for $\ell = 2, 3, \dots, n$, we define

$$r_A = \sum_{k=2}^{n-2} \binom{k}{2} + \sum_{\ell=1}^{n-2} d_\ell \cdot \binom{n-\ell}{2}.$$

Theorem 3.3. *Suppose that $m \geq n > 2$ and $A = i_1 i_2 \dots i_n \in \mathcal{P}([m])$. Then the complex $\mathcal{VR}(\mathcal{F}^m_{\preceq A}; 2)$ is homotopy equivalent to a wedge sum of S^3's.*

More specifically, if A is the vertex $\{1, 2, 3\} \subset [m]$,

$$\mathcal{VR}(\mathcal{F}^m_{\preceq A}; 2) \simeq S^3.$$

And for any other vertex A with $\{1, 2, 3\} \prec A$,

$$\mathcal{VR}(\mathcal{F}^m_{\preceq A}; 2) \simeq (\bigvee_{r_A} S^3) \vee \mathcal{VR}(\mathcal{F}^m_{\prec A}; 2).$$

Therefore if $A \in \mathcal{P}([m])$ with $|A| \geq 3$, $\mathcal{VR}(\mathcal{F}^m_{\preceq A}; 2)$ is homotopy equivalent to the wedge sum of $\sum\{r_B : \{1, 2, 3\} \preceq B \preceq A\}$-many copies of S^3.

Proof. Let $K = \mathcal{VR}(\mathcal{F}^m_{\preceq A}; 2)$ and $L = \mathcal{VR}(\mathcal{F}^m_{\prec A}; 2)$. Suppose $A = \{1, 2, 3\}$. Then $r_A = 1$, hence $\text{lk}_K(A)$ is homotopic to S^2 by Lemma 3.7. Because the complex L is contractible, the complex K is homotopy equivalent to S^3 by Lemma 2.2.

Fix A with $\{1, 2, 3\} \prec A$ and suppose for induction that L is homotopy equivalent to a wedge sum of S^3's. Again by Lemma 3.7, $\text{lk}_K(A)$ is homotopic to a wedge sum of r_A-many S^2's. Hence the inclusion map from $\text{lk}_K(A)$ to L is null-homotopic. Therefore, the general result holds due to again Lemma 2.2. □

The following result is a direct application of Lemma 2.1, Lemma 3.7, and Theorem 3.3.

Theorem 3.4. *Suppose that $m \geq n > 2$. For each n, we define*

$$t_n = \sum_{A \subseteq [m] \text{ with } |A| = n} r_A.$$

Then,

$$\mathcal{VR}(\mathcal{F}^m_{\le n}; 2) \simeq \bigvee_{t_n} S^3 \vee \mathcal{VR}(\mathcal{F}^m_{\le n-1}; 2).$$

Therefore, $\mathcal{VR}(\mathcal{F}^m_{\le n}; 2)$ is homotopy equivalent to the wedge sum of $(\sum_{k=3}^n t_k)$-many copies of S^3.

Adamaszek and Adams in [2] proved that $\mathcal{VR}(Q_m; 2) = \mathcal{VR}(\mathcal{F}^m_{\le m}; 2) \simeq \bigvee_{c_m} S^3$ for any $m > 2$, where $c_m = \sum_{0\le j<i<m}(j+1)(2^{m-2} - 2^{i-1})$. By Theorem 3.4, $c_m = \sum_{k=3}^m t_k$ where t_n is defined as in the statement of Theorem 3.4.

3.4 Vietoris-Rips Complex $\mathcal{VR}(\mathcal{F}^m_n \cup \mathcal{F}^m_{n'}; 2)$

In this section, we'll investigate the homotopy types of $\mathcal{VR}(\mathcal{F}^m_n \cup \mathcal{F}^m_{n'}; 2)$ with $n, n' \in \mathbb{N}$. Clearly when $|n - n'| \ge 3$, then $\mathcal{VR}(\mathcal{F}^m_n \cup \mathcal{F}^m_{n'}; 2)$ is a disjoint union of $\mathcal{VR}(\mathcal{F}^m_n; 2)$ and $\mathcal{VR}(\mathcal{F}^m_{n'}; 2)$; then by the discussion in Section 3.2, its homotopy type is clear. The homotopy types of the complex $\mathcal{VR}(\mathcal{F}^m_n \cup \mathcal{F}^m_{n+1}; 2)$ are discussed in Section 3.3 (see Theorem 3.2).

In the following, we'll find the homotopy types of the Vietoris-Rips complexes $\mathcal{VR}(\mathcal{F}^m_n \cup \mathcal{F}^m_{n+2}; 2)$ for $n + 2 \le m$. Clearly for $m \ge 3$, $\mathcal{VR}(\mathcal{F}^m_0 \cup \mathcal{F}^m_2; 2)$ and $\mathcal{VR}(\mathcal{F}^m_m \cup \mathcal{F}^m_{m-2}; 2)$ are contractible because both of them are cones. Next, we'll discuss the complexes $\mathcal{VR}(\mathcal{F}^m_n \cup \mathcal{F}^m_{n+2}; 2)$ in general.

The next result can be obtained by applying the proof of Lemma 3.7 with small modifications; next we'll go through the difference of the proofs. For each $A = i_1 i_2 \dots i_n \in \mathcal{F}^m_n$ with $c_1 = i_1 - 1$ and $c_\ell = i_\ell - (i_{\ell-1} + 1)$ for $\ell = 2, 3, \dots, n$, we define

$$s_A = \sum_{k=2}^{n-2} \binom{k}{2} + \sum_{\ell=1}^{n-2} c_\ell \binom{n-\ell}{2}.$$

Note that for any $A \subset [m]$ with $|A| = n$, $r_A = s_A + \binom{n-1}{2}$.

Lemma 3.8. *Suppose that $4 \le n < m - 1$ and $A = i_1 i_2 \dots i_n \subset [m]$ with $i_1 \ge 2$. Let $K = \mathcal{VR}(\mathcal{F}^m_{n-2} \cup \mathcal{F}^m_n; 2) \cap \mathcal{VR}(\mathcal{F}^m_{\preceq A}; 2)$.*

Then,

$$lk_K(A) \simeq \bigvee_{s_A} S^2$$

Proof. As in the proof of Lemma 3.7, we divide the vertices in $\mathrm{lk}_K(A)$ into pairwise disjoint collections $\mathcal{G}_k$ for $k = 0, 1, \ldots, i_{n-1}$. For $k \geq 1$, $\mathcal{G}_k$ is exactly defined in the same way as in the proof of Lemma 3.7. Note that the vertices in K have either size $n-2$ or n. Then $\mathcal{G}_0$ contains all the subsets of A with size $n-2$. The complexes K_k and $K_{\leq k}$ are defined in the same ways as in the proof of Lemma 3.7 for $k = 0, 1, \ldots, i_{n-1}$. Hence K_0 is homeomorphic to $\mathcal{VR}(\mathcal{F}^n_{n-2}; 2)$; hence by Corollary 3.1, it is homotopy equivalent to $\sum_{k=2}^{n-2} \binom{k}{2}$-many copies of S^2's. Then the rest of the proof is same as the proof of Lemma 3.7. $\square$

Theorem 3.5. *Suppose that* $1 \leq n < m-3$. *Then the complex* $\mathcal{VR}(\mathcal{F}^m_n \cup \mathcal{F}^m_{n+2}; 2)$ *is homotopy equivalent to a wedge sum of* S^3*'s.*

More specifically,

$$\mathcal{VR}(\mathcal{F}^m_1 \cup \mathcal{F}^m_3; 2) \simeq \bigvee_{\binom{m}{4}} S^3;$$

and for $n \geq 2$, *we define* $o_{m,n} = \sum\{s_A : A \in \mathcal{F}^m_{n+2}$ *with* $\min A \geq 2\}$ *and then*

$$\mathcal{VR}(\mathcal{F}^m_n \cup \mathcal{F}^m_{n+2}; 2) \simeq \mathcal{VR}(\mathcal{F}^{m-1}_{n-1} \cup \mathcal{F}^{m-1}_{n+1}; 2) \vee \bigvee_{o_{m,n}} S^3.$$

Therefore, $\mathcal{VR}(\mathcal{F}^m_n \cup \mathcal{F}^m_{n+2}; 2)$ *is homotopy equivalent to* $(\sum_{k=2}^{n} o_{m+k-n,k} + \binom{m+1-n}{4})$*-many copies of* S^3.

Proof. We firstly prove that $K = \mathcal{VR}(\mathcal{F}^m_1 \cup \mathcal{F}^m_3; 2) \simeq \bigvee_{\binom{m}{4}} S^3$. Let $L_0 = \mathcal{VR}(\mathcal{F}^m_1; 2)$ which is a complex generated by a simplex because each pair of singleton subsets of $[m]$ has distance 2. Hence by Lemma 3.1, $\mathrm{SC}_K(L_0)$ is contractible. By Lemma 3.4, there are two types of maximal simplices in $\mathcal{VR}(\mathcal{F}^m_3; 2)$, namely $N[i_1, i_2]$ and $L[i_1, i_2, i_3, i_4]$ for some $i_1, \ldots, i_4 \in [m]$; clearly $\{i_1\} \cup N[i_1, i_2]\}$ is a simplex in K. Hence $N[i_1, i_2] \in \mathrm{SC}_K(L_0)$ for each $i_1, i_2 \in [m]$. Within $\mathcal{VR}(\mathcal{F}^m_3; 2)$, there are $\binom{m}{4}$-many simplices in the form $L[i_1, i_2, i_3, i_4]$ and the intersection of each pair of such simplices contains at most one vertex. We list such simplices as

$\{\sigma_\ell : \ell = 1, 2, \dots, \binom{m}{4}\}$ and define $L_\ell = \mathrm{SC}_K(L_0) \cup \bigcup_{i=1}^{\ell} \sigma_\ell$. We see that $\sigma_\ell \notin \mathrm{SC}_K(L_0)$ for each $\ell = 1, 2, \dots, \binom{m}{4}$ because otherwise there is a number in $\cap\sigma_\ell$ which is a contradiction; and because each of σ_ℓ's proper faces has an nonempty intersection, we get that $\sigma_\ell^{(2)} \subset \mathrm{SC}_K(L_0)$. Hence $L_{\ell-1} \cap \sigma_\ell = \sigma_\ell^{(2)} \simeq S^2$. Therefore, by Lemma 2.1, $L_1 \simeq S^3$ and inductively $L_\ell \simeq \bigvee_\ell S^3$. This finishes the proof of first part.

Now we assume that $n \geq 2$ and $\mathcal{VR}(\mathcal{F}_{n-1}^{m-1} \cup \mathcal{F}_{n+1}^{m-1}; 2)$ is homotopy equivalent to a wedge sum of S^3's. Let $\mathcal{G}_0 = \{B \in \mathcal{F}_n^m \cup \mathcal{F}_{n+2}^m : 1 \in B\}$ and $K_0 = \mathcal{VR}(\mathcal{G}_0; 2)$; by a straightforward isometric mapping, we see that $K_0 \cong \mathcal{VR}(\mathcal{F}_{n-1}^{m-1} \cup \mathcal{F}_{n+1}^{m-1}; 2)$ which is homotopy equivalent to a wedge sum of S^3's by the assumption. Let $\mathcal{G}_1 = \{B \in \mathcal{F}_n^m \cup \mathcal{F}_{n+2}^m : |B| = n \text{ or } 1 \in B\}$ and $K_1 = \mathcal{VR}(\mathcal{G}_1; 2)$.

Next, we show that $K_1 = \mathrm{SC}_{K_1}(K_0) \simeq K_0$. Let σ be a simplex in K_1 consisting of vertices not containing 1. Then σ is a face of either $N[i_1, i_2, \dots, i_{n-1}]$ or $L[i_1, i_2, \dots, i_{n+1}]$ with all the numbers > 1. Since $1i_1i_1 \dots i_{n-1} \in N[i_1, i_2, \dots, i_{n-1}]$, $N[i_1, i_2, \dots, i_{n-1}] \in \mathrm{SC}_{K_1}(K_0)$. Also notice that $\{1i_1i_2 \dots i_{n+1}\} \cup L[i_1, i_2, \dots, i_{n+1}]$ is a simplex in K_1; hence $L[i_1, i_2, \dots, i_{n+1}] \in \mathrm{SC}_{K_1}(K_0)$. Therefore, $K_1 = \mathrm{SC}_{K_1}(K_0)$.

Next we show that the condition in Lemma 3.1 is satisfied which implies that $\mathrm{SC}_{K_1}(K_0) \simeq K_0$. Pick a vertex $B = i_1i_2 \dots i_n$ in $\mathcal{F}_n^m$ not containing 1 such that $B \in \mathrm{st}_{K_1}(D_1) \cap \mathrm{st}_{K_1}(D_2)$ with $D_1, D_2 \in \mathcal{G}_0$. There are three cases to discuss.

Case 1: Suppose $|D_1| = |D_2| = n + 2$. Then by Lemma 3.6, $B \subset D_1$ and $B \subset D_2$. Since both D_1 and D_2 contain 1, $|D_1 \cap D_2| = n + 1$ and therefore $\{D_1, D_2\} \in K_0$.

Case 2: Suppose $|D_1| = n$ and $|D_2| = n + 2$. Then D_1 contains an $(n-1)$-subset of B and 1; hence $D_1 \subset D_2$. By Lemma 3.6, $d(D_1, D_2) = 2$ and therefore $\{D_1, D_2\} \in K_0$.

