Partial Saturation Numbers of Graphs

by

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Abstract

Given a fixed simple graph H, a simple graph G is called H-saturated if G is H-free, but the addition of any edge $e \in E(\overline{G})$ creates a copy of H. The saturation number of H, denoted $\mathrm{sat}(n, H)$, is the minimum number of edges of an H-saturated graph G on n vertices. If G is not necessarily H-free, but the addition of any edge $e \in E(\overline{G})$ creates a *new* copy of H, then G is called partially H-saturated. The partial saturation number of H, denoted $\mathrm{psat}(n, H)$, is the minimum size of a partially H-saturated graph on n vertices. In this dissertation, we explore the relationship between $\mathrm{sat}(H, n)$ and $\mathrm{psat}(n, H)$ and determine $\mathrm{psat}(n, H)$ for various classes of graphs H.

We first show that $\mathrm{psat}(n, H) = \mathrm{sat}(n, H)$ for every graph H of order at most 4, with only one exception. In the case $H = C_4$, we characterize all minimum partially C_4-saturated graphs. For a double star on $s + t$ vertices, with $3 \le s < t$, we completely determine $\mathrm{psat}(n, S_{s,t})$ when n is large enough. We study the partial saturation number of triangle-free graphs and provide a nearly sharp lower bound. For a path P_k, we establish the exact value of $\mathrm{psat}(n, P_k)$ when $n \ge \left\lfloor \frac{3k-3}{2} \right\rfloor$. We observe that for $k \ge 6$, $\lim_{n\to\infty} \frac{\mathrm{sat}(n, P_k) - \mathrm{psat}(n, P_k)}{n} > 0$. Finally, we characterize all triangle-free graphs H such that $\lim_{n\to\infty} \frac{\mathrm{psat}(n, H)}{n}$ is minimized.

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Table of Contents

List of Figures

Chapter 1

Introduction

1.1 Definitions

We will consider only finite graphs that are simple and undirected. Our notation will be standard, generally following the notation of [19]. Given a graph G, we will use $V(G)$ to denote the vertex set of G and $E(G)$ to denote the edge set of G. The *order* of G, written $n(G)$, is the number vertices in G, and the *size* of G, written $e(G)$, is the number of edges in G. We use $\overline{G}$ to denote the complement of G. For any graph G, we use $c(G)$ to denote the number of components in G. For two vertex-disjoint graphs G_1 and G_2, we will use $G_1 + G_2$ to represent the union of G_1 and G_2, and $G_1 \vee G_2$ to represent the join of G_1 and G_2.

Given $A, B \subseteq V(G)$, we write $E(A, B)$ for the set of edges in G having one endpoint in A and the other in B. Given a vertex v in a graph G, the *open neighborhood of* v, denoted $N_G(v)$ or $N(v)$, is the set of vertices in G that are adjacent to v. The degree of v in G is $d_G(v) = |N(v)|$. We will use $\Delta(G)$ to denote the maximum degree of G and $\delta(G)$ to denote the minimum degree of G. For any two vertices u and v in G, we use $C_G(u, v)$, or $C(u, v)$, to denote the set of all common neighbors of u and v in G. We also let $c_G(u, v) = |C_G(u, v)|$.

Now let S be a set of vertices in G. We define the *degree sum of* S to be $\sigma_G(S) = \sigma(S) = \sum_{v \in S} d_G(v)$. We abbreviate $\sigma(V(G))$ as $\sigma(G)$. Note that $\sigma(G) = 2|E(G)|$. For any $v \in V(G)$, we write $v \sim S$ if v is adjacent to at least one vertex in S. We then define the *neighborhood of* S to be $N_G(S) = N(S) = \{v \in V(G) \mid v \sim S\}$. The *neighborhood of* v *with respect to* S, denoted by $N_{G,S}(v)$, is defined as the set of vertices in S adjacent to v. So $N_{G,S}(v) = N_G(v) \cap S$. Then the degree of v with respect to S is given by $d_{G,S}(v) = |N_{G,S}(v)|$. The *distance between two vertices* u *and* v, written $d_G(u, v)$ or simply $d(u, v)$, is the least length of a u, v-path. The *eccentricity* of v, written $\epsilon_G(v)$ or $\epsilon(v)$, is given as $\max_{u \in V(G)} d(u, v)$. We define the *distance between v and S* as $d(v, S) = \min\{d(v, x) \mid x \in S\}$.

A graph G is called *partly k-regular* if $\Delta(G) = k$ and $\delta(G) \geq k - 1$. A vertex of degree $k-1$ in a partly k-regular graph G is called a *minor vertex*. A partly k-regular graph with at most one minor vertex is called *almost k-regular*. We let K_n denote the complete graph on n vertices, P_k denote the path on k vertices, and S_k denote the star on k vertices. In a star S_k with $k \geq 3$, the unique vertex of degree $k-1$ is called the *central vertex* of the star. (If $S_2 = K_2$, either vertex can be considered the central vertex). We now define a *double star*, denoted $S_{s,t}$, to be a graph on $s+t$ vertices constructed by adding an edge between the central vertices of a star on s vertices and a star on t vertices. We say that $S_{s,t}$ is *balanced* if $s = t$ and *unbalanced* if $s < t$.

A complete l-partite graph is a simple graph whose vertices can be partitioned into l partite sets so that $u \sim v$ if and only if u and v belong to different partite sets. The Turán graph $T_{n,l}$ is the complete l-partite graph with n vertices whose l-partite sets differ in order by at most 1. Note that every partite set in $T_{n,l}$ has order $\lfloor n/l \rfloor$ or $\lceil n/l \rceil$, and that $n - \lceil n/l \rceil = \delta(T_{n,l}) \leq \Delta(T_{n,l}) = n - \lfloor n/l \rfloor$.

Now let H be a nonempty graph and $n \geq |V(H)|$. We say that a graph G on n vertices is *H-saturated* if G is H-free, but for any edge $e \in E(\overline{G})$, $G + e$ contains a copy of H. The *saturation number of H*, denoted $\text{sat}(n, H)$, is the minimum size of an H-saturated graph on n vertices. If G is not necessarily H-free, but for any edge $e \in E(\overline{G})$, $G + e$ contains at least one new copy of H, then we say that G is *partially H-saturated*. The *partial saturation number of H*, denoted $\text{psat}(n, H)$, is the minimum size of a partially H-saturated graph on n vertices.

The function $\text{psat}(n, H)$, in general, is not monotone with respect to n or H. First, we observe that $\text{psat}(n, S_k + e) \leq \text{sat}(n, S_k + e) \leq n - 1$, since S_n is $(S_k + e)$-saturated.

So by Theorem 1.2', we have $\text{psat}(n, S_k) > \text{psat}(n, S_k + e)$ for $k \geq 5$. Thus, the psat-function is not, in general, monotone with respect to subgraphs. To see that the psat-function is not, in general, monotone in n, consider $H = P_4$. By Theorem 2.12, we have $\text{psat}(2n-1, P_4) = n + 1 > \text{psat}(2n, P_4) = n$ for $n \geq 4$. For the remainder of this chapter, we list some known results and briefly introduce a related concept called weak saturation.

In 1972, L.T. Ollman [15] determined that $\text{sat}(n, C_4) = \left\lfloor \frac{3n-5}{2} \right\rfloor$ for $n \geq 5$ and also found all minimum C_4-saturated graphs. Later, in 1989, Zsolt Tuza [17] gave a shorter proof. In Chapter 2, we prove that $\text{psat}(n, C_4) = \text{sat}(n, C_4)$ for all $n \geq 5$ and characterize all minimum partially C_4-saturated graphs by modifying the techniques used in [17]. We also show that $\text{psat}(n, H) = \text{sat}(n, H)$ for every graph H of order at most 4, with the exception that $\text{psat}(5, P_4) = 3$ and $\text{sat}(5, P_4) = 4$.

In Chapter 3, we study the partial saturation number of double stars. For a double star on $s + t$ vertices, with $3 \leq s < t$, we completely determine $\text{psat}(n, S_{s,t})$ when n is large enough.

In Chapter 4, we study the partial saturation number of triangle-free graphs and provide a nearly sharp lower bound. We also give the complete formula for $\text{psat}(n, P_k)$ when $k \geq 5$ and $n \geq \left\lfloor \frac{3k-3}{2} \right\rfloor$. Finally, we discuss the topic of psat-sharp graphs and characterize all graphs H such that $\lim_{n\to\infty} \dfrac{\text{psat}(n, H)}{n}$ is minimized.

1.2 Some known results

In 1964, Erdös, Hajnal, and Moon first introduced the concept of partial saturation numbers (though not using that terminology) and determined the partial saturation number for complete graphs.

Theorem 1.1 (Erdös, Hajnal, and Moon [8]). *If $n \geq k \geq 2$, then*

$$\text{psat}(n, K_k) = \binom{n}{2} - \binom{n-k+2}{2}.$$

In addition, $K_{k-2} \vee \overline{K}_{n-k+2}$ *is the unique minimum partially* K_k*-saturated graph of order* n*.*

Since $K_{k-2} \vee \overline{K}_{n-k+2}$ is K_k-saturated as well, the following result follows directly.

Theorem 1.1'. *If* $n \geq k \geq 2$*, then*

$$\operatorname{sat}(n, K_k) = \binom{n}{2} - \binom{n-k+2}{2}.$$

In addition, $K_{k-2} \vee \overline{K}_{n-k+2}$ *is the unique minimum* K_k*-saturated graph of order* n*.*

In 1986, Kászonyi and Tuza [13] determined the saturation number for stars in theorem below.

Theorem 1.2 (Kászonyi and Tuza [13])**.** *Let* $n \geq k \geq 3$ *and* $r = \min\left\{\left\lfloor \frac{k}{2} \right\rfloor, n-k+1\right\}$*. Then*

$$\operatorname{sat}(n, S_k) = \left\lceil \frac{(k-2)(n-r)}{2} + \binom{r}{2} \right\rceil$$

In addition, for every tree T *of order* k *such that* $T \neq S_k$*, we have* $\operatorname{sat}(n, T) < \operatorname{sat}(n, S_k)$ *when* n *is large enough.*

The proof of the above theorem applies for the partial saturation number as well, and thus, we have the following theorem.

Theorem 1.2'. *Let* $n \geq k \geq 3$ *and* $r = \min\left\{\left\lfloor \frac{k}{2} \right\rfloor, n-k+1\right\}$*. Then*

$$\operatorname{psat}(n, S_k) = \left\lceil \frac{(k-2)(n-r)}{2} + \binom{r}{2} \right\rceil$$

In addition, for every tree T *of order* k *such that* $T \neq S_k$*, we have* $\operatorname{psat}(n, T) < \operatorname{psat}(n, S_k)$ *when* n *is large enough.*

Faudree, Faudree, Gould, and Jacobson [9] studied saturation numbers for trees, including the next two results.