Case 3: Suppose $|D_1| = |D_2| = n$. Then both D_1 and D_2 contains an $(n-1)$-subset of B and 1 and hence $|D_1 \cap D_2| = n - 1$, i.e., $d(D_1, D_2) = 2$. Therefore $\{D_1, D_2\} \in K_0$.

Now fix $A \in \mathcal{F}^m_{n+2}$ with $\min A \geq 2$ and assume for induction that $\mathcal{VR}(\{B \in \mathcal{F}^m_n \cup \mathcal{F}^m_{n+2} : B \prec A\}; 2)$ is homotopy equivalent to

$$\mathcal{VR}(\mathcal{F}^{m-1}_{n-1} \cup \mathcal{F}^{m-1}_{n+1}; 2) \vee \bigvee_{\sum_{B \in \mathcal{F}^m_{n+2} \text{ with } \min B \geq 2 \text{ and } B \prec A} s_B} S^3$$

which is a wedge sum of S^3's. If $A = \min_{\prec}\{C : C \in \mathcal{F}^m_{n+2} \text{ and } \min C = 2\}$, then the set $\{B : B \in \mathcal{F}^m_{n+2} \text{ with } \min B \geq 2 \text{ and } B \prec A\}$ is empty; so the inductive assumption holds since $\mathcal{VR}(\{B \in \mathcal{F}^m_n \cup \mathcal{F}^m_{n+} : B \prec A\}; 2) \simeq \mathcal{VR}(\mathcal{F}^{m-1}_{n-1} \cup \mathcal{F}^{m-1}_{n+1}; 2)$ by the discussion above.

Let $L = \mathcal{VR}(\{B \in \mathcal{F}^m_{n+2} \cup \mathcal{F}^m_n : B \preceq A\}; 2)$. Then by Lemma 3.8, $\mathrm{lk}_L(A)$ is homotopy equivalent to $\bigvee_{s_A} S^2$ which is clearly contractible in $L \setminus \{A\}$. Hence by Lemma 2.2, L is homotopy equivalent to

$$\mathcal{VR}(\mathcal{F}^{m-1}_{n+1} \cup \mathcal{F}^{m-1}_{n+1}; 2) \vee \bigvee_{\sum_{B \in \mathcal{F}^m_{n+2} \text{ with } \min B \geq 2 \text{ and } B \prec A} s_B} S^3 \vee \sum(\bigvee_{s_A} S^2),$$

i.e.

$$\mathcal{VR}(\mathcal{F}^{m-1}_{n+1} \cup \mathcal{F}^{m-1}_{n+1}; 2) \vee \bigvee_{\sum_{B \in \mathcal{F}^m_{n+2} \text{ with } \min B \geq 2 \text{ and } B \preceq A} s_B} S^3.$$

This finishes the proof. □

We conclude this section by showing that the vertices $\mathcal{F}^m_{n+1}$ in the complex $\mathcal{VR}(\mathcal{F}^m_n \cup \mathcal{F}^m_{n+1} \cup \mathcal{F}^m_{n+2}; 2)$ don't contribute to its homotopy type which means that it is homotopy equivalent to $\mathcal{VR}(\mathcal{F}^m_n \cup \mathcal{F}^m_{n+2}; 2)$.

Theorem 3.6. *Suppose that* $1 \leq n < m - 3$ *with* $m \geq 4$*. Then,*

$$\mathcal{VR}(\mathcal{F}^m_n \cup \mathcal{F}^m_{n+1} \cup \mathcal{F}^m_{n+2}; 2) \simeq \mathcal{VR}(\mathcal{F}^m_n \cup \mathcal{F}^m_{n+2}; 2).$$

Proof. Let $K = \mathcal{VR}(\mathcal{F}^m_n \cup \mathcal{F}^m_{n+1} \cup \mathcal{F}^m_{n+2}; 2)$ and $K_0 = \mathcal{VR}(\mathcal{F}^m_n \cup \mathcal{F}^m_{n+2}; 2)$. Then we claim that $K = \mathrm{SC}_K(K_0)$ and $\mathrm{SC}_K(K_0) \simeq K_0$.

It is clear that $\mathrm{SC}_K(K_0) \subseteq K$. Take a simplex σ in K such that none of its vertices is in K_0; hence all its vertices are in $\mathcal{F}^m_{n+1}$. By Lemma 3.4, σ is a face of either $N[i_1, i_2, \ldots, i_n]$ or

$L[i_1, i_2, \ldots, i_{n+2}]$. Note that $\{i_1 i_2 \ldots i_n\} \cup N[i_1, i_2, \ldots, i_n]$ is a simplex in K with $i_1 i_2 \ldots i_n \in K_0$; therefore $N[i_1, i_2, \ldots, i_n] \in \mathrm{SC}_K(K_0)$. Also $\{i_1 i_2 \ldots i_{n+2}\} \cup L[i_1, i_2, \ldots, i_{n+2}]$ is a simplex in K with $i_1 i_2 \ldots i_{n+2} \in K_0$; hence, $L[i_1, i_2, \ldots, i_{n+2}] \in \mathrm{SC}_K(K_0)$. Therefore, $\mathrm{SC}_K(K_0) = K$.

Take $D \in \mathcal{F}^m_{n+1}$ with $D \in \mathrm{st}_K(B_1) \cap \mathrm{st}_K(B_2)$ where B_1, B_2 are vertices in K_0. Using a similar discussion as in the proof of Theorem 3.5, $\{B_1, B_2\} \in K_0$. Hence the condition of Lemma 3.1 is satisfied which implies that $\mathrm{SC}_K(K_0) \simeq K_0$.

Therefore, we conclude that $K \simeq K_0$. □

Chapter 4

$\check{C}$ech complex with Scale 2

In this chapter, we investigate the homotopy types of $\check{C}$ech complexes, $\mathcal{N}(\mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}}; 2)$ and $\mathcal{N}(\mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}}; 2)$. We will show that the complex $\mathcal{N}(\mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}}; 2)$ is either contractible or homotopy equivalent to a wedge sum of S^1's and that the complex $\mathcal{N}(\mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}}; 2)$ is either contractible or homotopy equivalent to a wedge sum of S^2's. Using these results, we provide a different proof of the homotopy types of the $\check{C}$ech complexes on the hypercube graph with scale 2.

We start with the complex $\mathcal{N}(\mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}}; 2)$.

4.1 $\check{C}$ech Complex $\mathcal{N}(\mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}}; 2)$

The vertices of the $\check{C}$ech complex $\mathcal{N}(\mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}}; 2)$ are the subsets of $[m]$ with size n or $n+1$. By the definition, the facets of this complex are generated by the following collections:

$$N_1[A] = \{B : A \subseteq B\} \text{ when } A \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n}},$$

$$N_1[A] = \{B : B \subseteq A\} \text{ when } A \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}}.$$

For scale $r = 2$, we deal with the closed 1-neighborhood of vertices and for convenience, we omit 1 in the subscript here on. We denote that $N(A) = N[A] \setminus \{A\}$ for any $A \subseteq [m]$. We use $N[A]$ to denote the maximal simplex in $\mathcal{N}(\mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}}; 2)$ generated by itself. We also use $N[A]$ to denote the simplicial complex generated by this simplex. Notice that the complex $\mathcal{N}(\mathcal{F}^{m}_{0} \cup \mathcal{F}^{m}_{1}; 2)$ is an m-simplex and hence is contractible.

An inductive formula for the homotopy type of the complex $\mathcal{N}(\mathcal{F}_{\rm n}^{\rm m} \cup \mathcal{F}_{\rm n+1}^{\rm m}; 2)$ is given in the result below.

Theorem 4.1. *Fix $m \in \mathbb{N}$ and $n \geq 0$ with $m \geq n+1$. The Čech complex $\mathcal{N}(\mathcal{F}_{\rm n}^{\rm m} \cup \mathcal{F}_{\rm n+1}^{\rm m}; 2)$ is either contractible or homotopy equivalent to a wedge sum of circles. Specifically,*

$$\mathcal{N}(\mathcal{F}_{\rm n}^{\rm m} \cup \mathcal{F}_{\rm n+1}^{\rm m}; 2) \simeq \mathcal{N}(\mathcal{F}_{n-1}^{m-1} \cup \mathcal{F}_{n}^{m-1}; 2) \vee \left(\bigvee_{n \cdot \binom{m-1}{n+1}} S^1 \right).$$

Proof. The result holds for $n = 0$ since $\mathcal{N}(\mathcal{F}_0^m \cup \mathcal{F}_1^m; 2)$ is contractible.

Now we fix $n \geq 1$ and assume that the complex $\mathcal{N}(\mathcal{F}_{n-1}^{m-1} \cup \mathcal{F}_{n}^{m-1}; 2)$ is either contractible or homotopy equivalent to a wedge sum of S^1's. And we prove that the inductive formula holds for the complex $\mathcal{N}(\mathcal{F}_{\rm n}^{\rm m} \cup \mathcal{F}_{\rm n+1}^{\rm m}; 2)$.

We define $\mathcal{B}_1 = \{A \in \mathcal{F}_{\rm n}^{\rm m} \cup \mathcal{F}_{\rm n+1}^{\rm m} : 1 \in A\}$. It is straightforward to see that $\mathcal{B}_1$ is isometric to $\mathcal{F}_{n-1}^{m} \cup \mathcal{F}_{n}^{m}$ and hence by the inductive assumption, the complex $\mathcal{N}(\mathcal{B}_1; 2)$ is either contractible or homotopy equivalent to a wedge sum of S^1's.

Now fix $A \in \mathcal{F}_{\rm n}^{\rm m}$ with $1 \notin A$ and let A_- be the predecessor of A in the ordering $\prec$. Denote $\mathcal{P}_A = \mathcal{B}_1 \cup \{B \in \mathcal{F}_{\rm n}^{\rm m} : B \preceq A\}$ and the collection $\mathcal{P}_{A_-}$ is defined in the same way. Note that if $1 \in A_-$, then $\mathcal{P}_A \setminus \{A\} = \mathcal{B}_1$. We claim that $\mathcal{N}(\mathcal{P}_A; 2) \simeq \mathcal{N}(\mathcal{P}_{A_-}; 2)$. This implies that $\mathcal{N}(\mathcal{B}_1 \cup \mathcal{F}_{\rm n}^{\rm m}; 2) \simeq \mathcal{N}(\mathcal{B}_1; 2)$ by induction.

We assume that $\mathcal{P}_{A_-}$ is either contractible or homotopy equivalent to a wedge sum of S^1's. This is true when $1 \in A_-$. In the complex $\mathcal{N}(\mathcal{P}_A; 2)$, there are two maximal simplices containing A, namely $N[A]$ and $N[1A]$. Notice that $N[A] = \{A, 1A\}$; hence $\mathcal{N}(\mathcal{P}_{A_-}; 2) \cup \{N_{\mathcal{P}_A}[A] \setminus \{A\}\} = \mathcal{N}(\mathcal{P}_{A_-}; 2)$ which means that the maximal simplex introduced by A doesn't change the complex $\mathcal{N}(\mathcal{P}_{A_-}; 2)$. Also, it is straightforward to verify that the link of A in the complex $\mathcal{N}(\mathcal{P}_A; 2)$ is $N(1A)$ which is clearly contractible. Hence, the inclusion map is null-homotopic. By Lemma-2.1,

$$\mathcal{N}(\mathcal{P}_A; 2) \simeq \mathcal{N}(\mathcal{P}_{A_-}; 2).$$

Hence, we obtain that $\mathcal{N}(\mathcal{B}_1 \cup \mathcal{F}_{\rm n}^{\rm m}; 2)$ is also contractible or homotopy equivalent to a wedge sum of S^1's. Now we consider the complex $\mathcal{N}(\mathcal{F}_{\preceq A}^{m}; 2)$ with $A \in \mathcal{F}_{\rm n+1}^{\rm m}$ and $\mathcal{F}_{\preceq A}^{m} =$

$\{B \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} : B \preceq A\}$. Fix $A \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}}$ and A_- to be the predecessor of A. Assume that $\mathcal{N}(\mathcal{F}^m_{\preceq A_-}; 2)$ is contractible or homotopy equivalent to a wedge sum of S^1's. This is true when $1 \in A_-$ in which case $\mathcal{N}(\mathcal{F}^m_{\preceq A_-}; 2) = \mathcal{N}(\mathcal{B}_1 \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n}}; 2)$. In the complex $\mathcal{N}(\mathcal{F}^m_{\preceq A}; 2)$ the following maximal simplices contain the vertex A: $N(A)$ and $N(B)$ for each $B \in N(A)$. Note that in this case $N(B) = \{C \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} : B \subset C \text{ and } C \preceq A\}$ and A has $(n+1)$-many subsets with size n. It is clear that $\mathcal{N}(\mathcal{F}^m_{\preceq A}; 2) \setminus A = \mathcal{N}(\mathcal{F}^m_{\preceq A_-}; 2) \cup \{N(A)\}$. It is straightforward to verify that $\mathcal{N}(\mathcal{F}^m_{\preceq A_-}; 2) \cap N(A)$ is a complex with $(n+1)$-many isolated vertices, hence homotopy equivalent to $\vee_n S^0$. By the assumption $\mathcal{N}(\mathcal{F}^m_{\preceq A_-}; 2)$ is either contractible or homotopy equivalent to a wedge sum of S^1's, the inclusion map from $\mathcal{N}(\mathcal{F}^m_{\preceq A_-}; 2) \cap N(A)$ to $\mathcal{N}(\mathcal{F}^m_{\preceq A_-}; 2)$ is null-homotopic. By Lemma 2.1 and $N(A)$ being contractible,

$$\begin{aligned}
\mathcal{N}(\mathcal{F}^m_{\preceq A}; 2) \setminus A &\simeq \mathcal{N}(\mathcal{F}^m_{\preceq A_-}; 2) \vee N(C) \vee \sum (\mathcal{N}(\mathcal{F}^m_{\preceq A_-}; 2) \cap N(C)) \\
&\simeq \mathcal{N}(\mathcal{F}^m_{\preceq A_-}; 2) \vee \sum \left(\bigvee_n S^0 \right) \\
&\simeq \mathcal{N}(\mathcal{F}^m_{\preceq A_-}; 2) \vee \left(\bigvee_n S^1 \right).
\end{aligned}$$