Lemma 1.3 (Faudree et. al [9]). *If there exist trees T_k and T'_k each of order k such that T'_k is T_k-saturated, then $k \geq 4$, $T_k = S_{2,k-2}$, and $T'_k = S_k$.*

The following result is obtained directly from Lemma 1.3.

Theorem 1.4 (Faudree et. al [9]). *For any tree T_k of order $k \geq 5$ and any $n \geq k+2$,*

$$\mathrm{sat}(n, T_k) \geq n - \left\lfloor \frac{n+k-2}{k} \right\rfloor.$$

Moreover, $S_{2,k-2}$ is the only tree of order k attaining this minimum for all n.

Kászonyi and Tuza [13] found the best known general upper bound on the saturation number (and thus on the partial saturation number) using the vertex cover number of a graph, which we define here. A *vertex cover* of a graph H is a vertex subset of H that contains at least one endpoint of every edge. The *vertex cover number* of H, denoted $\beta(H)$, is the minimum size of a vertex cover of H.

Theorem 1.5 (Kászonyi and Tuza [13]). *Let β be the vertex cover number of H, and define $d = \min\{\,|N_H(x) \setminus C| : x \in C, C$ is a minimum vertex cover of $H\}$. Then,*

$$\mathrm{sat}(n, H) \leq (\beta - 1)n + \frac{(d-1)(n-\beta+1)}{2} - \binom{\beta}{2}.$$

We now provide a few examples on how to apply Theorem 1.5. If $H = K_k$, then $\beta = k-1$, $d = 1$, and $\mathrm{sat}(n, K_k) \leq (k-2)n - \binom{k-1}{2}$. If $H = S_k$, then $\beta = 1$, $d = k-1$, and $\mathrm{sat}(n, S_k) \leq \frac{k-2}{2}n$. If $H = S_{s,t}$ with $s \leq t$, then $\beta = 2$, $d = s-1$, and $\mathrm{sat}(n, S_{s,t}) \leq \frac{s}{2}(n-1)$.

In 2022, Cameron and Puleo gave a lower bound on $\mathrm{sat}(n, H)$ using the concept of the weight of a graph H, which we introduce here. Let uv be an edge in a nonempty graph H. We define the *weight of the edge uv* as $\mathrm{wt}_H(uv) = \mathrm{wt}(uv) = |N(u) \cap N(v)| + \max\{d_H(u), d_H(v)\}$. We define the *weight of the graph H* as $\mathrm{wt}(H) = \min_{uv \in E(H)} \mathrm{wt}(uv)$. Clearly, for every nonempty graph H, we must have that $\mathrm{wt}(H) \geq 1$, with equality if and only if H contains

K_2 as a component. The remark below holds because adding edges to a graph does not decrease its weight, and $\mathrm{wt}(K_k) = 2k - 3$ for $k \geq 2$.

Remark 1.6. *For every graph H with $|V(H)| \geq 2$, we have $\mathrm{wt}(H) \leq 2|V(H)| - 3$.*

Theorem 1.7 (Cameron and Puleo [3])**.** *Let H be a graph with weight $t \geq 1$. Then*

$$\mathrm{sat}(n, H) \geq \frac{t-1}{2}n - \frac{t^2 - 4t + 5}{2}.$$

It turns out that the proof of the above theorem does not make use of the condition that an H-saturated graph must be H-free. Thus, we conclude that this lower bound also applies to the partial saturation number. We give an altered version of Cameron and Puleo's proof below.

Theorem 1.7'. *Let H be a graph with weight $t \geq 1$. Then*

$$\mathrm{psat}(n, H) \geq \frac{t-1}{2}n - \frac{t^2 - 4t + 5}{2}.$$

Proof. Let G be a minimum partially H-saturated graph of order n and x^* be a vertex of minimum degree in G. Let $B = N_G(x^*)$ and $\overline{B} = V(G) \setminus B$. If $\delta(G) = t - 1$, then $\mathrm{psat}(n, H) = |E(G)| \geq \frac{(t-1)n}{2}$, and we are done. So assume $d_G(x^*) \leq t - 2$.

Let $y \in \overline{B} \setminus \{x^*\}$. Then $G + x^*y$ contains a new copy of H, say H'. So

$$\begin{aligned} t = \mathrm{wt}(H) = \mathrm{wt}(H') &\leq wt_{H'}(x^*y) \\ &= c_{H'}(x^*, y) + \max\{d_{H'}(x^*), d_{H'}(y)\} \\ &\leq c_G(x^*, y) + \max\{d_G(x^*) + 1, d_G(y) + 1\} \\ &= c_G(x^*, y) + d_G(y) + 1 \\ &= d_{G,B}(y) + d_G(y) + 1. \end{aligned}$$

Thus we have shown that $d_{G,B}(y) + d_G(y) \geq t-1$ for every vertex $y \in \overline{B} \setminus \{x^*\}$. Recall that $d_G(x^*) \leq t-2$. It then follows that

$$\begin{aligned}
\sigma(G) &= \sum_{x\in B} d_G(x) + \sum_{y\in\overline{B}} d_G(y) \\
&\geq \sum_{x\in B} d_{G,\overline{B}}(x) + \sum_{y\in\overline{B}} d_G(y) \\
&= \sum_{y\in\overline{B}} d_{G,B}(y) + \sum_{y\in\overline{B}} d_G(y) \\
&= \sum_{y\in\overline{B}} (d_{G,B}(y) + d_G(y)) \\
&\geq 2d_G(x^*) + (t-1)(n-1-d_G(x^*)) \\
&= (t-1)n - [(t-3)d_G(x^*) + t - 1] \\
&\geq (t-1)n - [(t-3)(t-2) + t - 1] \\
&= (t-1)n - (t^2 - 4t + 5).
\end{aligned}$$

This proves that $\mathrm{psat}(n, H) = |E(G)| \geq \frac{t-1}{2}n - \frac{t^2-4t+5}{2}$, and we are done. $\square$

1.3 Weak saturation

We now discuss the related notion of weakly saturated graphs. Let H be a nonempty graph and $n \geq |V(H)|$. A graph G of order n is *weakly H-saturated* if the missing edges of G can be added one at a time so that each added edge creates at least one new copy of H. The *weak saturation number of H*, denoted $\mathrm{wsat}(n, H)$, is the minimum size of a weakly H-saturated graph on n vertices. Clearly, we have that $\mathrm{wsat}(n, H) \geq e(H) - 1$. We refer the reader to [2] and [11] for general bounds on $\mathrm{wsat}(n, H)$. We also note here that $\mathrm{wsat}(n, H) \leq \mathrm{psat}(n, H) \leq \mathrm{sat}(n, H)$, since any partially H-saturated graph is also weakly H-saturated.

In 1977 Lovász [14] proved the following result, which was earlier conjectured by Bollobás and verified for $3 \leq k < 7$ in [1].

Theorem 1.8 (Lovász [14]). *For integers n and k,*

$$\text{wsat}(n, K_k) = \binom{n}{2} - \binom{n-k+2}{2}.$$

By Theorem 1.1, the graph $K_{k-2} \vee \overline{K}_{n-k+2}$ is the unique minimum partially K_k-saturated graph of order n. However, this is not the case for weak saturation. For example, when $k = 3$, every tree of order n is weakly K_3-saturated.

In 2002, Borowiecki and Sidorowicz [2] considered the weak saturation number of cycles and proved the following result.

Theorem 1.9 (Borowiecki and Sidorowicz [2]). *We have*

(i) $\text{wsat}(n, C_k) = n - 1$ *when k is odd and $n > k$.*

(ii) $\text{wsat}(n, C_k) = n$ *when k is even and $n \geq k$.*

For any tree T of order k, we have

$$k - 2 \leq \text{wsat}(n, T) \leq \binom{k-1}{2} \tag{1.1}$$

since $K_{k-1} + \overline{K}_{n-k+1}$ is weakly T-saturated. Note that the lower bound in (1.1) is sharp since $P_{k-1} + \overline{K}_{n-k+1}$ is weakly P_k-saturated, and thus $\text{wsat}(n, P_k) = k - 2$. The upper bound in (1.1) is sharp as well, due to the following result.

Theorem 1.10 (Borowiecki and Sidorowicz [2]). *If $n \geq k \geq 3$, then* $\text{wsat}(n, S_k) = \binom{k-1}{2}$.

The precise value of $\text{wsat}(n, H)$ was determined in [11] for many families of sparse graphs, and in particular, for many trees. This includes the result for double stars given below.

Theorem 1.11 (Faudree, Gould, and Jacobson [11]). *If $2 \leq s \leq t$ and $n \geq 2s + 2t$, then*

$$\mathrm{wsat}(n, S_{s,t}) = s + t - 2 + \binom{s-2}{2}.$$

It is easy to see that $S_{s,t-1} + K_{s-2} + \overline{K}_{n-2s-t+3}$ is weakly $S_{s,t}$-saturated when $n \geq 2s + 2t$. However, we should point out here that $\mathrm{wsat}(n, S_{s,t})$ is unknown when $n < 2s + 2t$.

Faudree, Gould, and Jacobson [11] showed that nearly all trees of order k have weak saturation number $k - 2$. On the other hand, Theorem 3.4 shows that for any tree T_k of order $k \geq 5$ and any $n \geq k + 2$, $\mathrm{psat}(n, T_k) \geq n - \lfloor (n + k - 2)/k \rfloor$. Thus, in general, it is far from true that $\mathrm{wsat}(n, H) = \mathrm{psat}(n, H)$.

Chapter 2

Graphs of small order

2.1 4-cycles

We first state the following two remarks without proof.

Remark 2.1. *Every partially C_4-saturated graph is connected and has diameter at most 3.*

Remark 2.2. *We have* $\mathrm{psat}(4, C_4) = \mathrm{sat}(4, C_4) = 4$. *In addition,* $K_4 - P_3$ *is the unique minimum C_4-saturated (and partially C_4-saturated) graph of order* 4.

(a) Type I triangle T (b) Type II triangle T

Figure 2.1: Type I and II triangles

A triangle $T = a_1a_2b$ in a graph G is said to be of *type I* if $d_G(a_1) = d_G(a_2) = 2$ and $d_G(b) > 2$, where b is called the *base vertex* of T. A triangle $T = ab_1b_2$ in a graph G is said to be of *type II* if $d_G(a) = 2$, $d_G(b_1) > 2$, and $d_G(b_2) > 2$, where b_1b_2 is called the *base edge* of T (with *base vertices* b_1 and b_2). Both types of triangles are shown in Figure 2.1. In each case, we say that G is obtained from G_0 by attaching T.

Remark 2.3. *If a graph G is obtained from G_0 by attaching a type I triangle at base vertex b, then G is partially C_4-saturated if and only if G_0 is partially C_4-saturated and* $\epsilon_{G_0}(b) \leq 2$.