To get the homotopy type of $\mathcal{N}(\mathcal{F}^m_{\preceq A}; 2)$, we investigate the link of A. We denote $K = \mathcal{N}(\mathcal{F}^m_{\preceq A}; 2)$ for convenience in this part. It is clear that

$$\mathrm{lk}_K(A) = N(A) \cup \Big(\bigcup_{B \in N(A)} (N[B] \setminus A) \Big).$$

We claim that $\mathrm{lk}_K(A)$ is contractible. List the vertices in $N(A)$ as $B_1, B_2, \ldots, B_{n+1}$. Notice that for $i \neq i'$, $N[B_i] \cap N[B_{i'}] = \{A\}$, hence $(N[B_i] \setminus A) \cap (N[B_{i'}] \setminus A) = \emptyset$; and also for each i, $N(A) \cap (N[B_i] \setminus A) = B_i$. It is straightforward to verify that $N(A) \cup (\bigcup_{j \leq i} (N[B_j)] \setminus A))$ is homotopy equivalent to $N(A) \cup (\bigcup_{j \leq i-1} (N[B_j)] \setminus A))$ by Lemma 2.1. Hence by induction $\mathrm{lk}_K(A)$ is contractible. Therefore the inclusion map from $\mathrm{lk}_K(A)$ to $\mathcal{N}(\mathcal{F}^m_{\preceq A}; 2)$ is null-homotopic.

Then applying Lemma 2.2, we obtain that

$$\mathcal{N}(\mathcal{F}^m_{\preceq A};2) \simeq \mathcal{N}(\mathcal{F}^m_{\preceq A_-};2) \vee \left(\bigvee_n S^1\right).$$

There are $\binom{m-1}{n+1}$-many vertices in $\mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}}$ not containing 1, hence, by induction

$$\mathcal{N}(\mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}};2) \simeq \mathcal{N}(\mathcal{F}^{m-1}_{n-1} \cup \mathcal{F}^{m-1}_{n};2) \vee \left(\bigvee_{n\cdot\binom{m-1}{n+1}} S^1\right).$$

□

Then we get the following result for the homotopy type of the complex $\mathcal{N}(\mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}};2)$.

Corollary 4.1. *Suppose that $n \geq 1$, and $m \geq n+2$. The Čech complex $\mathcal{N}(\mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}};2)$ is homotopy equivalent to a wedge sum of $\sum_{k=1}^{n} k \cdot \binom{m-n+k-1}{k+1}$-many copies of circles.*

4.2 Čech Complex $\mathcal{N}(\mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}};2)$

The vertices of the Čech complex $\mathcal{N}(\mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}};2)$ are the subsets of $[m]$ with size n, $n+1$, or $n+2$. By the definition of Čech complex, the facets of this complex are the ones generated by the following collections:

$$N[A] = \{B : A \subset B \text{ and } B \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}}\} \text{ when } A \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n}},$$

$$N[A] = \{B : A \subseteq B \text{ or } B \subseteq A \text{ and } B \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}}\} \text{ when } A \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}},$$

$$N[A] = \{B : B \subseteq A \text{ and } B \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}}\} \text{ when } A \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}}.$$

Recall that $N(A) = N[A] \setminus \{A\}$. We start with obtaining the homotopy type of the complex $\mathcal{N}(\mathcal{F}^m_0 \cup \mathcal{F}^m_1 \cup \mathcal{F}^m_2;2)$ in the following result.

Theorem 4.2. *For $m \geq 2$, the Čech complex $\mathcal{N}(\mathcal{F}^m_0 \cup \mathcal{F}^m_1 \cup \mathcal{F}^m_2;2)$ is homotopy equivalent to the wedge sum of spheres. More specifically,*

$$\mathcal{N}(\mathcal{F}^m_0 \cup \mathcal{F}^m_1 \cup \mathcal{F}^m_2;2) \simeq \bigvee_{\binom{m}{2}} S^2.$$

Proof. Recall that the complex $\mathcal{N}(\mathcal{F}_0^m \cup \mathcal{F}_1^m; 2)$ is contractible. For each $A \in \mathcal{F}_2^m$, define $\mathcal{F}_{\preceq A}^m = \{B : B \preceq A \text{ and } B \in \mathcal{F}_0^m \cup \mathcal{F}_1^m\}$ and $\mathcal{F}_{\prec A}^m = \mathcal{F}_{\preceq A}^m \setminus \{A\}$. Fix $A \in \mathcal{F}_2^m$. We will prove the result holds by an induction on the total ordering $\prec$. For induction, we assume $\mathcal{N}(\mathcal{F}_{\prec A}^m; 2)$ is either contractible or homotopy equivalent to a wedge sum of spheres. This is true when $A = \{1, 2\}$ since $\mathcal{F}_{\prec\{1,2\}}^m = \mathcal{F}_0^m \cup \mathcal{F}_1^m$. We claim that

$$\mathcal{N}(\mathcal{F}_{\preceq A}^m; 2) = \mathcal{N}(\mathcal{F}_{\prec A}^m; 2) \vee S^2.$$

Since there are $\binom{m}{2}$-many vertices with size 2, by induction the result in the theorem holds. Next we prove that the claim holds.

Denote $A = i_1 i_2$ and $A_1 = \{i_1\}$, $A_2 = \{i_2\}$. In the complex $\mathcal{N}(\mathcal{F}_{\preceq A}^m; 2)$, the maximal simplices that contain the element A are $N[A_1]$, $N[A_2]$, and $N[A]$. Note that for $j = 1, 2$, in the simplex $N[A_j]$ the vertices with size 2 are the ones $\preceq A$ and contain A_j as a subset.

The complex $\mathcal{N}(\mathcal{F}_{\preceq A}^m; 2) \setminus A = \mathcal{N}(\mathcal{F}_{\prec A}^m; 2) \cup \{N(A)\}$. Notice that the $\mathcal{N}(\mathcal{F}_{\prec A}^m; 2) \cap \{N(A)\}$ is the 1-simplex with vertices A_1 and A_2 which is clearly contractible. Hence by Lemma 2.1, $\mathcal{N}(\mathcal{F}_{\preceq A}^m; 2) \setminus A$ is homotopy equivalent to $\mathcal{N}(\mathcal{F}_{\prec A}^m; 2)$.

Now we denote that $K = \mathcal{N}(\mathcal{F}_{\preceq A}^m; 2)$. The link of A in K is

$$\mathrm{lk}_K(A) = (N[A_1] \setminus A) \cup (N[A_2] \setminus A) \cup N(A).$$

It is straightforward to verify that $(N[A_1] \setminus A) \cup (N[A_2] \setminus A)$ is contractible directly using Lemma 2.1. Also the intersection of $(N[A_1] \setminus A) \cup (N[A_2] \setminus A)$ and $N(A)$ is two isolated vertices A_1 and A_2 which clearly homotopy equivalent to S_0. Again by Lemma 2.1, $\mathrm{lk}_K(A)$ is homotopy equivalent to the suspension of S^0, i.e. S^1. Then using Lemma 2.2, the complex K is homotopy equivalent to $\mathcal{N}(\mathcal{F}_{\prec A}^m; 2) \vee S^2$. This finishes the proof of the claim.

Then by induction, we obtain that:

$$\mathcal{N}(\mathcal{F}_0^m \cup \mathcal{F}_1^m \cup \mathcal{F}_2^m; 2) = \mathcal{N}(\mathcal{F}_{\preceq\{(m-1),m\}}^m; 2) \simeq \bigvee_{\binom{m}{2}} S^2.$$

$\square$

Next, we determine the homotopy types of the complex, $\mathcal{N}(\mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}}; 2)$, for general n and $m \geq n+2$. To do this, we need two lemmas.

Lemma 4.1. *Fix $n \geq 1$ and $m \geq n+2$. For any $A \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}}$, define $\mathcal{P}_A = \{B \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}} : 1 \in B$ or $B \preceq A\}$. Then the following holds true for A with $1 \notin A$:*

1. *the complex $\mathcal{N}(\mathcal{P}_A; 2) \setminus A$ is homotopy equivalent to $\mathcal{N}(\mathcal{P}_{A_-}; 2)$ where A_- is the predecessor of A in the total ordering $\prec$;*
2. *the link of A in the complex $\mathcal{N}(\mathcal{P}_A; 2)$ is homotopy equivalent to $\bigvee_{n+1} S^1$.*

Proof. Note that if $1 \in A$, then $\mathcal{P}_A = \{B \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}} : 1 \in B\} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n}}$. Fix $A \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}}$ with $1 \notin A$. Observe that compared to the complex $\mathcal{N}(\mathcal{P}A_-; 2)$, there is exactly one new simplex, namely $N(A)$, in the complex $\mathcal{N}(\mathcal{P}_A; 2) \setminus A$. Hence,

$$\mathcal{N}(\mathcal{P}_A; 2) \setminus A = \mathcal{N}(\mathcal{P}_{A_-}; 2) \cup N(A).$$

Here, $N(A)$ represents the simplicial complex generated by itself. Notice that in $\mathcal{P}_A$, there is only one vertex containing A, namely $1A$. Hence in the metric space $\mathcal{P}_A$, $N(A) = \{1A\} \cup \{B \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n}} : B \subseteq A\}$.

The vertices in the intersection $\mathcal{N}(\mathcal{P}_{A_-}; 2) \cap N(A)$ are the vertices of $N(A)$. Clearly any pair of different n-subsets of A is disjoint in the complex $\mathcal{N}(\mathcal{P}_{A_-}; 2)$. Any n-subset B of A is connected to $1A$ by the maximal simplex $N[1B]$ in $\mathcal{N}(\mathcal{P}_{A_-}; 2)$. As a result $\mathcal{N}(\mathcal{P}_{A_-}; 2) \cap N(A)$ is a graph with $n+2$ vertices of which one is connected to all the others by one edge, implying, it is contractible. Therefore, by Lemma 2.1,

$$\begin{aligned}
&\mathcal{N}(\mathcal{P}_A; 2) \setminus A = \mathcal{N}(\mathcal{P}_{A_-}; 2) \cup N(A) \\
&\mathcal{N}(\mathcal{P}_A; 2) \setminus A \simeq \mathcal{N}(\mathcal{P}_{A_-}; 2) \vee N(A) \vee \sum \big(\mathcal{N}(\mathcal{P}_{A_-}; 2) \cap N(A)\big) \\
&\mathcal{N}(\mathcal{P}_A; 2) \setminus A \simeq \mathcal{N}(\mathcal{P}_{A_-}; 2).
\end{aligned}$$

Next, we investigate the homotopy type of the link of A in $\mathcal{N}(\mathcal{P}_A; 2)$. For convenience, we denote $K = \mathcal{N}(\mathcal{P}_A; 2)$. By the definition of link,

$$\mathrm{lk}_K(A) = N(A) \cup \left(\bigcup_{B \subset A \text{ and } B \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n}}} (N[B] \setminus A) \right) \cup (N[1A] \setminus A).$$

We claim that $\mathrm{lk}_K(A)$ is homotopy equivalent to a wedge sum of $(n+1)$-many copies of S^1. List the vertices in $N(A) \cap \mathcal{F}^{\mathrm{m}}_{\mathrm{n}}$ as $\{B_i : i = 1, 2, \ldots, n+1\}$ in the ordering of $\prec$. It is straightforward to see that $N[B_i] \cap N[B_j] = \{A\}$ for any pair $i \neq j$. Hence, $(N[B_i] \setminus A) \cap (N[B_j] \setminus A) = \emptyset$. Also, $N(A) \cap (N[B_i] \setminus A) = \{B_i\}$. Hence, by induction and Lemma 2.1, the complex $N(A) \cup \left(\bigcup_{B \subset A \text{ and } B \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n}}} N[B] \setminus A \right)$ is contractible. For convenience, denote $L = N(A) \cup \left(\bigcup_{B \subset A \text{ and } B \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n}}} N[B] \setminus A \right)$. To get the homotopy type of $\mathrm{lk}_K(A)$, we need to determine the homotopy type of the complex $L \cap (N[1A] \setminus A)$. Clearly, the vertices in the complex $L \cap (N[1A] \setminus A)$ are $\{1A\} \cup \{1B_i : i = 1, 2, \ldots, n+1\}$. It is straightforward to see that all these vertices are disjoint from each other in L. Hence $L \cap (N[1A] \setminus A)$ is homotopy equivalent to a wedge sum of $(n+1)$-many S^0. Therefore, by Lemma 2.1,

$$\mathrm{lk}_K(A) \simeq \sum (\bigvee_{n+1} S^0) \simeq \bigvee_{n+1} S^1.$$

$\square$

Lemma 4.2. *Fix $n \geq 1$ and $m \geq n+2$. For any $A \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}}$, define $\mathcal{P}_A = \{B \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}} : 1 \in B$ or $B \preceq A\}$. Then the following are true for A with $1 \notin A$:*

1. *the complex $\mathcal{N}(\mathcal{P}_{A_-}; 2) \cap N(A)$ is homotopy equivalent to a wedge sum of $\binom{n+1}{2}$-many copies of S^1 where A_- is the predecessor of A in the total ordering $\prec$;*
2. *the link of A in the complex $\mathcal{N}(\mathcal{P}_A; 2)$ is homotopy equivalent to $\bigvee_{\binom{n+2}{2}} S^1$.*

Proof. Fix $A \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}}$ with $1 \notin A$ and A_- being the predecessor of A in the total ordering $\prec$. In the metric space $\mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}}$, $N(A) = \{B \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} : B \subseteq A\}$ and we list the vertices in $N(A)$ as $\{B_i : i = 1, 2, \ldots, n+2\}$ in the ordering $\prec$. Clearly, $\{B_i : i = 1, 2, \ldots, n+2\}$ is the collection of vertices in the complex $\mathcal{N}(\mathcal{P}_{A_-}; 2) \cap N(A)$. Notice that for any pair $i, j \in [n+2]$, $B_i \cap B_j$ is an n-subset of $[m]$, i.e. $N[B_i \cap B_j]$ is a simplex containing both B_i and B_j. Furthermore, the intersection of any subcollection of $\{B_i : i = 1, 2, \ldots, n+2\}$ with more than 2 vertices contains $<$ n-many numbers which means any subcollection of $\{B_i : i = 1, 2, \ldots, n+2\}$ with more than 2 vertices can only belong to the simplex $N(A)$ which is not in $\mathcal{N}(\mathcal{P}_{A_-}; 2)$. Hence, the complex $\mathcal{N}(\mathcal{P}_{A_-}; 2) \cap N(A)$ is a complete graph with $(n+2)$-many vertices which is homotopy equivalent to a wedge sum of $\binom{n+1}{2}$-many S^1's.

For convenience, we denote $K = \mathcal{N}(\mathcal{P}_A; 2)$. By the definition of link,

$$\mathrm{lk}_K(A) = N(A) \cup \left(\bigcup_{i=1}^{n+2} N(B_i) \setminus A \right).$$

Next, we prove the second statement using induction. We claim that

$$N(A) \cup \left(\bigcup_{i=1}^{k} N[B_i] \setminus A \right) \simeq \bigvee_{\sum_{i=1}^{k}(i-1)} S^1.$$

It is straightforward to see that $N(A) \cap (N[B_1] \setminus A) = \{B_1\}$. Hence by Lemma 2.1, $N(A) \cup (N[B_1] \setminus A)$ is contractible. This shows that the claim hold when $k = 1$. Now we assume that the claim holds for $k-1$, i.e. $N(A) \cup \left(\bigcup_{i=1}^{k-1} N[B_i] \setminus A \right) \simeq \bigvee_{\sum_{i=1}^{k-1}(i-1)} S^1$. Denote that $L = N(A) \cup \left(\bigcup_{i=1}^{k-1} N[B_i] \setminus A \right)$. To determine the homotopy type of $L \cup N[B_k] \setminus A$, we need to know the homotopy type of the complex $L \cap (N[B_k] \setminus A)$. It is straightforward to verify that the vertices of the complex $L \cap (N[B_k] \setminus A)$ are $B_1 \cap B_k$, …, $B_{k-1} \cap B_k$, and B_k. Notice that any pair of these vertices are disjoint in L. Hence, the complex $L \cap (N[B_k] \setminus A)$ is homotopy equivalent to a wedge sum of $(k-1)$-many S^0. Clearly, the inclusion map from $L \cap (N[B_k] \setminus A)$ to L is null-homotopic and same is true for the inclusion map from $L \cap (N[B_k] \setminus A)$ to $N[B_k] \setminus A$. By Lemma 2.1, $L \cup (N[B_k] \setminus A)$ is homotopy equivalent to $\bigvee_{\sum_{i=1}^{k}(i-1)} S^1$.

Since $\sum_{i=1}^{n+2}(i-1) = \sum_{i=1}^{n+1} i = \binom{n+2}{2}$, we obtain that

$$\mathrm{lk}_K(A) \simeq \bigvee_{\binom{n+2}{2}} S^1.$$

□

Now we are ready to discuss the homotopy type of the complex $\mathcal{N}(\mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}}; 2)$.

Theorem 4.3. *For $n \geq 1$ and $m \geq n+2$, the Čech complex $\mathcal{N}(\mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}}; 2)$ is homotopy equivalent to the wedge sum of spheres. In particular,*

$$\mathcal{N}(\mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}}; 2) \simeq \mathcal{N}(\mathcal{F}^{m-1}_{n-1} \cup \mathcal{F}^{m-1}_{n} \cup \mathcal{F}^{m-1}_{n+1}; 2) \vee \left(\bigvee_{\binom{m-1}{n+1}\cdot(n+1)+\binom{m-1}{n+2}\cdot(n+1)^2} S^2 \right).$$

Proof. Fix $n \geq 1$ and $m \geq n+2$. Assume that $\mathcal{N}(\mathcal{F}^{m-1}_{n-1} \cup \mathcal{F}^{m-1}_{n} \cup \mathcal{F}^{m-1}_{n+1}; 2)$ is homotopy equivalent to a wedge sum of spheres. This holds when $n = 1$ by Theorem 4.2.

Denote $\mathcal{B}_1 = \{A \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}} : 1 \in A\}$. By an isometric mapping, $\mathcal{N}(\mathcal{B}_1; 2) \simeq \mathcal{N}(\mathcal{F}^{m-1}_{n-1} \cup \mathcal{F}^{m-1}_{n} \cup \mathcal{F}^{m-1}_{n+1}; 2)$. Hence the complex $\mathcal{N}(\mathcal{B}_1; 2)$ is homotopy equivalent to a wedge sum of spheres. Firstly, we will show that $\mathcal{N}(\mathcal{B}_1 \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n}}; 2)$ is homotopy equivalent to the complex $\mathcal{N}(\mathcal{B}_1; 2)$. We list the vertices not containing 1 in $\mathcal{F}^{\mathrm{m}}_{\mathrm{n}}$ as $\{A_i\}$ in the total ordering $\prec$. It is sufficient to show that $\mathcal{N}(\mathcal{B}_1 \cup \{A_i : i \leq k\}; 2)$ is homotopy equivalent to $\mathcal{N}(\mathcal{B}_1 \cup \{A_i : i < k\}; 2)$. Notice that in the space $\mathcal{B}_1 \cup \{A_i : i \leq k\}$, $N(A_k) = \{1A_k\}$ and $N[1A_k]$ is the only other maximal simplex containing A_k. Applying Lemma 2.1 and Lemma 2.2, the result holds.

For any $A \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}}$ with $1 \notin A$, by Lemma 4.1, the complex $\mathcal{N}(\mathcal{B}_1 \cup \{B : B \preceq A\}; 2) \setminus A$ is homotopy equivalent to $\mathcal{N}(\mathcal{B}_1 \cup \{B : B \prec A\}; 2)$ and the link of A in $\mathcal{N}(\mathcal{B}_1 \cup \{B : B \preceq A\}; 2)$ is homotopy equivalent to a wedge sum of $(n+1)$-many copies of S^1. For induction we assume that $\mathcal{N}(\mathcal{B}_1 \cup \{B : B \prec A\}; 2)$ is homotopy equivalent to a wedge sum of spheres. This is true when $1 \in A_-$. The reason being, when $1 \in A_-$, $\mathcal{N}(\mathcal{B}_1 \cup \{B : B \prec A\}; 2) = \mathcal{N}(\mathcal{B}_1 \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n}}; 2) \simeq \mathcal{N}(\mathcal{B}_1; 2)$. Then by Lemma 2.2,

$$\mathcal{N}(\mathcal{B}_1 \cup \{B : B \preceq A\}; 2) \simeq \mathcal{N}(\mathcal{B}_1 \cup \{B : B \prec A\}; 2) \vee \bigvee_{n+1} S^2.$$

Note that there are $\binom{m-1}{n+1}$-many vertices in $\mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}}$ which doesn't contain 1. Hence by induction,

$$\mathcal{N}(\mathcal{B}_1 \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}}; 2) \simeq \mathcal{N}(\mathcal{B}_1; 2) \vee \bigvee_{\binom{m-1}{n+1}\cdot(n+1)} S^2.$$

Lastly, we discuss the homotopy type of $\mathcal{N}(\mathcal{B}_1 \cup \{B : B \preceq A\}; 2)$ with $A \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}}$ and $1 \notin A$. For each $A \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}}$ with $1 \notin A$, by Lemma 4.2, we know that: 1) $\mathcal{N}(\mathcal{B}_1 \cup \{B : B \prec A\}; 2) \cap N(A)$ is homotopy equivalent to a wedge sum of $\binom{n+1}{2}$-many S^1's; 2) the link of A in $\mathcal{N}(\mathcal{B}_1 \cup \{B : B \preceq A\}; 2)$ is homotopy equivalent to a wedge sum of $\binom{n+2}{2}$-many S^1's. For induction we assume that $\mathcal{N}(\mathcal{B}_1 \cup \{B : B \prec A\}; 2)$ is homotopy equivalent to a wedge sum of spheres. Clearly by the result, this is true when $1 \in A_-$. Then by Lemma 2.1 and 1),

$$\mathcal{N}(\mathcal{B}_1 \cup \{B : B \preceq A\}; 2) \setminus A = \mathcal{N}(\mathcal{B}_1 \cup \{B : B \prec A\}; 2) \cup N(A)$$
$$\mathcal{N}(\mathcal{B}_1 \cup \{B : B \preceq A\}; 2) \setminus A \simeq \mathcal{N}(\mathcal{B}_1 \cup \{B : B \prec A\}; 2) \vee \Big(\bigvee_{\binom{n+1}{2}} S^2\Big).$$

Then by Lemma 2.2 and 2),

$$\mathcal{N}(\mathcal{B}_1 \cup \{B : B \preceq A\}; 2) \simeq \mathcal{N}(\mathcal{B}_1 \cup \{B : B \prec A\}; 2) \vee \Big(\bigvee_{\binom{n+1}{2}} S^2\Big) \vee \Big(\bigvee_{\binom{n+2}{2}} S^2\Big).$$

Notice that there are $\binom{m-1}{n+2}$-many vertices in $\mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}}$ not containing 1. Using induction, we obtain that

$$\mathcal{N}(\mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}}; 2) \simeq \mathcal{N}(\mathcal{B}_1; 2) \vee \Big(\bigvee_{\binom{m-1}{n+1}\cdot(n+1)+\binom{m-1}{n+2}\cdot(\binom{n+1}{2}+\binom{n+2}{2})} S^2\Big).$$

Notice that $\binom{n+1}{2} + \binom{n+2}{2} = (n+1)^2$. This finishes the proof. $\square$

By an inductive calculation, we have the following corollary:

Corollary 4.2. *For $n \geq 0$ and $m \geq n+2$, the Čech complex $\mathcal{N}(\mathcal{F}^m_n \cup \mathcal{F}^m_{n+1} \cup \mathcal{F}^m_{n+2}; 2)$ is homotopy equivalent to the wedge of $\binom{m-n}{2} + \sum_{i=1}^{n} \left(\binom{m+i-1-n}{i+1} \cdot (i+1) + \binom{m+i-1-n}{i+2} \cdot (i+1)^2\right)$-many copies of spheres.*

Now we are ready discuss the homotopy types of the Čech complex of the hypercube graph Q_m with $m \geq 2$. We can decompose $\mathcal{N}(Q_m; 2)$ as a union of subcomplexes that we have discussed.

Theorem 4.4. *For $m \geq 2$,*

$$\mathcal{N}(Q_m; 2) = \bigcup_{n=0}^{m-2} \mathcal{N}(\mathcal{F}_n^m \cup \mathcal{F}_{n+1}^m \cup \mathcal{F}_{n+2}^m; 2).$$

Also, for any $n \in [m-3]$ with $m \geq 3$,

$$\mathcal{N}(\mathcal{F}_0^m \cup \mathcal{F}_1^m \cup \ldots \cup \mathcal{F}_{n+2}^m; 2) \cap \mathcal{N}(\mathcal{F}_{n+1}^m \cup \mathcal{F}_{n+2}^m \cup \mathcal{F}_{n+3}^m; 2) = \mathcal{N}(\mathcal{F}_{n+1}^m \cup \mathcal{F}_{n+2}^m; 2).$$

Then using Lemma 2.1 and induction, we obtain the homotopy type of the complex $\mathcal{N}(Q_m; 2)$ in a different representation. Fix $m \geq 2$. For any n with $0 \leq n \leq m-2$, we define that

$$\gamma_n = \binom{m-n}{2} + \sum_{i=1}^{n} \left(\binom{m+i-1-n}{i+1} \cdot (i+1) + \binom{m+i-1-n}{i+2} \cdot (i+1)^2 \right).$$

And for any n with $1 \leq n \leq m-2$, we define that

$$\delta_n = \sum_{i=1}^{n} i \cdot \binom{m-n+i-1}{i+1}.$$

Corollary 4.3. *For $m \geq 3$, the complex $\mathcal{N}(Q_m; 2)$ is homotopy equivalent to a wedge sum of $(\sum_{n=0}^{m-2} \gamma_n + \sum_{n=1}^{m-2} \delta_n)$-many copies of spheres.*

Chapter 5

$\check{C}$ech complex with Scale 3

In this chapter, we explore the homotopy types of $\check{C}$ech complexes, $\mathcal{N}(\mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}}; 3)$ and $\mathcal{N}(\mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+3}}; 3)$. With the scale being an odd number, there is a different form of the collections of vertices which generate the maximal simplices in the complex. The maximal simplices are generated by the union of the closed neighborhoods of adjacent vertices. We construct the complex on the graph $\bigcup\limits_{i=n}^{n+k} \mathcal{F}^m_i$, where vertices are all the elements of the union, and any two vertices are adjacent if their symmetric distance is 1. For example, in the graph $\mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}}$, vertex $A \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}}$ is adjacent to vertices $B \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}}$ with $B \supset A$ and vertices $C \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n}}$ with $C \subset A$.