It is easy to check that each graph G^i in Figure 2.2 is partially C_4-saturated, with size $\left\lfloor \frac{3|V(G^i)|-5}{2} \right\rfloor$. Our next lemma characterizes all partially C_4-saturated graphs of order at least 5 that are unicyclic.

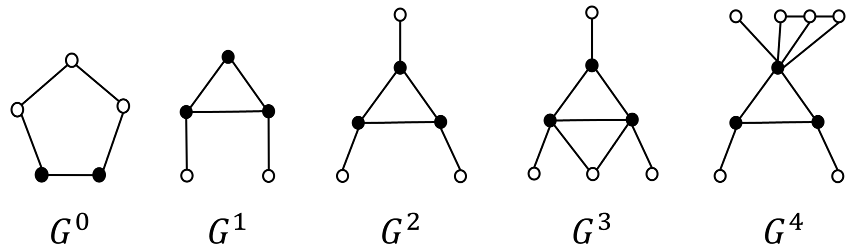

Figure 2.2: Partially C_4-saturated graphs of small order

Lemma 2.4. *Let G be a partially C_4-saturated graph of order $n \geq 5$.*

(i) Let u, v, and w be three vertices in G such that $u \sim v$, $v \sim w$, and $d_G(u) = 1$. Then the edge vw must be contained in a triangle.

(ii) If $|E(G)| \leq n$, then $G = G^0$, G^1, or G^2.

Proof. Part (i) can be shown by considering $G + uw$. This implies that G is not a tree. Now assume $|E(G)| \leq n$. Then $|E(G)| = n$ (i.e., G is unicyclic). If $\delta(G) \geq 2$, then $G = C_n$, which implies that $G = G^0$. Next assume $\delta(G) = 1$. Then by part (i), we have (1) the unique cycle in G must be a triangle, (2) every vertex in G must be adjacent to some vertex in the triangle, and (3) every vertex in the triangle must have degree either 2 or 3. Thus, $G = G^1$ or G^2. □

For each i, where $0 \leq i \leq 4$, we define $\mathcal{G}^i$ to be the collection of all graphs obtained from G^i in Figure 2.2 by attaching some number of type I triangles based at the solid-marked vertices only (any two adjacent vertices when $i = 0$, or the three vertices in the central triangle when $i \neq 0$). Our next remark lists some simple properties of graphs in $\mathcal{G}^i$.

Remark 2.5. *Let G be any graph of order n in $\mathcal{G}^i$, where $0 \leq i \leq 4$. Then*

(i) G is partially C_4-saturated with $|E(G)| = \left\lfloor \frac{3n-5}{2} \right\rfloor$.

(ii) For any vertex v in G, $\epsilon_G(v) = 2$ if v is a solid-marked vertex, and $\epsilon_G(v) = 3$ otherwise.

(iii) G has even order if and only if $i = 2$.

(iv) For every $n \geq 5$, there exists a graph of order n in $\mathcal{G}^1 \cup \mathcal{G}^2$.

(v) G is C_4-saturated if and only if $0 \leq i \leq 2$.

Theorem 2.6. *For all $n \geq 5$, $\operatorname{psat}(n, C_4) = \left\lfloor \dfrac{3n-5}{2} \right\rfloor$. In addition, a graph G is minimum partially C_4-saturated if and only if $G \in \mathcal{G}^i$ for some i, $0 \leq i \leq 4$.*

Proof. Let $n \geq 5$. Then, by Remark 2.5, there exists a partially C_4-saturated graph G of order n in $\mathcal{G}^1 \cup \mathcal{G}^2$ with $|E(G)| = \left\lfloor \frac{3n-5}{2} \right\rfloor$, which implies that $\operatorname{psat}(n, C_4) \leq \left\lfloor \frac{3n-5}{2} \right\rfloor$. Our proof is by contradiction. Suppose there exists a partially C_4-saturated graph G of order n such that $|E(G)| \leq \left\lfloor \frac{3n-5}{2} \right\rfloor$ and $G \notin \mathcal{G}^i$ for any i, $0 \leq i \leq 4$. For convenience, let G be such a graph with the minimum number of vertices. Since $\left\lfloor \frac{3n-5}{2} \right\rfloor = n$ when $n = 5$ or $n = 6$, it follows that $n \geq 7$, by Lemma 2.4(ii). Also, since $|E(G)| < \frac{3n}{2}$, we must have that $\delta(G) \leq 2$.

First we claim that G contains no type I triangle. Suppose to the contrary, that G is obtained from a graph G_0 by attaching a type I triangle at base vertex b. It then follows by Remark 2.3 that G_0 must be partially C_4-saturated with size $|E(G_0)| \leq \left\lfloor \frac{3|V(G_0)|-5}{2} \right\rfloor$. Thus, $G_0 \in \mathcal{G}^i$ for some i by the minimality of $|V(G)|$. Note that $\epsilon_{G_0}(b) \leq 2$, since $diam(G) \leq 3$. Then b must be a solid-marked vertex in $\mathcal{G}^i$, by Remark 2.5(ii). Therefore, by the definition of $\mathcal{G}^i$, we have $G \in \mathcal{G}^i$ as well, which contradicts our choice of G. Thus, we have shown that G cannot contain a type I triangle. The rest of the proof is divided into three cases.

Case 1: $\delta(G) = 1$.

Let L_0 be the set of all degree 1 vertices in G. For $1 \leq i \leq 3$, define $L_i = \{v \in V(G) \mid d(v, L_0) = i\}$. Then $V(G) = L_0 \cup L_1 \cup L_2 \cup L_3$, since $d(G) \leq 3$. Let $l_i = |L_i|$ for $0 \leq i \leq 3$. For $i = 2, 3$, let L_{i2} be the set of vertices in L_i with at least two neighbors in L_{i-1}, where $l_{i2} = |L_{i2}|$. We now list some observations about the structure of G.

(1) For $i = 2, 3$, $|E(L_{i-1}, L_i)| \geq |L_i \setminus L_{i2}| + 2|L_{i2}| = l_i + l_{i2}$.

(2) Since L_0 forms an independent set and $G+uv$ contains a new copy of C_4 for any two vertices u and v in L_0, it follows that $E(L_0, L_1)$ forms a matching of size $l_0 = l_1$ and that L_1 forms a clique.

(3) If $v_2 \in L_2 \setminus L_{22}$ and v_1 is the sole neighbor of v_2 in L_1, then there must exist a vertex in L_2 that is adjacent to both v_2 and v_1.

To prove (3), we let v_0 be the sole neighbor of v_1 in L_0 and H be a new copy of C_4 in $G+v_0v_2$. Then H must contain the edges v_0v_1 and v_0v_2 since $d_G(v_0)=1$. Hence, the fourth vertex of H, say v_2', must be a common neighbor of both v_1 and v_2, so we must have $v_2' \in L_2$.

It follows from (3) that

$$|E(\langle L_2\rangle)| = \frac{\sigma_{\langle L_2\rangle}(L_2 \setminus L_{22}) + \sigma_{\langle L_2\rangle}(L_{22})}{2} \geq \frac{\sigma_{\langle L_2\rangle}(L_2 \setminus L_{22})}{2} \geq \frac{l_2 - l_{22}}{2}.$$

We also note that $|E(\langle L_3\rangle)| \geq \frac{l_3 - l_{32}}{2}$ since every vertex in $L_3 \setminus L_{32}$ has degree at least 2 and thus has at least one neighbor in L_3. For $i=2$ and 3, we let E_{i1} denote the set of vertices in $L_i \setminus L_{i2}$ with at least two neighbors in L_i, and let E_{i2} denote the number of vertices in L_{i2} with at least one neighbor in L_i. We then write $\epsilon_{i1} = |E_{i1}|$ and $\epsilon_{i2} = |E_{i2}|$. Then we have the following improved estimate.

(4) For $i = 2, 3$, $|E(\langle L_i\rangle)| \geq \dfrac{l_i - l_{i2} + \epsilon_{i1} + \epsilon_{i2}}{2}$.

Therefore, by (1), (2), and (4), we have

$$\begin{aligned}
\frac{3n-5}{2} \geq |E(G)| &= |E(L_0, L_1)| + |E(\langle L_1\rangle)| + |E(L_1, L_2)| + |E(\langle L_2\rangle)| + |E(L_2, L_3)| + |E(\langle L_3\rangle)| \\
&\geq l_0 + \binom{l_1}{2} + (l_2 + l_{22}) + \frac{l_2 - l_{22} + \epsilon_{21} + \epsilon_{22}}{2} + (l_3 + l_{32}) + \frac{l_3 - l_{32} + \epsilon_{31} + \epsilon_{32}}{2} \\
&= \frac{3}{2}(l_0 + l_1 + l_2 + l_3) + \binom{l_0}{2} - 2l_0 + \frac{l_{22} + l_{32} + \epsilon_{21} + \epsilon_{22} + \epsilon_{31} + \epsilon_{32}}{2} \\
&= \frac{3n}{2} + \frac{{l_0}^2 - 5l_0}{2} + \frac{l_{22} + l_{32} + \epsilon_{21} + \epsilon_{22} + \epsilon_{31} + \epsilon_{32}}{2} \\
&= \frac{3n-6}{2} + \frac{(l_0-2)(l_0-3)}{2} + \frac{l_{22} + l_{32} + \epsilon_{21} + \epsilon_{22} + \epsilon_{31} + \epsilon_{32}}{2} \\
&\geq \frac{3n-6}{2}.
\end{aligned}$$

Note that the left and right quantities in the above inequality differ by $\frac{1}{2}$ only. Thus, our next four observations (5) through (8) follow immediately.

(5) $l_0 = 2$ or 3.

(6) By (1), for $i = 2, 3$, we have $|E(L_{i-1}, L_i)| = |L_i \setminus L_{i2}| + 2|L_{i2}| = l_i + l_{i2}$. In addition, the unique vertex in L_{i2}, if it exists, has exactly two neighbors in L_{i-1}.

(7) By (4), $|E(\langle L_i \rangle)| = \dfrac{l_i - l_{i2}}{2}$ or $\dfrac{l_i - l_{i2} + 1}{2}$ for $i = 2, 3$.

(8) $l_{22} + l_{32} + \epsilon_{21} + \epsilon_{22} + \epsilon_{31} + \epsilon_{32} \leq 1$.

(9) $\epsilon_{22} = \epsilon_{32} = 0$, since $\epsilon_{i2} \geq 1$ implies that $l_{i2} \geq 1$. So $|I| \leq 1$, where $I = L_{22} \cup L_{32} \cup E_{21} \cup E_{31}$.