In this chapter, for convenience, we denote

$$\mathcal{K}_3 = \mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}} \text{ and } \mathcal{K}_4 = \mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}} .$$

We will show that $\mathcal{N}(\mathcal{K}_3; 3)$ is either contractible or homotopy equivalent to a wedge sums of S^2's and that $\mathcal{N}(\mathcal{F}^m_0 \cup \mathcal{F}^m_1 \cup \mathcal{F}^m_2 \cup \mathcal{F}^m_3; 3)$ is either contractible or homotopy equivalent to a wedge sums of S^3's. To find these homotopy types, we will use similar technique as the proofs in the previous chapters but with the help of two new total orders $\ltimes, \lhd$ where $\ltimes$ is defined on the space $\mathcal{K}_3$ and $\lhd$ is defined on $\mathcal{K}_4$.

5.1 $\check{C}$ech Complex $\mathcal{N}(\mathcal{K}_3;3)$

In this section we determine the homotopy types of the complex $\mathcal{N}(\mathcal{K}_3;3)$. For convenience, we also consider $\mathcal{K}_3$ as a graph with vertices to be the elements in $\mathcal{K}_3$ and any pair of vertices being adjacent if their symmetric difference distance is 1.

In the $\check{C}$ech complex $\mathcal{N}(\mathcal{K}_3;3)$, the maximal simplices are generated by the following collections of vertices:

1) for any $A \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n}}$ and $B \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}}$ with $A \sim B$, $N_1[A,B] = \{C \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n}} : C \subset B\} \cup \{C \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} : A \subset C\} \cup \{C \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}} : B \subset C\}$;
2) for any $A \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}}$ and $B \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}}$ with $A \sim B$, $N_1[B,A] = \{C \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n}} : C \subset B\} \cup \{C \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} : C \subset A\} \cup \{C \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}} : B \subset C\}$.

Recall that for $\check{C}$ech complex with scale 2, the 1-neighborhoods of the vertices generates the maximal simplices. Therefore, another way of representing these collection which generates maximal simplex in $\mathcal{N}(\mathcal{K}_3;3)$ is:

$$N_1[A,B] = \{C \in \mathcal{K}_3 : C \in N_1[A] \text{ or } N_1[B]\},$$

where $A\Delta B = 1$.

From here on, for convenience, we will omit 1 in the subscript. Also, we denote a maximal simplex in $\mathcal{N}(\mathcal{K}_3;3)$ by $N[A,B]$ with A and B being adjacent. The same notation is also used to denote the simplicial complex generated by the simplex. Denote $N[A,B) = N[A,B]\setminus\{B\}$, $N(A,B] = N[A,B] \setminus \{A\}$, and $N(A,B) = N[A,B] \setminus \{A,B\}$.

Let us define the total order $\ltimes$ on the collection $\mathcal{K}_3$. For any pair $A, B \in \mathcal{K}_3$ with $A = i_1i_2\cdots i_p$, and $B = j_1j_2\cdots j_q$, we say $A \ltimes B$ if one of the following holds:

1. $|A| = |B|$ and $A \prec B$;
2. $A \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n}}$ and $B \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}}$;
3. $A, B \notin \mathcal{F}^{\mathrm{m}}_{\mathrm{n}}$

(i) $i_k = j_k$ for all $1 \le k \le n$ and $i_{n+1} \dots i_p \prec j_{n+1} \dots j_q$;

(ii) $i_k \ne j_k$ for some $1 \le k \le n$ and $i_1 i_2 \dots i_n \prec j_1 j_2 \dots j_n$.

It is straightforward to verify that $\ltimes$ is a well-defined total ordering on $\mathcal{K}_3$. To determine the homotopy type of the respective complex, we use a similar method as in the previous chapter. We track the change of the homotopy types when bringing in new vertex in the total order $\ltimes$. This total order is crucial in our method because bringing in vertices in this manner facilitates an inductive approach to achieving the homotopy types of the complex. Here, we give an example of this order. In the space $\mathcal{F}_2^m \cup \mathcal{F}_3^m \cup \mathcal{F}_4^m$, consider the element $\{2,3,4,5\} \in \mathcal{F}_4^m$. Though $\{2,3,4\}, \{2,3,5\}$, $\{2,4,5\}, \{3,4,5\}$ are all subsets of $\{2,3,4,5\}$. However, $\{2,3,4\} \ltimes \{2,3,5\} \ltimes \{2,3,4,5\} \ltimes \{2,4,5\} \ltimes \{3,4,5\}$. Hence, we introduce $\{2,3,4,5\}$ before $\{2,4,5\}, \{3,4,5\}$.

Lemma 5.1. *For any definable m, the Čech complex $\mathcal{N}(\mathcal{F}_0^m \cup \mathcal{F}_1^m \cup \mathcal{F}_2^m; 3)$ is contractible.*

Proof. Notice that in the complex $\mathcal{N}(\mathcal{F}_0^m \cup \mathcal{F}_1^m \cup \mathcal{F}_2^m; 3)$, for any $A \in \mathcal{F}_1^m$ and $B \in \mathcal{F}_2^m$ with $A \sim B$, $N[A, B] \subset N[\emptyset, A]$. Hence the complex $\mathcal{N}(\mathcal{F}_0^m \cup \mathcal{F}_1^m \cup \mathcal{F}_2^m; 3)$ is generated by the collection of maximal simplices $\{N[\emptyset, A] : A \in \mathcal{F}_1^m\}$. This means that the complex is a cone with vertex $\emptyset$ and is therefore, contractible. $\square$

Lemma 5.2. *Consider the space $\mathcal{K}_3$ for $m \ge n+3$. For $A \in \mathcal{F}_{\mathrm{n}}^{\mathrm{m}}$, denote A_- as the predecessor of A in the total ordering $\ltimes$. Define $\mathcal{B}_1 = \{B \in \mathcal{K}_3 : 1 \in B\}$, $\mathcal{B}_{A_-} = \mathcal{B}_1 \cup \{B \in \mathcal{F}_{\mathrm{n}}^{\mathrm{m}} : B \ltimes A\}$ and $\mathcal{B}_A = \mathcal{B}_{A_-} \cup \{A\}$. Then, the complex $\mathcal{N}(\mathcal{B}_A; 3)$ is homotopy equivalent $\mathcal{N}(\mathcal{B}_{A_-}; 3)$.*

Proof. Notice that in the metric space $\mathcal{B}_A$, $1A$ is the only element whose distance from A is 1. Hence, there is only one simplex generated by A in the complex $\mathcal{N}(\mathcal{B}_A; 3)$ which is $N[A, 1A]$. Notice that in $\mathcal{B}_A$, $N[A, 1A] = N[1A]$. Other maximal simplices which include A are $N[B, 1A]$ and $N[1A, C]$ where $B \in \mathcal{F}_{\mathrm{n}}^{\mathrm{m}}$ with $B \subset 1A$ and $C \in \mathcal{F}_{\mathrm{n+2}}^{\mathrm{m}}$ with $1A \subset C$. Hence in $\mathcal{B}_A$, $N[A, 1A] = N[1A]$ is a subset of any other simplex with $1A$ being a generator. Therefore, $\mathcal{N}(\mathcal{B}_A; 3) \setminus A = \mathcal{N}(\mathcal{B}_{A_-}; 3) \cup N(A, 1A] = \mathcal{N}(\mathcal{B}_{A_-}; 3)$.

Next, we denote $L = \mathcal{N}(\mathcal{B}_A; 3)$ and determine the homotopy type of the link of A in K. We define

$$L_1 = \bigcup_{B \subset 1A \text{ and } B \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n}}} N[B, 1A] \setminus A \text{ and } L_2 = \bigcup_{C \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}} \text{ and } 1A \subset C} N[1A, C] \setminus A.$$

Then by the definition of link, $\mathrm{lk}_L(A) = L_1 \cup L_2$.

First, we claim that L_1 is contractible. Notice that for any pair B and B' in $\mathcal{F}^{\mathrm{m}}_{\mathrm{n}}$, in the space $\mathcal{B}_{A_-}$, $N[B, 1A] \cap N[B', 1A] = N[1A]$. This holds because the only element in $\mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}}$ which is adjacent to both B and B' is $1A$. Therefore, by an inductive argument using Lemma 2.1, L_1 is contractible.

Then, we claim that $\mathrm{lk}_L(A) = L_1 \cup L_2$ is contractible. By Lemma 2.2, this implies that the complex $\mathcal{N}(\mathcal{B}_A; 3)$ is homotopy equivalent $\mathcal{N}(\mathcal{B}_{A_-}; 3)$. To show that $L_1 \cup L_2$ is contractible, we list all the supersets of $1A$ in $\mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}}$ as $\{C_1, C_2, \ldots, C_q\}$ in the ordering of $\ltimes$. Notice that $q = m - n - 1$. We prove that $L_1 \cup (\bigcup_{i=1}^{k} N[1A, C_i]) \setminus A$ is contractible for any $k \leq q$. Since the only vertex adjacent to both C_i and $C_{i'}$ is $1A$, so $N[1A, C_i] \cap N[1A, C_{i'}] = N[1A]$ in $\mathcal{B}_A$. Assume that the complex $L_1 \cup (\bigcup_{i=1}^{k-1} N[1A, C_i]) \setminus A$ is contractible. We denote $K = L_1 \cup (\bigcup_{i=1}^{k-1} N[1A, C_i]) \setminus A$ for convenience. Now, we look at the intersection of K and $N[1A, C_k] \setminus A$. The common vertices are all the ones in $N[1A, C_k] \setminus A$. Notice that for each vertex D in $N[1A, C_k] \setminus N[1A]$, D contains 1 as $D \in \mathcal{B}_A$ and D is adjacent to a subset $B = D \cap 1A$ of $1A$; each pair of such vertices are not adjacent. Hence the intersection of K and $N[1A, C_k] \setminus A$ consists of a simplex $N[1A]$ and all the vertices being subsets of C_k containing 1 each of which is connected to $N[1A]$ uniquely by a subset of $1A$. Hence this complex obtained by intersection is contractible. And this shows that $K \cup N[1A, C_k]$ is contractible by Lemma 2.1. Hence $L_1 \cup L_2$ is contractible.

$\square$

The following corollary is clear using the lemma above.

Corollary 5.1. *For $m \geq n+2$, denote $\mathcal{B}_1$ to be $\{D \in \mathcal{K}_3 | 1 \in D\}$. Then the complex $\mathcal{N}(\mathcal{B}_1; 3)$ is homotopy equivalent to $\mathcal{N}(\mathcal{B}_1 \cup \mathcal{F}_{\mathrm{n}}^{\mathrm{m}}; 3)$. Furthermore, by an isometric mapping,*

$$\mathcal{N}(\mathcal{B}_1 \cup \mathcal{F}_{\mathrm{n}}^{\mathrm{m}}; 3) \simeq \mathcal{N}(\mathcal{F}_{n-1}^{m-1} \cup \mathcal{F}_{n}^{m-1} \cup \mathcal{F}_{n+1}^{m-1}; 3).$$

We also need the following result to determine the homotopy type of the complex $\mathcal{N}(\mathcal{K}_3; 3)$.