(10) Assume $I = \{z^*\}$ when $I \neq \emptyset$. If $z^* \in L_{i2}$ for $i = 2$ or 3, then z^* has exactly two neighbors in L_{i-1} by (6) and no neighbor in L_i by (9). If $z^* \in E_{i1}$ for $i = 2$ or 3, then z^* has exactly one neighbor in L_{i-1} by definition and exactly two neighbors in L_i by (7). If z is any vertex in $L_2 \cup L_3$ other than z^*, then z has exactly one neighbor in L_{i-1} and exactly one neighbor in L_i by definition.

(11) Let $F = \langle L_2 \rangle + \langle L_3 \rangle$. Then it follows by (10) that every component in F is a copy of K_2, with exactly one exception when $l_{22} + l_{32} + \epsilon_{21} + \epsilon_{31} = 1$. If $l_{i2} = 1$ for $i = 2$ or 3, then the exceptional component in F is a copy of K_1 in $\langle L_{i2} \rangle$. If $\epsilon_{i1} = 1$ for $i = 2$ or 3 then the exceptional component in F is a copy of K_1 in $\langle L_i \setminus L_{i2} \rangle = \langle L_i \rangle$.

(12) If T is a nontrivial component in $\langle L_2 \rangle$ (so that $V(T) \subseteq L_2 \setminus L_{22}$ and $T \cong K_2$ or P_3), then all vertices in T have the same neighbor in L_1 by (3).

We now consider two subcases.

Case 1.1: $L_3 = \emptyset$.

Since G contains no type I triangle, then no component in $\langle L_2 \rangle$ is isomorphic to K_2. Thus, $l_2 = 0$, 1, or 3. Recall that $l_0 = 2$ or 3, by (5). If $l_2 = 0$, then $G = G^2$ since $G \neq P_4$. If $l_2 = 1$, then $L_2 = L_{22} = \{z^*\}$ and z^* has exactly two neighbors in L_1, by (10). So $G = G^1$ if $l_0 = 2$, and $G = G^3$ if $l_0 = 3$. If $l_2 = 3$, then $\langle L_2 \rangle \cong P_3$ by (11). In addition, all three vertices in P_3 have the same neighbor in L_1 by (12). Then it can be easily checked that $l_0 \neq 2$, since

G is partially C_4-saturated. Thus, $l_0 = 3$ and $G = G^4$. However, none of these cases are possible since $G \notin \mathcal{G}^i$ for any i, $0 \leq i \leq 4$. Thus, $L_3 \neq \emptyset$.

Case 1.2: $L_3 \neq \emptyset$.

Let x be an arbitrary vertex in L_3. It follows by Remark 2.1 that for every vertex $v_0 \in L_0$, there exists a v_0x-path of length 3. So there exist at least l_0 different paths of length 3 from x to L_0, where $l_0 = 2$ or 3, by (5).

By (9), we have that $|L_{22} \cup L_{32}| \leq 1$. If $z^* \in L_{22} \cup L_{32}$, then it follows by (6) that z^* has *exactly two neighbors* in L_{i-1}. If $z \in L_i \setminus (L_{22} \cup L_{32})$, where $1 \leq i \leq 3$, then z has *exactly one neighbor* in L_{i-1}, by (2) and the definition of $L_i \setminus L_{i2}$. Upon inspection, we see that there can be at most two different paths of length 3 from x to L_0. Thus, from the previous paragraph, $l_0 = 2$ and there are exactly two different paths of length 3 from x to L_0.

Now assume $L_0 = \{u_0, v_0\}$ and $L_1 = \{u_1, v_1\}$ so that $u_0 \not\sim v_1$. Then $G + u_0v_1$ must contain a new copy of C_4, say H. Clearly, $\{u_0u_1, u_0v_1\} \subseteq E(H)$, which indicates that the fourth vertex of H, say z^*, must be adjacent to both u_1 and v_1. Thus, $L_{22} = \{z^*\}$. Then by (11), every component in $\langle L_3 \rangle$ must be a copy of K_2.

If $x \in L_3$, then x has exactly one neighbor in L_2 since $L_{32} = \emptyset$. Recall that there are exactly two different paths of length 3 from x to L_0. Thus, $x \sim z^*$. Since x was chosen arbitrarily, we have $L_3 \subseteq N(z^*)$. Now let u_3 and v_3 be two adjacent vertices in L_3. Then $u_3v_3z^*$ forms a type I triangle in G, which is a contradiction. This proves Case 1.

Case 2: $\delta(G) = 2$, and there exists a vertex v_0 of degree 2 in G whose neighbors are nonadjacent.

Define $L_0 = \{v_0\}$, $L_1 = N(v_0) = \{x_1, y_1\}$, and L_2, L_3, L_{22}, L_{32} as in Case 1. By applying an argument similar to that in Case 1, we can see that every vertex in $L_2 \setminus L_{22}$ has at least one other neighbor in L_2. The same holds for vertices in $L_3 \setminus L_{32}$, by a different argument, also to be found in Case 1. We now have

$$\begin{aligned}\frac{3n-5}{2} \geq |E(G)| &\geq 2+0+(l_2+l_{22})+\frac{l_2-l_{22}}{2}+(l_3+l_{32})+\frac{l_3-l_{32}}{2}\\ &\geq 2+\frac{3(l_2+l_3)}{2}+\frac{l_{22}+l_{32}}{2}\\ &=\frac{3n-5}{2}+\frac{l_{22}+l_{32}}{2}.\end{aligned}$$

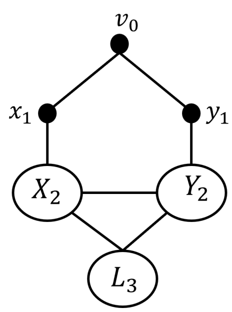

Figure 2.3: $\delta(G) = 2$

This implies that $\langle L_2\rangle$ and $\langle L_3\rangle$ are both 1-regular, and $l_{22} = l_{32} = 0$. In particular, every vertex in L_2 is adjacent to exactly one of x_1 and y_1. Now let X_2 and Y_2 be the sets of neighbors of x_1 and y_1 in L_2, respectively. Then X_2 and Y_2 are disjoint, nonempty sets. See Figure 2.3. We claim that $L_3 = \emptyset$. Suppose, to the contrary, that there exist two adjacent vertices u and v in L_3. Let u' and v' be the sole neighbors of u and v in L_2, respectively. Then we must have $u' \neq v'$, since G contains no type I triangles. For convenience, we assume $u' \in X_2$. We can then easily see that $v' \in X_2$ as well by considering $G + u'v$. Let $z \in Y_2$. Then z is adjacent to at most one of u' and v'. So we assume $z \nsim v'$. But then $G + uz$ would not contain a new copy of C_4. This proves our claim that $L_3 = \emptyset$. So X_2 and Y_2 are both independent sets, since G contains no type 1 triangles. Thus, $|X_2| = |Y_2| = 1$, by considering $G + uv$, where $\{u, v\} \subseteq X_2$ or $\{u, v\} \subseteq Y_2$. Therefore, $G = C_5$.

Case 3: $\delta(G) = 2$, and every vertex of degree 2 in G is contained in a type II triangle.

Recall that if $T = ab_1b_2$ is a type II triangle in G with $d_G(a) = 2$, then b_1b_2 is the base edge of T, and b_1, b_2 are the base vertices of T. Let G'_0 be the subgraph of G induced by all the base edges in G, and $F'_1, ..., F'_r$ be the components of G'_0. For each i, $1 \leq i \leq r$, let t_i be the number of type II triangles in G whose base edge is in F_i', and F_i be the subgraph of G obtained from F_i' by attaching the t_i type II triangles. Then we have $t_i \geq e(F_i') \geq n(F_i') - 1$ and $n(F_i) = t_i + n(F_i')$. Thus,

$$e(F_i) = 2t_i + e(F_i') \geq \frac{3}{2}\big(t_i + e(F_i')\big) \geq \frac{3}{2}\big(t_i + n(F_i') - 1\big) = \frac{3}{2}\big(n(F_i) - 1\big).$$

For $1 \leq i \neq j \leq r$, there exists at least one edge joining F_i and F_j, by considering $G+uv$, where u is a degree 2 vertex in F_i and v is a degree 2 vertex in F_j. Hence, $\sigma_G(V(F_i)) \geq 2e(F_i) + r - 1 \geq 3n(F_i) + r - 4$. Now let $G_0 = \bigcup_{i=1}^r F_i$ and $F_0 = G - V(G_0)$. Then $\sigma_G(V(F_0)) \geq 3n(F_0)$ because $d_G(v) \geq 3$ for all $v \in V(F_0)$. So

$$\sigma(G) = \sum_{i=0}^{r} \sigma_G(V(F_i)) \geq 3\sum_{i=0}^{r} n(F_i) + r(r-4) \geq 3n - 4,$$

which is impossible because $|E(G)| \leq \frac{3n-5}{2}$. □

The corollary below follows directly from Remark 2.5 and Theorem 2.6.

Corollary 2.7 (Ollmann [15])**.** *For all $n \geq 5$, $\mathrm{sat}(n, C_4) = \left\lfloor \frac{3n-5}{2} \right\rfloor$. In addition, a graph G is minimum C_4-saturated if and only if $G \in \mathcal{G}^i$ for some i, $0 \leq i \leq 2$.*

In 1995, Fisher, Fraughnaugh, and Langley [12] gave an upper bound of $\left\lceil \frac{10}{7}(n-1) \right\rceil$ for the graph C_5. Later, in [5] and [6], Chen proved that this upper bound serves as the lower bound as well for all $n \geq 21$ and also characterized all minimum C_5-saturated graphs of order n.

2.2 All other graphs of order 4 or less

Remark 2.8. *Let H be a graph where every edge is contained in a triangle. Then $diam(G) \leq 2$ for every partially H-saturated graph G.*

For $n \geq 4$, the *friendship graph* F_n is defined as follows:

$$F_n = \begin{cases} K_1 \vee \frac{n-1}{2} K_2 & \text{if } n \text{ is odd} \\ K_1 \vee (\frac{n-2}{2} K_2 + K_1) & \text{if } n \text{ is even} \end{cases}.$$

Chen, Faudree, and Gould [4] studied the saturation number of generalized books. In particular, they showed that $\text{sat}(n, K_4 - K_2) = \lceil \frac{3n-4}{2} \rceil$ for $n \geq 10$. Our next result is obtained by using a similar proof technique to the one used in [4].

Theorem 2.9. *For $n \geq 4$, $\text{psat}(n, K_4 - K_2) = \text{sat}(n, K_4 - K_2) = \lceil \frac{3n-4}{2} \rceil$.*

Proof. Let $H = K_4 - K_2$. It is easily seen that F_n is H-saturated and $e(F_n) = \lceil \frac{3n-4}{2} \rceil$. Now let G be a partially H-saturated graph of order n. It then suffices to show that $\sigma(G) \geq 3n-4$.

Since every edge in H is contained in a triangle and H is 2-connected, it follows that $diam(G) \leq 2$, G is connected, and G contains at most one vertex of degree 1.

Let A be the set of vertices of degree at most 2 in G, and let $a = |A|$. Then we have that $\sigma(A) \geq 2a-1$ since all but at most one vertex in G has degree 2. If there exists a vertex in G adjacent to every vertex in A, then $\sigma(G) \geq a+(2a-1)+3(n-a-1) = 3n-4$. Thus, we will assume that no vertex in G is adjacent to every vertex in A. Since $diam(G) \leq 2$, G cannot contain a degree 1 vertex, since the unique neighbor of such a vertex would have degree $n-1$. Thus, $\delta(G) \geq 2$. So if $a \leq 4$, then we have $\sigma(G) \geq 2a + 3(n-a) = 3n - a \geq 3n - 4$.

Thus we shall assume that $a \geq 5$. If $n \leq 5$, then $n = a = 5$ and $G = C_5$. However, C_5 is not partially H-saturated, so we must have that $n \geq 6$. We now divide the rest of the proof into two cases.

Case 1. There exist two adjacent vertices in A, say u and v.

Assume $u \sim u'$ and $v \sim v'$. Since $diam(G) \leq 2$, every vertex in $V(G) \setminus \{u, v, u', v'\}$ must be adjacent to both u' and v'. Recall our earlier assumption that no vertex in G is adjacent to every vertex in A. So $u' \neq v'$. We then have $\sigma(G) = \sigma\big(\{u', v'\}\big) + \sigma\big(V(G) \setminus \{u', v'\}\big) \geq 2(n-3) + 2(n-2) = 4n - 10 \geq 3n - 4$, since $n \geq 6$.

Case 2. A forms an independent set in G.

Since $diam(G) \leq 2$, every pair of vertices in A must have a common neighbor. Let $v_1 \in A$ be such that $N(v_1) = \{x_2, x_3\}$. Since no vertex in G is adjacent to every vertex in A, there exist vertices $v_2, v_3 \in A$ such that $v_2 \not\sim x_2$ and $v_3 \not\sim x_3$. Then the common neighbor of v_1 and v_2 must be x_3 and the common neighbor of v_1 and v_3 must be x_2. Let x_1 be a common neighbor of v_2 and v_3. Clearly, $x_1 \notin \{x_2, x_3\}$. This situation is depicted in Figure 2.4.

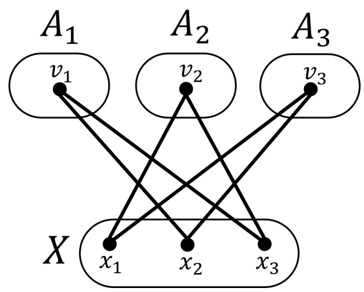

Figure 2.4: Theorem 2.9 Case 2

Now let $X = \{x_1, x_2, x_3\}$. It is easily seen that every vertex in A must be adjacent to exactly two vertices in X. For $1 \leq i \leq 3$, define $A_i = \{v \in A \mid N(v) = X \setminus \{x_i\}\}$, and let $a_i = |A_i|$. Then we have $a = a_1 + a_2 + a_3$, and $\sigma(G) = \sigma(A) + \sigma(X) + \sigma\big(V(G) \setminus (A \cup X)\big) \geq 2a + 2(a_1 + a_2 + a_3) + 3(n - a - 3) = 3n + a - 9 \geq 3n - 4$, since $a \geq 5$. □

Remark 2.10. *Let G be a graph with c_0 tree components, each of which has at least n_0 vertices. Then $e(G) \geq n(G) - c_0 \geq n(G) - \frac{n(G)}{n_0}$.*

Remark 2.11. *Assume $H = H' + K_1$ and G has order $n \geq |V(H)|$. Then*

(i) G is H'-saturated if and only if G is H-saturated. Thus, $\mathrm{sat}(n, H) = \mathrm{sat}(n, H')$.

(ii) G is partially H'-saturated if and only if G is partially H-saturated. Thus, $\mathrm{psat}(n, H) = \mathrm{psat}(n, H')$.

(iii) If $\mathrm{psat}(n, H') = \mathrm{sat}(n, H')$, *then* $\mathrm{psat}(n, H) = \mathrm{sat}(n, H)$.

Table 2.1 gives the exact values of $\text{sat}(n, H)$ with the corresponding references for every graph H of order 4 or less with no isolated vertices.

H	$\text{sat}(n, H)$	Minimum graph G	Reference
K_2	0	$\overline{K}_n$	KT[13]
S_3	$\lfloor \frac{n}{2} \rfloor$	$\frac{n}{2}K_2$ or $\frac{n-1}{2}K_2 + K_1$	KT[13]
K_3	$n-1$	S_n	EHM[8]
$2K_2$	3	$K_3 + \overline{K}_{n-3}$	KT[13]
S_4	$n-1$	$K_3 + K_1$ or $C_{n-2} + K_2$	KT[13]
P_4	$\frac{n}{2}$ or $\frac{n+3}{2}$	$\frac{n}{2}K_2$ or $\frac{n-3}{2}K_2 + K_3$	KT[13]
$K_4 - P_3$	$n-1$	S_n	FG[10]
C_4	$\lfloor \frac{3n-5}{2} \rfloor$ for $n \geq 5$	$\mathcal{G}^1 \cup \mathcal{G}^2$	Ollmann[15]
$K_4 - K_2$	$\lceil \frac{3n-4}{2} \rceil$	F_n	Thm 2.9
K_4	$2n-3$	$K_2 \vee \overline{K}_{n-2}$	EHM[8]

Table 2.1: Saturation numbers for graphs of order 4 or less

Theorem 2.12. *Let H be any nontrivial graph of order 4 or less. Then* $\text{psat}(n, H) = \text{sat}(n, H)$ *for every $n \geq |V(H)|$, with the exception that* $\text{psat}(5, P_4) = 3$ *and* $\text{sat}(5, P_4) = 4$.

Proof. First, we address the case where $H = P_4$ and $n = 5$. It is known that $\text{sat}(5, P_4) = 4$. We also have that $\text{psat}(5, P_4) = 3$, since $K_1 + P_4$ is partially P_4-saturated, and no graph of order 5 and size at most 2 can be partially P_4-saturated.

Now let G be a partially H-saturated graph of order $n \geq |V(H)|$, where $n \neq 5$ when $H = P_4$. It then suffices to show that $e(G) \geq \text{sat}(n, H)$. In addition, by Remark 2.11, we may assume that H contains no isolated vertices and is thus one of the ten graphs listed in Table 2.1.

Our result holds when $H = K_2$, K_3, or K_4 by Theorem 1.1', when $H = S_3$ or S_4 by Theorem 1.2', when $H = C_4$ by Theorem 2.6, and when $H = K_4 - K_2$ by Theorem 2.9. If

$H = 2K_2$, then $e(G) \geq 3 = sat(n, H)$, since no graph of order n and size at most 2 can be partially $2K_2$-saturated.

Now assume $H = K_4 - P_3$. Then $\text{sat}(n, H) = n - 1$ and $\text{wt}(H) = 3$. Suppose $e(G) < n - 1$. Then by Remark 2.10, G contains at least two different tree components, say T_1 and T_2. For $i \in \{1, 2\}$, we select a vertex $v_i \in V(T_i)$ such that $d_{T_i}(v_i) = 1$. Then $G + v_1v_2$ contains a new copy of H, say H^*. However, $2 \geq wt_{H^*}(v_1v_2) \geq \text{wt}(H^*) = \text{wt}(H) = 3$, which is impossible. Thus, $e(G) \geq n - 1 = \text{sat}(n, K_4 - P_3)$.

Next assume $H = P_4$. Since G is partially P_4-saturated, it follows that G does not contain P_3 as a component. Also, if K_1 is a component in G, then every other component in G has order at least 3. Furthermore, we have $\sigma(G) \geq n - 1$ since G contains at most one isolated vertex. So $e(G) \geq \lceil \frac{n-1}{2} \rceil = \lfloor \frac{n}{2} \rfloor$, and we are done if n is even. Since the $n = 5$ case was already covered at the beginning of this proof, we now assume n is odd and $n \geq 7$. We want to show that $e(G) \geq \frac{n+3}{2} = \text{sat}(n, P_4)$.

Let G_0 be an odd component in G with a minimum number of vertices, and let $G_1 = G - V(G_0)$. Then $\delta(G_1) \geq 1$ since G contains at most one isolated vertex. Hence,

$$e(G) = e(G_0) + e(G_1) \geq e(G_0) + \frac{n - n(G_0)}{2}.$$

If $n(G_0) \geq 5$, then $e(G) \geq (n(G_0) - 1) + \frac{n-n(G_0)}{2} = \frac{n+n(G_0)-2}{2} \geq \frac{n+3}{2}$. If $n(G_0) = 3$, then $G_0 = K_3$ since G does not contain P_3 as a component. So $e(G) \geq 3 + \frac{n-3}{2} = \frac{n+3}{2}$. Now assume $n(G_0) = 1$ so that G contains K_1 as a component. Then every other component in G has order at least 3. If a tree component is of order 3, then some missing edge can be added to the tree without creating a copy of P_4. So every tree component in G_1 has order at least 4. Remark 2.10 then implies that $e(G) = e(G_1) \geq n(G_1) - \frac{n(G_1)}{4} = \frac{3}{4}(n - 1)$. Thus, $e(G) \geq \lceil \frac{3(n-1)}{4} \rceil \geq \frac{n+3}{2}$ since $n \geq 7$ is odd. □

Chapter 3

Double stars

Recall that the double star $S_{s,t}$ is a graph on $s+t$ vertices, where $s \leq t$, and is constructed by adding an edge between the central vertices of a star on s vertices and a star on t vertices. We will refer to this added edge as the *central edge* of the double star. Note that every edge of $S_{s,t}$ is incident to at least one central vertex. See Figure 3.1. We say that $S_{s,t}$ is *balanced* if $s = t$ and *unbalanced* if $s < t$.

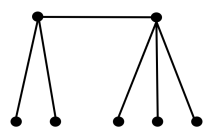

Figure 3.1: $S_{3,4}$

Remark 3.1. *Let u and v be two adjacent vertices in a graph G such that $d(u) \leq d(v)$. Then G contains a copy of $S_{s,t}$ with central edge uv if and only if $d(u) \geq s$, $d(v) \geq t$, and $c(u,v) \leq d(u) + d(v) - s - t$.*

In 2009, Faudree, Faudree, Gould, and Jacobson [9] proved the following result on double stars.