Lemma 5.3. *Consider the space $\mathcal{K}_3$ for $m \geq n+2$. For any $D \in \mathcal{F}_{\mathrm{n+1}}^{\mathrm{m}}$ with $1 \notin D$, define $\mathcal{B}_{D_-} = \mathcal{F}_{\mathrm{n}}^{\mathrm{m}} \cup \{D' \in \mathcal{F}_{\mathrm{n+1}}^{\mathrm{m}} \cup \mathcal{F}_{\mathrm{n+2}}^{\mathrm{m}} | D' \ltimes D\}$ and $\mathcal{B}_D = \mathcal{B}_{D_-} \cup \{D\}$ where D_- is the predecessor of D. Suppose that the complex $\mathcal{N}(\mathcal{B}_{D_-}; 3)$ is homotopy equivalent to a wedge sum of spheres.*

Then, for any $D = i_1 i_2 \ldots i_{n+1} \in \mathcal{F}_{\mathrm{n+1}}^{\mathrm{m}}$ with $1 \notin D$, the following holds:

$$\mathcal{N}(\mathcal{B}_D; 3) \simeq \mathcal{N}(\mathcal{B}_{D_-}; 3) \vee \bigvee_{\alpha} S^2,$$

where,

$$\alpha = \sum_{k=1}^{n} i_k - \frac{n(n+1)}{2} - n$$

Proof. Fix $D = i_1 i_2 \ldots i_{n+1} \in \mathcal{F}_{\mathrm{n+1}}^{\mathrm{m}}$ with $1 \notin D$ which means $i_1 > 1$. First we give notations for all the vertices in $\mathcal{K}_3$ which are adjacent to D. Define $C_p = i_1 i_2 \ldots \hat{i}_{n+2-p} \ldots i_{n+1} = D \setminus \{i_{n+2-p}\},$ for $p \in \{1, 2, \ldots, n+1\}$. Then $C_p \in \mathcal{F}_{\mathrm{n}}^{\mathrm{m}}$ and each C_p would be adjacent to D. Note that $C_1 \ltimes C_2 \ltimes \ldots \ltimes C_n \ltimes C_{n+1}$. For each $q \in \{1, 2, \ldots, i_n - n\}$, let j_q be an integer between 1 and $i_n - 1$ with $j_q \notin D$ and define $D_q = D \cup \{j_q\}$. Then D_q is a vertex in $\mathcal{F}_{\mathrm{n+2}}^{\mathrm{m}}$ which is adjacent to D. Also in the total ordering $\ltimes$, $D_1 \ltimes D_2 \ltimes \ldots \ltimes D_{i_n - n} \ltimes D$. Therefore, the 1-closed neighborhood of D in $\mathcal{B}_D$ is $N[D] = \{C_1, \ldots C_{n+1}, D, D_1 \ldots, D_{i_n - n}\}$. Then the maximal simplices that contain D in the $\check{C}$ech complex $\mathcal{N}(\mathcal{B}_D, 3)$ are:

1) for all $p \in \{1, 2, \ldots, n+1\}, N[C_p, D] = N_{\mathcal{B}_D}[C_p] \cup N_{\mathcal{B}_D}[D]$ where, $N_{\mathcal{B}_D}[C_p] = \{A \in \mathcal{F}_{\mathrm{n+1}}^{\mathrm{m}} | C_p \subset A \vee A \ltimes D$ or $A = D\}$

2) for all $q \in \{1, 2, \ldots, i_n - n\}, N[D, D_q] = N_{\mathcal{B}_D}[D_q] \cup N_{\mathcal{B}_D}[D]$ where $N_{\mathcal{B}_D}[D_q] = \{B \in \mathcal{F}_{\mathrm{n+1}}^{\mathrm{m}} | B \subset D_q \vee B \ltimes = D\}$

3) for all p and for each $A \in N_{\mathcal{B}_D}[C_p] \setminus D : N_{\mathcal{B}_D}[C_p, A] = N_{\mathcal{B}_{D_-}}[C_p, A] \cup \{D\}$

4) for all q and for each $B \in N_{\mathcal{B}_D}[D_q] \setminus D : N_{\mathcal{B}_D}[B, D_q] = N_{\mathcal{B}_{D_-}}[B, D_q] \cup \{D\}$

Note that the maximal simplices in the forms of 1) and 2) are the new maximal simplices with the vertex D as a generator. And the maximal simplices in the forms of 3) and 4) are the maximal simplices containing D but generated by other vertices.

Define $L_1 = \bigcup_{p=1}^{n+1} N[C_p, D)$ and $L_2 = \bigcup_{q=1}^{i_n-n} N(D, D_q]$. Then, we can write

$$\mathcal{N}(\mathcal{B}_D; 3) \setminus D = \mathcal{N}(\mathcal{B}_{D_-}; 3) \cup (L_1 \cup L_2).$$

We claim that $L_1 \cup L_2$ is contractible. For any pair $p_1, p_2 \in [n+1]$, it is straightforward to see that $N[C_{p_1}, D) \cap N[C_{p_2}, D) = N(D)$. This implies that L_1 is contractible. Based on the order in which the vertices were added, for any $q \in \{1, 2, \ldots, i_n - n\}$, the closed neighborhood of D_q in $\mathcal{N}(\mathcal{B}_D; 3)$ is $N(D_q) = \{D_q^p \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} : D_q^p = D_q \setminus \{i_{n+2-p}\}$ and $D_p \ltimes D$ for $p \in [n+1]\} \cup \{D\}$. And it is straightforward to see that $C_p \subset D_q^p$, which implies that $D_q^p \in N_{\mathcal{B}_D}[C_p, A)$ for each $p \in [n+1]$ with $D_q^p \ltimes D$. As a result, for any arbitrary q, the intersection would be $L_1 \cap N(D, D_q] = \bigcup_{p \in [n+1] \text{ and } D_q^p \ltimes D} N(D) \cup \{D_q^p\}$. Using a similar approach as in the proof of Lemma-5.2, we get that the complex $L_1 \cup L_2$ is contractible.

Denote the intersection $\mathcal{N}(\mathcal{B}_{D_-}; 3) \cap (L_1 \cup L_2)$ by $\mathcal{I}_D$. Notice that for any $p, p' \in [n+1]$ the only kind of maximal simplicies containing both C_p and $C_{p'}$ are the ones whose generators including the vertex D. Also for any $q, q' \in [i_n - n]$ the only kind of maximal simplicies containing both D_q and $D_{q'}$ are the ones whose generators including the vertex D.

Notice that the vertices of the intersection $\mathcal{I}_D$ are the vertices in the complexes $N(D)$, $\bigcup\{N[C_p] \setminus D\}_{p=1}^{n+1}$ and $\bigcup\{N[D_q] \setminus D\}_{q=1}^{i_n-n}\}$. We denote S_D to be full subcomplex of $\mathcal{I}_D$ with vertices from the complex $N(D)$, i.e. $\{C_p\}_{p=1}^{n+1} \cup \{D_q\}_{q=1}^{i_n-n}$.

We claim that, for any $p \in \{2, \ldots, n, n+1\}$, C_p is connected to the first $i_{n+2-p} - (n+2-p)$ many D_q's in the complex $\mathcal{N}(\mathcal{B}_{D_-}; 3)$. By the definition of maximal simplex in Čech complexes with scale 3, a vertex in $\mathcal{F}^{\mathrm{m}}_{\mathrm{n}}$ is connected to another vertex in $\mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}}$ only through a maximal simplex one of whose generators is an vertex in $\mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}}$. Therefore, let us look at the elements in $\mathcal{B}_{D_-}$ that would connect them. For some p, let C_p be adjacent to $A \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}}$ such that $C_p \subset A$ and $A \ltimes D$. Say, $A = C_p \cup \{j\}$. Recall that $C_p = D \setminus \{i_{n+2-p}\}$. Since $A \ltimes D$,

j has to be less than i_{n+2-p} and $j \in D$. Hence, there are $i_{n+2-p} - (n+2-p)$ many such A's that are less than D in the ordering of $\ltimes$. For each such j, we define $A^p_j = C_p \cup \{j\}$ and then $A^p_j \cup \{i_{n+2-p}\}$ would be some D_q with $i_q = j$. Thus, C_p shares an edge with $i_{n+2-p} - (n+2-p)$ many D_q's in $\mathcal{N}(\mathcal{B}_{D_-}; 3)$ by the maximal simplices $N[C_p, A^p_j]$.

Observe that as p keeps increasing, it is the initial i_k's that are being removed which are smaller, implying, the number of options for j decrease. Also, one of the j values would always be 1 as $1 \notin D$. Based on our notation, $D_1 = 1D$. So, when $j = 1$, each C_p is connected to D_1. And on arranging the j values for each C_p in an increasing order, it is easy to see that the C_p is connected to the first $(i_{n+2-p} - (n+2-p))$-many D_q's.

For C_1, which is the smallest of all C_p's, j has to be less than i_n and not i_{n+1}. This is because based on the order $\ltimes$, every element in $\mathcal{B}_D$ must have the element in their n^{th} position less than or equal to i_n. And using a similar reasoning as above, C_1 is connected to the first $(i_n - n)$-many D_q's.

Therefore, in S_D, each C_p is connected to $(i_{n+2-p} - (n+2-p))$-many D_q's while no pair of the C_p's are adjacent to each other and no pair of the D_q's are adjacent to each other. Hence the subcomplex S_D is a simple, connected graph with $n+1+i_n-n = i_n+1$ vertices, and $i_n - n + \sum\limits_{p=2}^{n+1}(i_{n+2-p} - (n+2-p))$ edges. The Euler's characteristic of the complex S_D is

$$\begin{aligned}
\chi_{\mathcal{I}_D} = 1 - (V - E) &= 1 - (i_n + 1 - i_n + n - \sum_{p=2}^{n+1}(i_{n+2-p} - (n+2-p))) \\
&= 1 - 1 - n + \sum_{k=1}^{n} i_k - \sum_{k=1}^{n} k \\
&= \sum_{k=1}^{n} i_k - \frac{n(n+1)}{2} - n,
\end{aligned}$$

It is a routine process to verify that the homotopy type doesn't change when we bring other maximal simplices in the intersection $\mathcal{I}_D$ to the subcomplex S_D. Hence, $\mathcal{I}_D$ is homotopy equivalent to S_D, i.e. homotopy equivalent to a wedge sum of S^1's. And then using the fact that that $L_1 \cup L_2$ is contractible and the assumption that $\mathcal{N}(\mathcal{B}_{D_-}; 3)$ homotopy equivalent to a wedge sum of spheres, we obtain that the inclusion maps $\iota_1 : \mathcal{I}_D \hookrightarrow \mathcal{N}(\mathcal{B}_{D_-}; 3)$, $\iota_2 : \mathcal{I}_D \hookrightarrow L_1 \cup L_2$

are null-homotopic. Therefore,

$$\mathcal{N}(\mathcal{B}_D;3) \setminus \{D\} \simeq \mathcal{N}(\mathcal{B}_{D_-};3) \vee (L_1 \cup L_2) \vee \sum(\mathcal{I}_D)$$

Since $L_1 \cup L_2$ is contractible,

$$\mathcal{N}(\mathcal{B}_D;3) \setminus \{D\} \simeq \mathcal{N}(\mathcal{B}_{D_-};3) \vee \bigvee_{\alpha} S^2$$

where α is as defined above.

Now, let us look at the link of D in the complex $\mathcal{N}(\mathcal{B}_D;3)$. Denote $K = \mathcal{N}(\mathcal{B}_D;3)$. Then the link of the vertex D in K, $\mathrm{lk}_K(D)$, is the complex whose maximal simplices are the ones from collections $\{(N_{\mathcal{B}_D}[C_p, A]) \setminus \{D\} : p \in [n+1], C_p \subset A$ and $A \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}}\}$ and $\{(N_{\mathcal{B}_D}[B, D_q]) \setminus \{D\} : q \in [i_n - n], B \subset D_q$ and $B \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}}\}$. For each $p \in [n+1]$, define M_p to be the complex generated by the maximal simplices $\{(N_{\mathcal{B}_D}[C_p, A]) \setminus \{D\} : C_p \subset A$ and $A \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}}\}$. Notice that pairwise intersection of complexes generated by these simplices is $N[C_p] \setminus D$. Therefore by an induction and Lemma 2.1, M_p is contractible for each p. Also, by the definition of M_p, for each pair $p, p' \in [n+1]$, the intersection of M_p and $M_{p'}$ is $N(D)$; hence using the same argument, $\bigcup_{p\in[n+1]} M_p$ is contractible.

For each $q \in [i_n - n]$, we define W_q to be the complex generated by the collection of maximal simplices $\{(N_{\mathcal{B}_D}[B, D_q]) \setminus \{D\} : B \subset D_q$ and $B \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}}\}$ which is contractible using the same arguments above. Notice that for each pair $q, q' \in [i_n - n]$, the intersection of W_q and $W_{q'}$ is $N(D)$ in $\mathcal{N}(\mathcal{B}_D;3)$ and the intersection of W_q and $\bigcup_{p\in[n+1]} M_p$ is the union of complexes generated by $N[B] \setminus D_q$ with $B \subset D_q$. This is a collection of simplies with only one common vertex D_q, hence their union is a cone which is clearly contractible. Therefore, by an induction and Lemma 2.1, the complex $\bigcup_{p\in[n+1]} M_p \cup \bigcup_{q\in[i_n-n]} W_q$ is contractible, i.e. $\mathrm{lk}_K(D)$ is contractible.

Therefor, by Lemma-2.2,

$$\mathcal{N}(\mathcal{B}_D;3) \simeq \mathcal{N}(\mathcal{B}_D;3) \setminus D \vee \sum(lk_{\mathcal{N}(\mathcal{B}_D;3)}(D))$$
$$\mathcal{N}(\mathcal{B}_D;3) \simeq \mathcal{N}(\mathcal{B}_{D_-};3) \vee \bigvee_{\alpha} S^2, \text{ where } \alpha = \sum_{k=1}^{n} i_k - \frac{n(n+1)}{2} - n.$$

□

Theorem 5.1. *For $m \geq n+2$ and $n \geq 1$, the Čech complexes $\mathcal{N}(\mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}};3)$ is homotopy equivalent to a wedge sum of spheres of dimension 2. More specifically,*

$$\mathcal{N}(\mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}};3) \simeq \mathcal{N}(\mathcal{F}^{m-1}_{n-1} \cup \mathcal{F}^{m-1}_{n} \cup \mathcal{F}^{m-1}_{n+1};3) \vee \bigvee_{\sum\limits_{A \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} \text{ and } 1 \notin A} \alpha_A} S^2,$$

where $\alpha_A = \sum\limits_{k=1}^{n} i_k - \dfrac{n(n+1)}{2} - n$ *for* $A = i_1 i_2 \dots i_{n+1} \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}}$.

Proof. The result trivial holds when $n = 0$ because $\mathcal{N}(\mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}};3)$ is contractible in this case.