Theorem 3.2 (Faudree et. al [9]). *Let $H = S_{s,t}$ where $3 \leq s \leq t$.*

(i) If $s < t$ and $n \geq s^3$, then

$$\binom{s-1}{2} n \leq sat(n, S_{s,t}) \leq \binom{s}{2} n - \frac{(s-1)^2 + 8}{8}.$$

(ii) If $s = t$ and $n \geq t^3$, then

$$\frac{(t-1)n}{2} \leq sat(n, S_{t,t}) \leq \frac{(t-1)n}{2} + \frac{(t-1)(t+1)}{2}.$$

3.1 Subdivided stars

In this section, we consider the case $H = S_{2,t}$, which is referred to as a subdivided star in [9]. Since P_4 is partially P_4-saturated, Lemma 1.3 holds for partial saturation only when $k \geq 5$, as given in the result below.

Lemma 3.3. *If there exist trees G and T_k each of order k such that G is partially T_k-saturated, then either $k \leq 4$, or $G = S_k$ and $T_k = S_{2,k-2}$.*

Proof. Let G and T_k be trees of order k such that G is partially T_k-saturated. If G is T_k-saturated, then we are done by Lemma 1.3. So assume G is not T_k-saturated. Then G must be isomorphic to T_k since G is a tree of order k and contains a copy of T_k. We want to show that $k \leq 4$. Suppose for a contradiction that $k \geq 5$.

We first claim that for any (u, x, v)-path of order 3 in G, it must be that either $d_G(u) = d_G(x) - 1$ or that $d_G(v) = d_G(x) - 1$. To show this, we consider $G + uv$. Without loss of generality, we assume that $T_k \cong G'$, where $G' = (G + uv) - ux$. Then $G' \cong T_k \cong G$. So $\big(d_{G'}(u) = d_G(u),\, d_{G'}(x) = d_G(x) - 1,\, d_{G'}(v) = d_G(v) + 1\big)$ is a reordering of $\big(d_G(u),\, d_G(x),\, d_G(v)\big)$. Thus, $\big(d_G(x) - 1,\, d_G(v) + 1\big)$ is a reordering of $\big(d_G(x),\, d_G(v)\big)$. So we must have $d_G(v) = d_G(x) - 1$.

It is easily seen that $G \neq S_k$, so $diam(G) \geq 3$. Let $l = diam(G)$, and $P = (v_0, v_1, ..., v_l)$ be a longest path in G. Let $S = \big(d_G(v_0),\, d_G(v_1),\, ...,\, d_G(v_l)\big)$ be the degree sequence of vertices in P. We claim that there exist positive integers a, b such that $a + b = l + 1$ and $S = (1, 2, 3, ..., a, b, b-1, ..., 1)$. For a proof, let i, $0 \leq i \leq l-1$, be the smallest index such that $d_G(v_i) \neq d_G(v_{i+1}) - 1$. Such an i must exist since $d_G(v_0) = d_G(v_l) = 1$. Now let $a = d_G(v_i)$ and $b = d_G(v_{i+1}) \neq a + 1$. We are done if $i = l - 1$, in which case $S = (1, 2, ..., a, b = 1)$. Now assume $i \leq l - 2$. By applying the previous claim on $(v_i,\, v_{i+1},\, v_{i+2})$, it follows that $d_G(v_{i+2}) = d_G(v_{i+1}) - 1 = b - 1$. Thus our claim follows by applying the same claim sequentially on $(v_j,\, v_{j+1},\, v_{j+2})$ for $j = i + 1,\, i + 2,\, ...,\, l - 2$, ending with $d_G(v_l) = 1$.

Assume $a \geq b$ for convenience. If $l = diam(G) = 3$, then the degree sequence of the path P is $S = (1, 2, 2, 1)$ or $(1, 2, 3, 1)$. Then $G = P_4$ or $S_{2,3}$, respectively. But $k \geq 5$, and $S_{2,3}$ is not $S_{2,3}$-saturated. Thus, we must have that $l \geq 4$. Since $a \geq b$, we have $d_G(v_0) = 1$, $d_G(v_1) = 2$, $d_G(v_2) = 3$, and $d_G(v_3) \geq 2$. Now consider $G + v_0v_3$. Then there exists an edge $e \in \{v_0v_1, v_1v_2, v_2v_3\}$ such that $G_1 = (G + v_0v_3) - e \cong G$. So $\big(d_{G_1}(v_0),\, d_{G_1}(v_1),\, d_{G_1}(v_2),\, d_{G_1}(v_3)\big)$ is a reordering of the sequence $S = \big(1, 2, 3, d_G(v_3)\big)$. It can then be verified that $d_G(v_3) = 2$. So $S = (1, 2, 3, 2, 1)$, and $l = diam(G) = 4$. Then G must have the form shown in Figure 3.2. Let u be the unique neighbor of v_2 outside of the path P. Then $G + v_1u$ contains no new copy of G. This is impossible, and thus concludes our proof.

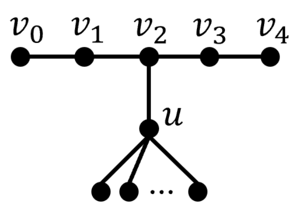

Figure 3.2: $S = (1,\ 2,\ 3,\ 2,\ 1)$

□

The following result is obtained directly from Lemma 3.3, parallel to the way Theorem 1.4 follows from Lemma 1.3. Thus, we omit the proof here.

Theorem 3.4. *For any tree T_k of order $k \geq 5$ and any $n \geq k + 2$,*

$$\text{psat}(n, T_k) \geq n - \left\lfloor \frac{n + k - 2}{k} \right\rfloor.$$

Moreover, $S_{2,k-2}$ is the only tree of order k attaining this minimum for all n.

In the remainder of this chapter, we provide the exact value for $\text{psat}(n, S_{s,t})$ when $3 \leq s < t$ and n is large enough.

3.2 Constructions

3.2.1 Extended $(k,\, l,\, n)$-graphs

Let $X_1, X_2, ..., X_l$ be l disjoint sets such that $\lfloor n/l \rfloor = |X_1| \le |X_2| \le ... \le |X_l| = \lceil n/l \rceil$, where $n \ge l \ge 2$. An almost k-regular graph G of order n with vertex set $V(G) = X_1 \cup X_2 \cup ... \cup X_l$ is said to be a $(k,\, l,\, n)$-*graph with partition sets* $X_1,\, X_2,\, ...,\, X_l$ if the following two conditions are satisfied:

(i) Each X_i is an independent set in G, except when $l = 2$ and n is odd, in which case each vertex in X_2 has at most one neighbor in X_2.

(ii) If l divides n and $a \in \{0, 1, 2, ..., n-1\}$, then there exists a matching M in G such that $G - M$ is partly k-regular with exactly a or $a+1$ minor vertices which are equitably divided among all partition sets in G.

Note that if $l \ge 3$, or $l = 2$ and n is even, then a $(k,\, l,\, n)$-graph with partition sets $X_1,\, X_2,\, ...,\, X_l$ is a spanning subgraph of $T_{n,l}$, the Turán graph with partite sets $X_1,\, X_2,\, ...,\, X_l$. Our next remark provides a sufficient condition on the existence of a $(k,\, l,\, n)$-graph.

Remark 3.5. *There exists a* $(k,\, l,\, n)$*-graph whenever* $l \ge 2$ *and* $n \ge kl$.

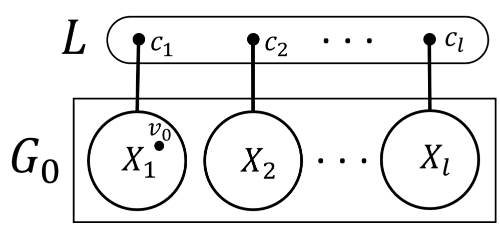

Figure 3.3: Extended $(k,\, l,\, n)$-graph

Let G_0 be a $(k,\, l,\, n)$-graph with partite sets $X_1, X_2, ..., X_l$, and $L = \{c_1,\, c_2,\, ... c_l\}$ be a set of size l disjoint from $V(G_0)$. For each i, $1 \le i \le l$, let G_i be the star with central vertex c_i and vertex set $X_i \cup \{c_i\}$. The graph $G = G_0 \cup \left(\sum_{i=1}^{l} G_i\right)$ is called an *extended*

$(k,\ l,\ n)$-graph with partition sets $L; X_1, X_2, ..., X_l$. We shall refer to G_0 as the *base subgraph* of G. See Figure 3.3, where v_0 is the unique minor vertex, if it exists.

For the remainder of Section 3.2, we let s and t be fixed integers so that $3 \leq s < t$.

3.2.2 $S_{s,t}$-saturated graphs

Let $n = (t+2)q_1 + r_1$, where $0 \leq r_1 \leq t+1$. Then

$$n - q_1 = n - \left\lfloor \frac{n}{t+2} \right\rfloor = \left\lceil \frac{(t+1)n}{t+2} \right\rceil = \frac{(t+1)n + r_1}{t+2}. \tag{3.1}$$

We now present the following upper bound on $sat(n, S_{s,t})$.

Theorem 3.6. *Assume $3 \leq s < t$ and $n \equiv r_1 \pmod{t+2}$ with $0 \leq r_1 \leq t+1$. If $n \geq (t+2)\lceil s/2 \rceil$, then there exists an $S_{s,t}$-saturated graph G of order n with $\delta(G) = s-1$ such that*

$$\sigma(G) \leq s \left\lceil \frac{(t+1)n}{t+2} \right\rceil - \min\{r_1, s\} + 1.$$

Proof. Assume $n = (t+2)q_1 + r_1$, where $0 \leq r_1 \leq t+1$ and $q_1 \geq \lceil s/2 \rceil$. Let $r' = \min\{r_1, s\}$, and $n_0 = n - q_1 - r'$. Then by assumption, we have $n_0 \geq n - q_1 - r_1 = (t+1)q_1 \geq (t+1)\lceil s/2 \rceil$. By Remark 3.5, there exists an extended $(s-2,\ q_1,\ n_0)$-graph G^* with base subgraph G_0 and partition sets $L = \{c_1, c_2, ..., c_{q_1}\}$; $X_1,\ X_2,\ ...,\ X_{q_1}$. Note that for each i, $1 \leq i \leq q_1$, we have $|X_i| \geq n_0/q_1 \geq t+1$. Let $K_{r'}$ be the complete graph with vertex set R' disjoint from G^*. We now construct the desired $S_{s,t}$-saturated graph G of order n from $G^* + K_{r'}$ so that every vertex in $X_1 \cup X_2 \cup ... \cup X_{q_1} \cup R'$ has degree $s-1$ in G.

Let $a = r'(s - r')$. Then $a \geq 0$. Note that if $a > 0$, then $0 < r' = r_1 < s$, and $|X_i| = n_0/q_1 = t+1$ for each i, $1 \leq i \leq q_1$. Also note that $a \leq s^2/4 < (t+2)\lceil s/2 \rceil \leq n_0$. By Remark 3.5, there exists a matching M in G_0 such that $G_0 - M$ is a partly $(s-2)$-regular graph with exactly a or $a+1$ minor vertices which are equitably divided among all partition sets in G_0.

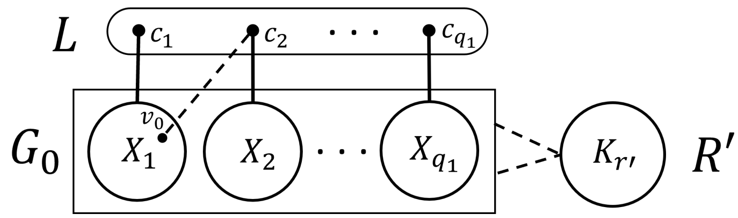

Figure 3.4: $S_{s,t}$-saturated graph of order $n = (t+2)q_1 + r_1$

We now construct the desired graph G of order n from $G^* + K_{r'}$ by adding $r'(s-r')$ new edges joining the minor vertices in $G_0 - M$ and R' so that (i) each of the $r'(s-r')$ minor vertices in $G_0 - M$ is adjacent to exactly one vertex in R', and (ii) each vertex in R' is adjacent to exactly $s - r'$ minor vertices in $G_0 - M$ and at most two minor vertices in each X_i, $1 \le i \le q_1$. In addition, when $G_0 - M$ has $a+1$ minor vertices, we also add one more edge joining the remaining minor vertex in $G_0 - M$, say $v_0 \in X_1$, with some vertex in $L \setminus \{c_1\}$, say c_2, so that every vertex in $V(G_0) \cup R'$ has degree exactly $s-1$ in G. See Figure 3.4.

We leave it to the reader to verify that G is an n-vertex $S_{s,t}$-saturated graph such that

$$\begin{aligned}\sigma(G) &= \sigma(L) + \sigma(V(G_0) \cup R') \\ &\le (n_0 + 1) + (s-1)(n_0 + r') \\ &= s(n - q_1) - r' + 1 \\ &= s\left\lceil \frac{(t+1)n}{t+2} \right\rceil - \min\{r_1, s\} + 1, \text{ by Equation 3.1.}\end{aligned}$$

This completes our proof of Theorem 3.6. □

3.2.3 Partially $S_{s,t}$-saturated graphs

Theorem 3.7. *Assume $n - \lceil s/2 \rceil = (t+1)q_2 + r_2$, where $1 \le r_2 \le t+1$ and $q_2 \ge 2$. Then there exists a partially $S_{s,t}$-saturated graph G of order n with $\delta(G) \le s-2$ such that*

$$\sigma(G) \le (st+1)q_2 + r_2 s + \lceil s/2 \rceil \big(\lceil s/2 \rceil - 1\big) + 1.$$

Proof. Let $n_0 = n - q_2 - \lceil s/2 \rceil = tq_2 + r_2$. Then by Remark 3.5, there exists an $(s-2,\, q_2,\, n_0)$-graph G^* with base subgraph G_0 and partition sets $L = \{c_1, c_2, ..., c_{q_2}\};\ X_1,\ X_2,\ ...,\ X_{q_2}$, where $|X_i| \geq n_0/q_2 \geq t$ for each i, $1 \leq i \leq q_2$. Let $v_0 \in X_1$ be the unique minor vertex of G_0, if it exists. If v_0 does exist, we also let u_0 be a vertex in X_{q_2} such that $v_0 \not\sim u_0$ in G_0.

Next, we define a graph G_L with vertex set L such that

$$E(G_L) = \begin{cases} \{c_1c_2,\ ...,\ c_{q_2-1}c_{q_2}\} & \text{if } q_2 \text{ is even} \\ \{c_1c_2,\ ...,\ c_{q_2-2}c_{q_2-1}\} & \text{if } q_2 \text{ is odd and } v_0 \text{ exists} \\ \{c_1c_2,\ ...,\ c_{q_2-2}c_{q_2-1}\} \cup \{c_{q_2-1}c_{q_2}\} & \text{if } q_2 \text{ is odd and } v_0 \text{ does not exist.} \end{cases}$$

Note that $\lfloor q_2/2 \rfloor \leq e(G_L) \leq \lceil q_2/2 \rceil$. Finally, we define $G = \big[G_L \cup (G^* + v_0u_0)\big] + K_{\lceil s/2 \rceil}$. See Figure 3.5.

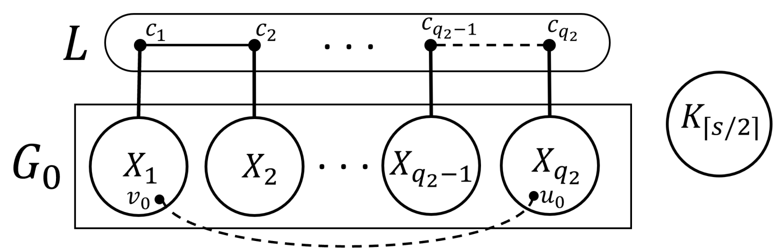

Figure 3.5: Partially $S_{s,t}$-saturated graph of order $n = (t+1)q_2 + r_2 + \lceil s/2 \rceil$

It can then be seen that G is partially $S_{s,t}$-saturated with

$$\begin{aligned} \sigma(G) &\leq n_0 + q_2 + 1 + (s-1)n_0 + \lceil s/2 \rceil\big(\lceil s/2 \rceil - 1\big) \\ &= s(tq_2 + r_2) + q_2 + 1 + \lceil s/2 \rceil\big(\lceil s/2 \rceil - 1\big) \\ &= (st+1)q_2 + r_2 s + \lceil s/2 \rceil\big(\lceil s/2 \rceil - 1\big) + 1. \end{aligned}$$

□

3.3 Some properties of partially $S_{s,t}$-saturated graphs

In this section, we fix G to be a partially $S_{s,t}$-saturated graph of order n. For each $i \geq 0$, we define D_i to be the set of vertices of degree i in G and d_i to be the cardinality of D_i. We also define the following three sets: $D_i^+ = \bigcup_{k \geq i} D_k$, $D_i^- = \bigcup_{k \leq i} D_k$, and $D_i^j = \bigcup_{i \leq k \leq j} D_k$. Thus, $V(G) = D_{t+1}^+ \cup D_t \cup D_s^{t-1} \cup D_{s-1} \cup D_{s-2}^-$. We now proceed to give a detailed partition of D_{s-1}, which will lead to a new partition of $V(G)$. For each vertex v in D_t, we define $N^*(v) = \{x \in D_{s-1} \mid N_{G,D_t^+}(x) = \{v\}\}$. In other words, $N^*(v)$ is the set of all vertices in D_{s-1} whose sole neighbor in D_t^+ is v itself. It can be seen that $N^*(v)$ forms a clique whenever $|N^*(v)| \geq 2$, as G is partially $S_{s,t}$-saturated. We then define $W = \bigcup_{|N^*(v)| \geq 2} N^*(v)$. See Figure 3.6. Note that a vertex w in D_{s-1} belongs to W if and only if there exists a vertex $v \in D_t$ and another vertex w' (distinct from w) such that $\{w, w'\} \subseteq N^*(v)$. In particular, $|W| \neq 1$.

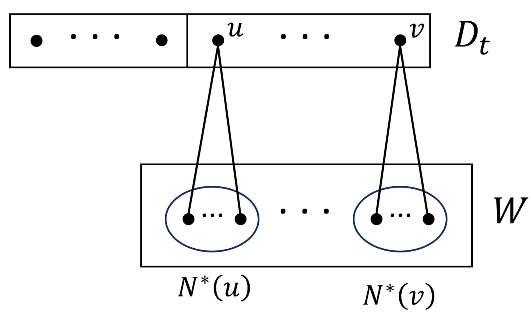

Figure 3.6: Partition of W

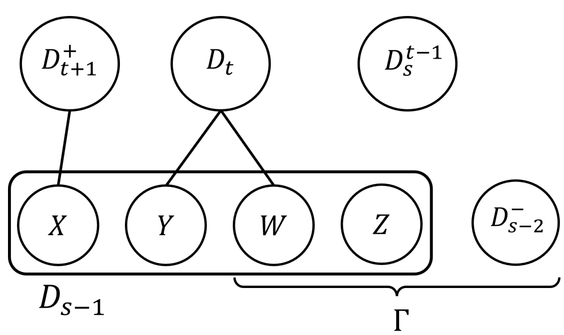

Figure 3.7: D-partition of $V(G)$

Next, we define the sets $X = \{v \in D_{s-1} \mid v \sim D^+_{t+1}\}$ and $Z = \{v \in D_{s-1} | v \not\sim D^+_t\}$. Clearly, $W \subseteq D_{s-1} \setminus (X \cup Z)$ by definition. Now define $Y = D_{s-1} \setminus (X \cup Z \cup W) = \{v \in D_{s-1} \setminus W \mid v \sim D_t, \text{but}\, v \not\sim D^+_{t+1}\}$ and $\Gamma = W \cup Z \cup D^-_{s-2}$. We also write $x = |X|$ and $y = |Y|$. Then $D_{s-1} = X \cup Y \cup W \cup Z$, and $V(G)$ is partitioned into the following four sets: $X \cup D^+_{t+1}$, $Y \cup D_t$, D^{t-1}_s, and Γ. See Figure 3.7.

Lemma 3.8. Γ *forms a clique of size at most* s *in* G.

Proof. Since $\Gamma \subseteq D^-_{s-1}$, it suffices to show that Γ is a clique. Suppose, for a contradiction, that Γ contains two nonadjacent vertices u_1 and u_2. Then $G + u_1u_2$ contains a copy, say H, of $S_{s,t}$. So either u_1 or u_2 must be a central vertex of H. Without loss of generality, assume that the central edge of H is u_1v, for some $v \in D^+_t$. Then clearly, $d_G(u_1) = s - 1$. Since $u_1 \notin D^-_{s-2}$, and no vertex in Z has a neighbor in D^+_t, we must have that $u_1 \in N^*(v) \subseteq W$ and $v \in D_t$. Then $N^*(v)$ is a clique of size at least two, as noted before. Let u'_1 be a vertex in $N^*(v)$ distinct from u_1. Then u'_1 is a common neighbor of both u_1 and v in G. But then $G + u_1u_2$ would not contain a copy of $S_{s,t}$ using u_1v as the central edge, by Remark 3.1. Thus, Γ must be a clique in G with $|\Gamma| \leq s$. □

In the next two lemmas, we provide upper bounds on the sizes of X and Y.

Lemma 3.9. *We have*

(i) $x \leq \sigma(D^+_{t+1})$

(ii) $y \leq (t+1)(d_t/2)$. *In particular, if* $d_t = 1$, *then* $y \leq 1$.