Let $\mathcal{B}_1 = \{C \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}} \,|1 \in C\}$ and consider its Čech complex $\mathcal{N}(\mathcal{B}_1;3)$. Then, the metric spaces $\mathcal{B}_1$ and $\mathcal{F}^{m-1}_{n-1} \cup \mathcal{F}^{m-1}_{n} \cup \mathcal{F}^{m-1}_{n+1}$ are isometric and hence by Lemma-2.4,

$$\mathcal{N}(\mathcal{B}_1;3) \simeq \mathcal{N}(\mathcal{F}^{m-1}_{n-1} \cup \mathcal{F}^{m-1}_{n} \cup \mathcal{F}^{m-1}_{n+1};3).$$

We assume that $\mathcal{N}(\mathcal{B}_1;3)$ is either contractible or homotopy equivalent to a wedge sum of spheres. This is true when $n = 1$ because $\mathcal{N}(\mathcal{B}_1;3)$ is contractible when $n = 1$.

We discuss the homotopy type of $\mathcal{N}(\mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}};3)$ by bringing vertex in following the total ordering $\ltimes$. By Lemma 5.1,

$$\mathcal{N}(\mathcal{B}_1 \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n}};3) \simeq \mathcal{N}(\mathcal{B}_1;3).$$

Then we proceed by adding vertices from $\mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}}$. Then by induction and Lemma 5.3,

$$\mathcal{N}(\mathcal{B}_1 \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}}; 3) \simeq \bigvee_{\sum_{A \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} \text{ and } 1 \notin A} \alpha_A} S^2.$$

Lastly, we claim that adding the vertices from $\mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}}$ to $\mathcal{B}_1 \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}}$ yields a complex whose homotopy doesn't change. This finishes the proof.

Fix $D \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}}$ with $1 \notin D$ and D_- is the predecessor of D based on the order $\ltimes$. Denote $\mathcal{B}_M = \mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \{A \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}} : 1 \in A\}$, $\mathcal{B}_{D_-} = \mathcal{B}_M \cup \{A \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}} : A \ltimes = D\}$ and $\mathcal{B}_D = \mathcal{B}_{D_-} \cup \{D\}$. We claim that $\mathcal{N}(\mathcal{B}_D; 3) \simeq \mathcal{N}(\mathcal{B}_{D_-}; 3)$.

We introduce a few notations about D first. Let $D = i_1 i_2 \ldots i_n i_{n+1} i_{n+2}$, $D_0 = i_1 i_2 \ldots i_n$, $D_1 := i_1 i_2 \ldots i_n i_{n+1}$, and $D_2 := i_1 i_2 \ldots i_n i_{n+2}$.

Based on the order $\ltimes$, $D_0 j$ is in $\mathcal{B}_{D_-}$ for any $j \in \{i_{n+1}, \ldots, m\}$. Note that $D_0 \ltimes D_1 \ltimes D_2 \ltimes D_-$. Notice that there are a total of $n+2$ subsets of D in $\mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}}$; but only D_1 and D_2 are $\ltimes D_-$. This implies $N[D] = \{D_1, D_2, D\}$ in B_D. We list the maximal simplices in the complex $\mathcal{N}(\mathcal{B}_D; 3)$ which contains D below:

1) $N[D_1, D] = N_{\mathcal{B}_D}[D_1] \cup N[D]$

2) $N[D_2, D] = N_{\mathcal{B}_D}[D_2] \cup N[D]$

3) $N[A, D_1]$ for all $A \in N(D_1) \setminus$ and $N[B, D_2]$ for all $B \in N(D_2) \setminus D$.

Notice that maximal simplices in form 1) and 2) are the only ones with D as one of the generators.

First, we show that the complex $\mathcal{N}(\mathcal{B}_D; 3) \setminus D$ is homotopy equivalent to $\mathcal{N}(B_{D_-}; 3)$. Note that

$$\mathcal{N}(\mathcal{B}_D; 3) \setminus D = \mathcal{N}(B_{D_-}; 3) \cup N[D_1, D) \cup N[D_2, D).$$

We define a new complex L_D to be the complex generated by the maximal simplices $\{N[D_1, D), N[D_2, D)\}$. The intersection of the complexes generated by these two simplices is the set $\{D_0, D_1, D_2\}$ which is simplex in both of the complexes. Therefore, by lemma-2.1, L_D is contractible. Therefore by Lemma-2.1,

$$\mathcal{N}(\mathcal{B}_D;3) \setminus D \simeq \mathcal{N}(\mathcal{B}_{D_-};3).$$

Then we determine the homotopy type of the link of D in $\mathcal{N}(\mathcal{B}_D;3)$. We start from defining the complexes

$$L_1 = \bigcup_{A \in N(D_1)} N[A, D_1] \setminus D \text{ and } L_2 = \bigcup_{A \in N(D_2)} N[A, D_2] \setminus D$$

Notice that here we use the simplex to represent the complex generated by this simplex. Then, the link of D in $\mathcal{N}(\mathcal{B}_D;3)$ is $L_1 \cup L_2$. Notice that D_0 is in all the maximal simplexes involved, hence $L_1 \cup L_2$ is a cone with vertex V_0. Therefore, $L_1 \cup L_2$ is contractible. Consequently, by Lemma 2.2,

$$\mathcal{N}(\mathcal{B}_D;3) \simeq \mathcal{N}(\mathcal{B}_D;3) \setminus D \vee \sum(\mathrm{lk}_{\mathcal{N}(\mathcal{B}_D;3)}(D)) \simeq \mathcal{N}(\mathcal{B}_{D_-};3).$$

□

5.2 Čech Complex $\mathcal{N}(\mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+3}};3)$

In this section, we will determine the homotopy type of the complex $\mathcal{N}(\mathcal{F}^m_0 \cup \mathcal{F}^m_1 \cup \mathcal{F}^m_2 \cup \mathcal{F}^m_3;3)$ for general $m \geq 3$. We'll introduce a new total order on the space, $\mathcal{K}_4 = \mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+3}}$ and with the help of this ordering, we prove our main result in this section.

In the complex $\mathcal{N}(\mathcal{K}_4;3)$, the maximal faces are generated by the following sets:

$$\text{for any } A, B \in \mathcal{K}_4, \text{ with } A \sim B, N_1[A,B] = N_1[A] \cup N_1[B]$$

where $N_1[A], N_1[B]$ are the closed 1-neighborhood maximal simplex generators. For convenience, we will omit 1 in the subscript. Also, we will use the same notation $N[A,B), N(A,B]$, and $N(A,B)$ as in the previous section.

Next, we define the total order $\lhd$ on $\mathcal{K}_4$. For any $A, B \in \mathcal{K}_4$ with $A = i_1 i_2 \cdots i_p,$ and $B = j_1 j_2 \cdots j_q$, we say $A \lhd B$ if only of the following holds:

1. $|A| = |B|$ and $A \prec B$;

2. $A \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n}}$ and $B \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+3}}$;

3. $A, B \notin \mathcal{F}^{\mathrm{m}}_{\mathrm{n}}$ or $\mathcal{F}^{\mathrm{m}}_{\mathrm{n+3}}$, and $A \ltimes B$ in $\mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}}$;

4. $B \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n+3}}$ and $A \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}}$.

We need the following result first.

Lemma 5.4. *Let $\{A_p\}_{p=1}^{6}$ be a collection of vertices. Define the complexes K_1, K_2, K_3 using their maximal simplices as follows:*

1) the maximal simplices of K_1 are $\{A_1, A_2, A_3, A_4\}, \{A_1, A_2, A_3, A_5\}$,and $\{A_2, A_3, A_4, A_5\}$,

2) the maximal simplice of K_2 are $\{A_1, A_2, A_4, A_6\}, \{A_1, A_2, A_5, A_6\}$ and $\{A_2, A_4, A_5, A_6\}$,

3) the maximal simplice of K_3 are $\{A_1, A_3, A_4, A_6\}, \{A_1, A_3, A_5, A_6\}$, and $\{A_3, A_4, A_5, A_6\}$.

Then, the complex $K_1 \cup K_2 \cup K_3$ is homotopy equivalent to S^3.

Proof. Notice that K_1 K_2, and K_3 are all cones hence they are contractible. Let us consider the space $K_1 \cup K_2$. The 0-simplices that are common in both the spaces are A_1, A_2, A_4 and A_5. Therefore, the maximal simplices in the complex $K_1 \cap K_2$ are

$$\{A_1, A_2, A_4\}, \{A_1, A_2, A_5\}, \{A_2, A_4, A_5\}$$

which is cone with vertex A_2 hence contractible. By Lemma 2.1, $K_1 \cup K_2$ contractible.

Let L_1, L_2 and L_3 be the complexes generated by the maximal simplices in K_3 respectively. Now, consider $(K_1 \cup K_2) \cup L_1$. Then their intersection is $\{\{A_1, A_3, A_4\}, \{A_1, A_4, A_6\}\}$ and it is contractible. This means that $(K_1 \cup K_2) \cup L_1$ is contractible. Similarly, it is straightforward to verify that $(K_1 \cup K_2) \cup L_1 \cup L_2$ is contractible. Next, notice that $(K_1 \cup K_2 \cup L_1 \cup L_2) \cap L_3$ is the complex generated by the collection $\{\{A_3, A_4, A_5\}, \{A_3, A_4, A_6\}, \{A_3, A_5, A_6\}, \{A_4, A_5, A_6\}\}$

which is a boundary of the 3-simplex $\{A_3, A_4, A_5, A_6\}$. Hence, the intersection is homotopy equivalent to the sphere S^2. The inclusion maps are also null-homotopic clearly. Then by Lemma 2.1, $K_1 \cup K_2 \cup K_3 \simeq \sum(S^2) \simeq S^3$. $\square$

We need the following result to determine the homotopy type of the complex $\mathcal{N}(\mathcal{F}_0^m \cup \mathcal{F}_1^m \cup \mathcal{F}_2^m \cup \mathcal{F}_3^m; 3)$.

Lemma 5.5. *Fix $m \geq 3$ and $A \in \mathcal{F}_3^m$. Denote $\mathcal{B}_A = \{B \in \mathcal{F}_0^m \cup \mathcal{F}_1^m \cup \mathcal{F}_2^m \cup \mathcal{F}_3^m :$ $B \triangleleft A$ or $B = A\}$ Denote $K = \mathcal{N}(\mathcal{B}_A; 3)$. Then,*

$$lk_K(A) \simeq \bigvee_2 S^3.$$

Proof. Represent A as $i_1 i_2 i_3$. Then denote $A_j = \{i_j\}$ for $j = 1, 2, 3$ and also $A_{12} = A_1 A_2$, $A_{13} = A_1 A_3$, $A_{23} = A_2 A_3$. Notice that $A_1 \triangleleft A_{12} \triangleleft A_{13} \triangleleft A_2 \triangleleft A_{23} \triangleleft A_3 \triangleleft A$.

Then the maximal simplices with A as a generator are $N[A_{12}, A]$, $N[A_{13}, A]$, and $[N_{23}, A]$. Other maximal simplices in K containing A are $N[A_1, A_{12}]$, $N[A_2, A_{12}]$, $N[A_1, A_{13}]$, $N[A_3, A_{13}]$, $N[A_2, A_{23}]$, $N[A_3, A_{23}]$. Then the link of A in K is the complex generated by all these simplices with the vertex A removed.

Let L_1 be the complex whose maximal simplices are $N[A_1, A_{12}]$, $N[A_2, A_{12}]$, $N[A_1, A_{13}]$, $N[A_3, A_{13}]$, $N[A_2, A_{23}]$, $N[A_3, A_{23}]$ with A removed. Notice that the emptyset $\emptyset$ is in all of them. Hence L_1 is a cone. So it is contractible.

Let M_1 be the complex generated by the simplex $N[A_{12}, A)$; M_2 be the complex generated by $N[A_{13}, A)$; and M_3 be the complex generated by $[N_{23}, A)$.

By going through simplices in the intersections of M_1 and L_1, $L_1 \cup M_1$ is contractible. Then using similar discussion, $L_1 \cup M_1 \cup M_2$ is homotopy equivalent to S^3 and then $(L_1 \cup M_1 \cup M_2) \cup M_3$ is homotopy equivalen to $\vee_2 S^3$. $\square$

Theorem 5.2. *For any $m \geq 3$, the Čech complex $\mathcal{N}(\mathcal{F}_0^m \cup \mathcal{F}_1^m \cup \mathcal{F}_2^m \cup \mathcal{F}_3^m; 3)$ is homotopy equivalent to a wedge sum of S^4's. Specifically,*

$$\mathcal{N}(\mathcal{F}_0^m \cup \mathcal{F}_1^m \cup \mathcal{F}_2^m \cup \mathcal{F}_3^m; 3) \simeq \bigvee_{\binom{m}{3} \cdot 3} S^4.$$

Proof. The complex $\mathcal{N}(\mathcal{F}_0^m \cup \mathcal{F}_1^m \cup \mathcal{F}_2^m; 3)$ is contractible by Lemma 5.1. Fix $A = i_1 i_2 i_3 \in \mathcal{F}_3^m$. We use the same notations, A_1, A_2, A_3, A_{12}, A_{13}, A_{23}, and $\mathcal{B}_A$ as in the proof of Lemma 5.5. We define $\mathcal{B}_{A_-} = \mathcal{B}_A \setminus \{A\}$.

We assume, for induction, that $\mathcal{N}(\mathcal{B}_{A_-}; 3)$ is either contractible or homotopy equivalent to S^4. This is true when $A_- \in \mathcal{F}_2^m$. Let M_1 be the complex generated by the simplex $N[A_{12}, A)$; M_2 be the complex generated by $N[A_{13}, A)$; and M_3 be the complex generated by $[N_{23}, A)$. Then

$$\mathcal{N}(\mathcal{B}_A; 3) \setminus A = \mathcal{N}(\mathcal{B}_{A_-}; 3) \cup M_1 \cup M_2 \cup M_3.$$

Going through the maximal simplices in $\mathcal{N}(\mathcal{B}_{A_-}; 3) \cap M_1$ and $(\mathcal{N}(\mathcal{B}_{A_-}; 3) \cup M_1) \cap M_2$, it is straightforward to see that both of them are contractible; therefore by Lemma 2.1, the complex $\mathcal{N}(\mathcal{B}_{A_-}; 3) \cup M_1 \cup M_2$ is homotopy equivalent to $\mathcal{N}(\mathcal{B}_{A_-}; 3)$. Then the intersection of the complex $\mathcal{N}(\mathcal{B}_{A_-}; 3) \cup M_1 \cup M_2$ and M_3 is homotopy equivalent to S^3 using Lemma 5.4. Hence by Lemma 2.1 again, the complex $\mathcal{N}(\mathcal{B}_A; 3) \setminus A$ is homotopy equivalent to $\mathcal{N}(\mathcal{B}_{A_-}; 3) \vee S^4$.

By Lemma 5.5, the link of A in the complex $\mathcal{N}(\mathcal{B}_A; 3)$ is homotopy equivalent to $\vee_2 S^3$ which is clearly null-homotopic in $\mathcal{N}(\mathcal{B}_A; 3)$ by the inductive assumption. By Lemma 2.2, the complex $\mathcal{N}(\mathcal{B}_A; 3)$ is homotopy equivalent to $\mathcal{N}(\mathcal{B}_{A_-}; 3) \vee \bigvee_3 S^4$.

Because there are $\binom{m}{3}$-many vertices in $\mathcal{F}_3^m$, by an inductive process, we obtain the homotopy type of $\mathcal{N}(\mathcal{F}_0^m \cup \mathcal{F}_1^m \cup \mathcal{F}_2^m \cup \mathcal{F}_3^m; 3)$ which is $\bigvee_{3 \cdot \binom{m}{3}} S^4$. This finishes the proof. $\square$

Chapter 6

Conclusion and Future Work

The Vietoris-Rips complexes and Čech complexes are commonly used in the fields of computational and applied topology. However, little is known about these complexes at larger scales and for different spaces. We end with some open questions that have been asked in [23] about the Vietoris-Rips complex of a finite metric space.

Suppose $2 < n < m - 2$. For any pair of subsets B_1, B_2 of $[m]$ with $|B_1| = |B_2| = n$, $d(B_1, B_2) \leq 2k + 1$ is equivalent to $d(B_1, B_2) \leq 2k$ for any non negative integer k. Hence, the Vietoris-Rips complex $\mathcal{VR}(\mathcal{F}_n^m; 3)$ is identical with $\mathcal{VR}(\mathcal{F}_n^m; 2)$. Little is known for $r \geq 4$. The complex $\mathcal{VR}(\mathcal{F}_3^6; 4)$ is the boundary of a cross-polytope on 20 vertices, hence, it is homotopy equivalent to S^9. Using polymake [25], we found that the reduced homology groups of $\mathcal{VR}(\mathcal{F}_3^7; 4)$ is trivial when $n \neq 6$ or 9; and, $\tilde{H}_6(\mathcal{VR}(\mathcal{F}_3^7; 4)) = \mathbb{Z}^{29}$ and $\tilde{H}_9(\mathcal{VR}(\mathcal{F}_3^7; 4)) = \mathbb{Z}^7$. Observe that the complex $\mathcal{VR}(\mathcal{F}_3^m; 4)$ is identical with $\mathcal{VR}(\mathcal{F}_3^m; 5)$; and both of them are equal to the independence complex of the Kneser graph $\mathrm{KG}_{3,m-6}$ with $m \geq 6$. Therefore, the complex $\mathcal{VR}(\mathcal{F}_n^m; 4)$ for general $2n < m$ is very likely to be homotopy equivalent to a wedge sum of spheres with different dimensions. Following are some open questions based on the above computations:

Question 6.1. *What are the homotopy types of the complex* $\mathcal{VR}(\mathcal{F}_{\mathrm{n}}^{\mathrm{m}}; r)$ *for* $r \geq 4$?

Question 6.2. *Assume that* $2n < m$. *Are the complexes* $\mathcal{VR}(\mathcal{F}_{\mathrm{n}}^{\mathrm{m}}; 4)$ *with* $2n < m$ *homotopy equivalent to a wedge sum of spheres* S^6*'s and* S^9*'s?*

Regarding $\check{C}$ech complexes, using the results that we proved so far, we can extend the result of Adams, Shukla and Singh in [8] about the Čech complexes of hypercube graphs

(Q_m) for $r \geq 2$. Assume $m = 3, r = 2$, then Q_m consists of $2^3 = 8$ vertices. There are two ways of representing these vertices: either $\{000, 001, 010, 011, 100, 101, 110, 111\}$ or $\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$. The first representation with the Hamming distance is the notation used by Adams, Shukla and Singh in [8]. Like mentioned in Chapter 1, each binary string of length m can be represented as a subset of $[m]$. In this example, 00 is represented either by $\emptyset$ or $0, 01$ is represented by $\{1\}, 10$ is represented by $\{2\}$, 11 by $\{1,2\}$, and so on. As the scale is 2 (even), by definition, the maximal simplices of the complex $\mathcal{N}(Q_3; 2)$ are generated by the sets $N_1[A]$ for any A in the vertex set. Meaning, elements which differ by 1 element are connected. Now consider the complexes $\mathcal{K}_1 = \mathcal{N}(\mathcal{F}_0^3 \cup \mathcal{F}_1^3 \cup \mathcal{F}_2^3; 2)$ and $\mathcal{K}_2 = \mathcal{N}(\mathcal{F}_1^3 \cup \mathcal{F}_2^3 \cup \mathcal{F}_3^3; 2)$. A union of these two complexes would also have the above mentioned 8 vertices and their maximal simplices are also 1-closed neighborhood of all the vertices. An element A from $\mathcal{F}_1^3$ in $\mathcal{K}_1$ is connected to $0 \in \mathcal{F}_0^3$ as well as to the elements in $\mathcal{F}_2^m$ which have A as a subset. While in the complex $\mathcal{K}_2, A$ is only connected to the elements in $\mathcal{F}_2^m$ which have A as a subset. So, when we take the union $\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2, N_{\mathcal{K}}[A] = N_{\mathcal{K}_1}[A] \cup N_{\mathcal{K}_2}[A] = N_{\mathcal{K}_1}[A]$. Based on a similar reasoning, for any $B \in \mathcal{F}_2^m, N_{\mathcal{K}}[B] = N_{\mathcal{K}_1}[B] \cup N_{\mathcal{K}_2}[B] = N_{\mathcal{K}_2}[B]$. And the other maximal simplices would be the same, that is, $N_{\mathcal{K}}[0] = N_{\mathcal{K}_1}[0]$ and $N_{\mathcal{K}}[\{123\}] = N_{\mathcal{K}_2}[\{123\}]$. As a result, one can say that $\mathcal{N}(Q_2, 3)$ is equivalent to $\mathcal{K}$, the union of a finite union metric spaces.

Next, let us look at the intersection space of the two complexes $\mathcal{I}_K = \mathcal{K}_1 \cap \mathcal{K}_2$. We have already seen that for any $A \in \mathcal{F}_1^m, N_{\mathcal{K}_2}[A] \subset N_{\mathcal{K}_1}[A]$ and for any $B \in \mathcal{F}_2^m, N_{\mathcal{K}_1}[B] \subset N_{\mathcal{K}_2}[B]$. While the element 0 is in $\mathcal{K}_1$ but not in $\mathcal{K}_2$ and the element $\{123\}$ is in $\mathcal{K}_2$ but not in $\mathcal{K}_1$. Hence, it is immediate that the intersection $\mathcal{N}(\mathcal{F}_0^3 \cup \mathcal{F}_1^3 \cup \mathcal{F}_2^3; 2) \cap \mathcal{N}(\mathcal{F}_1^3 \cup \mathcal{F}_2^3 \cup \mathcal{F}_3^3; 2)$ is homotopy equivalent to $\mathcal{N}(\mathcal{F}_1^3 \cup \mathcal{F}_2^3; 2)$. Using the inductive formulae in Theorems 4.1, 4.2, 4.3, we have:

$$\mathcal{N}(\mathcal{F}_1^3 \cup \mathcal{F}_2^3; 2) \simeq S^1$$
$$\mathcal{N}(\mathcal{F}_0^3 \cup \mathcal{F}_1^3 \cup \mathcal{F}_2^3; 2) \simeq \bigvee_3 S^2$$
$$\mathcal{N}(\mathcal{F}_1^3 \cup \mathcal{F}_2^3 \cup \mathcal{F}_3^3; 2) \simeq \bigvee_3 S^2.$$

Since the inclusion of S^1 into S^2 is null-homotopic, the inclusion maps

$\iota_1 : \mathcal{N}(\mathcal{F}^3_1 \cup \mathcal{F}^3_2; 2) \hookrightarrow \mathcal{N}(\mathcal{F}^3_0 \cup \mathcal{F}^3_1 \cup \mathcal{F}^3_2; 2)$ and $\iota_2 : \mathcal{N}(\mathcal{F}^3_1 \cup \mathcal{F}^3_2; 2) \hookrightarrow \mathcal{N}(\mathcal{F}^3_1 \cup \mathcal{F}^3_2 \cup \mathcal{F}^3_3; 2)$ are null-homotopic., implying, by Lemma 2.1,

$$\mathcal{N}(Q_3; 2) \simeq \mathcal{N}(\mathcal{F}^3_0 \cup \mathcal{F}^3_1 \cup \mathcal{F}^3_2; 2) \vee \mathcal{N}(\mathcal{F}^3_1 \cup \mathcal{F}^3_2 \cup \mathcal{F}^3_3; 2) \vee \sum \left(\mathcal{N}(\mathcal{F}^3_1 \cup \mathcal{F}^3_2; 2)\right)$$
$$\mathcal{N}(Q_3; 2) \simeq \bigvee_3 S^2 \vee \bigvee_3 S^2 \vee \sum \left(S^1\right) = \bigvee_7 S^2$$

In [8] Adams, Shukla and Singh have proved that the Čech complex $\mathcal{N}(Q_m; 2)$ is homotopy equivalent to a wedge sum of $2^{m-2}(m^2+3m-4)-1$ spheres of dimension 2. In particular, when $m = 3$, then, $2^{3-2}(3^2 + 3(3) - 4) - 1 = 7$. Based on the above example, it is natural to state that:

Conjecture 6.1. *Suppose* $m \geq 2$, *and* $m \geq r$, *then,*
$$\mathcal{N}(Q_m; r) = \bigcup_{i=0}^{m-r} \mathcal{N}\left(\mathcal{F}^m_i \cup \mathcal{F}^m_{i+1} \cup \ldots \cup \mathcal{F}^m_{i+r}; r\right).$$

Conjecture 6.2. *For* $m \geq 2$, *and* $m \geq r$,
$$\mathcal{N}(\mathcal{F}^m_i \cup \mathcal{F}^m_{i+1} \cup \ldots \cup \mathcal{F}^m_{i+r}; r) \cap \mathcal{N}(\mathcal{F}^m_{i+1} \cup \mathcal{F}^m_{i+2} \cup \ldots \cup \mathcal{F}^m_{i+r+1}; r) \simeq \mathcal{N}(\mathcal{F}^m_{i+1} \cup \ldots \cup \mathcal{F}^m_{i+r}; r).$$

These conjectures provide a new approach to investigate the homotopy type or other topological properties of the complexes $\mathcal{N}(Q_m; r)$ for general m and scale r.

Based on the computations provided in Chapter 1 and using similar proof techniques as used in Chapters 4, 5, we conjecture the following result for general n and $r = 3$.

Question 6.3. *For* $n \geq 1$ *and* $m \geq 4$, *the Čech complex on* $\mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+3}}$ *with scale* 3 *is homotopy equivalent to a wedge sum of* S^3*'s and* S^4*'s, that is;*

$$\mathcal{N}(\mathcal{F}^{\mathrm{m}}_{\mathrm{n}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+2}} \cup \mathcal{F}^{\mathrm{m}}_{\mathrm{n+3}}; 3) \simeq \mathcal{N}(\mathcal{F}^{m-1}_{n-1} \cup \mathcal{F}^{m-1}_{n} \cup \mathcal{F}^{m-1}_{n+1} \cup \mathcal{F}^{m-1}_{n-1}; 3) \vee \bigvee_a S^3 \vee \bigvee_{b+c} S^4,$$

where, $a = \binom{m-1}{n+3} \cdot \binom{n+2}{n-1}$, *and* $b = 3 \cdot \binom{m-1}{n+2} + 3 \cdot \binom{m-1}{n+3} \cdot \binom{n+3}{n}$

$c = 3 \cdot \underset{A \in \mathcal{F}^{\mathrm{m}}_{\mathrm{n+1}} \text{ and } 1 \notin A}{\;} \beta_A$

and $\beta_A = i_n - n + \sum_{k=1}^{n} i_k - \dfrac{(n+1)(n+2)}{2}$ *for* $A = i_1 i_2 \ldots i_{n+1}$.

More interesting questions related to the $\check{C}$ech complexes are:

Question 6.4. *In what dimensions, and for what scales is the reduced homology of the complex $\mathcal{N}(Q_m; r)$ non-trivial?*

Question 6.5. *What are the homotopy types of the complex $\mathcal{N}\left(\bigcup_{i=0}^{r+1} \mathcal{F}_i^m; r\right)$ for $r \geq 3$?*

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