Proof. Part (i) follows from counting the number of edges joining X and D^+_{t+1}. To prove part (ii), we first define $Y_1 = \{v \in Y \mid |N(v) \cap D_t| = 1\}$. Then $|Y_1| \leq d_t$ since $Y_1 \cap W = \emptyset$. In particular, if $d_t = 1$, then $y = |Y| = |Y_1| \leq 1$. By counting the number of edges joining Y and D_t, we obtain $2y - |Y_1| \leq td_t$. This yields $2y \leq |Y_1| + td_t \leq (t+1)d_t$, which proves part (ii). □

Lemma 3.10. *If* $\delta(G) \leq s - 2$, *then* *(i) every vertex $v \in D_t^+$ is adjacent to at most $d(v) - 1$ vertices in D_{s-1}*

(ii) $x \leq \sigma(D_{t+1}^+) - |D_{t+1}^+|$

(iii) $y \leq t(d_t/2)$.

Proof. Let $v \in D_t^+$ and $z \in D_{s-2}^-$. If $v \sim z$, then we are done. So assume $v \not\sim z$. Then $G + vz$ contains a copy H of $S_{s,t}$. Since $vz \in E(H)$ and $d_H(z) \leq d_{G+vz}(z) \leq s-1$, v must be a central vertex of H. Let v' be the other central vertex of H. Then $v \sim v'$ in G and $d_G(v') \geq d_H(v') \geq s$. So $v \sim D_s^+$, and thus, v is adjacent to at most $d(v) - 1$ vertices in D_{s-1}. This proves (i). Parts (ii) and (iii) follow easily by applying the same argument used in the proof of Lemma 3.9. □

Observe that $d(v)$ in G can be treated as a function from $V(G)$ to $\mathbb{Z}$, called the degree function of G. In order to give a better estimate of $\sigma(G)$, we define a new function f, called the score function of G, based on the degree function of G. First let X^* be a subset of X of size $|X^*| = \min\{|X| \, , \sigma(D_{t+1}^+) - (t+1)|D_{t+1}^+|\}$, D_t^* be a subset of D_t of size $\lfloor d_t/2 \rfloor$, and v^* be a fixed vertex in D_t^* when $d_t \geq 3$ and d_t is odd. For every vertex v in G, we define the *score of* v, denoted as $f(v)$, according to the following four cases.

(i) If $v \in D_{t+1}^+ \cup X$, then $$f(v) = \begin{cases} t+1 & \text{if } v \in D_{t+1}^+ \\ s & \text{if } v \in X^* \\ s-1 & \text{if } v \in X \setminus X^* \end{cases}$$

(ii) If $v \in D_t \cup Y$ and $d_t = 1$, then $$f(v) = \begin{cases} t & \text{if } v \in D_t \text{ and } y = 0 \\ t-1 & \text{if } v \in D_t \text{ and } y = 1 \\ s & \text{if } v \in Y \end{cases}$$

(iii) If $v \in D_t \cup Y$ and $d_t \geq 2$, then $$f(v) = \begin{cases} t+1 & \text{if } v \in D_t \setminus D_t^* \\ t-1 & \text{if } v \in D_t^* \\ s-1 & \text{if } v \in Y \end{cases}$$

with the exception that $f(v^*) = t-2 \geq s-1$ when d_t is odd.

(iv) If $v \in D_s^{t-1} \cup \Gamma$, then $f(v) = d(v)$.

It is easy to see that $0 \leq f(v) \leq t+1$ for each vertex v in G. For any $A \subseteq V(G)$, the *total score of* A is defined as $\sigma'(A) = \sum_{v \in A} f(v)$, and the *mean score of* A is given by $\mu'(A) = \frac{\sigma'(A)}{|A|}$. For each $i \geq 0$, let F_i be the set of vertices in G with score i, and let $f_i = |F_i|$. We also define the sets $F_{s-1}^* = F_{s-1} \cap (X \cup Y) = F_{s-1} \setminus \Gamma$ and $F_s^t = \{v \in V(G) \mid s \leq f(v) \leq t\}$, with $f_{s-1}^* = |F_{s-1}^*|$ and $f_s^t = |F_s^t|$. Then $V(G) = F_{t+1} \cup F_{s-1}^* \cup F_s^t \cup \Gamma$. See Figure 3.8.

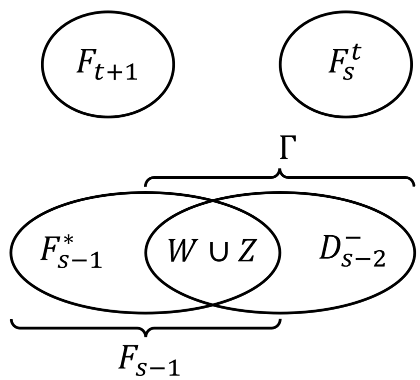

Figure 3.8: F-partition of $V(G)$

Lemma 3.11. *We have*

(i) $\sigma(G) \geq \sigma'(G)$

(ii) $f_{s-1}^* \leq (t+1) f_{t+1}$

(iii) If $\delta(G) \leq s-2$, *then* $f_{s-1}^* \leq t f_{t+1}$.

Proof. Observe that $d(v) = f(v)$ if $v \in \Gamma \cup D_s^{t-1} \cup (X \setminus X^*)$. We can also easily see that $\sigma(D_t \cup Y) = \sigma'(D_t \cup Y)$ (in both the case where $d_t = 1$ and the case where $d_t \geq 2$). In addition, $\sigma(D_{t+1}^+ \cup X^*) - \sigma'(D_{t+1}^+ \cup X^*) = [\sigma(D_{t+1}^+) - \sigma'(D_{t+1}^+)] - [\sigma'(X^*) - \sigma(X^*)] = \sigma(D_{t+1}^+) - (t+1)|D_{t+1}^+| - |X^*| \geq 0$. This proves part (i).

To prove (ii), we first observe that

$$F_{t+1} = \begin{cases} D^+_{t+1} & \text{if } d_t \le 1 \\ D^+_{t+1} \cup (D_t \setminus D^*_t) & \text{if } d_t \ge 2 \end{cases} \quad \text{and} \quad F^*_{s-1} = \begin{cases} X \setminus X^* & \text{if } d_t \le 1 \\ (X \setminus X^*) \cup Y & \text{if } d_t \ge 2 \end{cases}$$

with the exception that $F^*_{s-1} = (X \setminus X^*) \cup Y \cup \{v^*\}$ when $d_t \ge 3$ is odd and $t = s+1$. Note that $|X \setminus X^*| \le (t+1)|D^+_{t+1}|$ by Lemma 3.9(i), and $y \le (t+1)\frac{d_t}{2} \le (t+1)|D_t \setminus D^*_t|$ by Lemma 3.9(ii). In addition, if $d_t \ge 3$ is odd, then $y+1 \le (t+1)\frac{d_t}{2} + 1 \le (t+1)\frac{d_t+1}{2} = (t+1)|D_t \setminus D^*_t|$. This yields part (ii). Part (iii) can be proved similarly by applying Lemma 3.10(ii) and (iii). □

3.4 Lower bounds

We first present the following lemma.

Lemma 3.12. *Let $V^* = V(G) \setminus \Gamma = F_{t+1} \cup F^t_s \cup F^*_{s-1}$.*

(i) Assume $|V^*| = (t+2)q + r$, where $1 \le r \le t+2$. Then

$$\sigma'(V^*) \ge (st+s)q + rs + \min\{0,\, t-s+2-r\}.$$

(ii) Assume $|V^*| = (t+1)q + r$, where $1 \le r \le t+1$. If $\delta(G) \le s-2$, then

$$\sigma'(V^*) \ge (st+1)q + rs + \min\{0,\, t-s+2-r\}.$$

Proof. We prove part (i) only. We partition V^* into $q+1$ subsets $A_0, A_1, ..., A_q$ such that $|A_i| = t+2$ for $0 \le i \le q-1$ and $|A_q| = r$. The additional condition required for each A_i is dependent upon the following two cases.

Case 1. $f_{t+1} \ge q+1$.

In this case, we may assume that for each i, $0 \le i \le q$, A_i contains at least one vertex in F_{t+1}. Thus,

$$\begin{aligned}\sigma'(V^*) &= \sum_{i=0}^{q-1} \sigma'(A_i) + \sigma'(A_q) \\ &\ge \big[t+1+(t+1)(s-1)\big]q + t + 1 + (r-1)(s-1) \\ &= (st+s)q + rs + (t-s+2-r).\end{aligned}$$

Case 2. $f_{t+1} \le q$.

We assume $|A_i \cap F_{t+1}| = 1$ for $0 \le i \le f_{t+1} - 1 \le q - 1$. By Lemma 3.11(ii), we may also assume that for all $i \ge f_{t+1}$, every vertex in A_i has score at least s. Thus,

$$\begin{aligned}\sigma'(V^*) &= \sum_{i=0}^{f_{t+1}-1} \sigma'(A_i) + \sum_{i=f_{t+1}}^{q-1} \sigma'(A_i) + \sigma'(A_q) \\ &\ge (st+s)f_{t+1} + s(t+2)(q-f_{t+1}) + rs \\ &\ge (st+s)q + rs.\end{aligned}$$

This concludes the proof of part (i). The proof of part (ii) is similar by applying Lemma 3.11(iii). □

3.4.1 Minimum degree at least $s-1$

In this section, we assume that $\delta(G) \ge s-1$ and $n = (t+2)q_1 + r_1$, where $0 \le r_1 \le t+1$. Then we have

$$|V^*| = n - |\Gamma| = (t+2)q_1 + r_1 - |\Gamma|.$$

Theorem 3.13. *Let G be a partially $S_{s,t}$-saturated graph of order n with $\delta(G) \ge s-1$. Then*

$$\sigma(G) \ge s\left\lceil \frac{(t+1)n}{t+2} \right\rceil - \min\{r_1, s\},$$

where $n \equiv r_1 \pmod{t+2}$, with $0 \le r_1 \le t+1$.

Proof. By Equation 3.1, it suffices to prove that $\sigma(G) \geq (st+s)q_1 + r_1 s - \min\{r_1, s\}$. Recall that Γ forms a clique of size at most s. We also have $F_{s-1} = F^*_{s-1} \cup \Gamma$ since $\delta(G) \geq s-1$. Note that $\sigma(G) \geq \sigma'(G)$ by Lemma 3.11(i). We shall obtain our desired lower bound on $\sigma(G)$ by applying Lemma 3.12(i). The proof is divided into two cases.

Case 1. $|\Gamma| < r_1$.

In this case, we have

$$\begin{aligned}
\sigma(G) \geq \sigma'(G) = \sigma'(V^*) + \sigma'(\Gamma)& \\
&\geq (st+s)q_1 + (r_1 - |\Gamma|)s + \min\{0,\, t-s+2-(r_1-|\Gamma|)\} + |\Gamma|(s-1) \\
&= (st+s)q_1 + r_1 s + \min\{-|\Gamma|,\, t-s+2-r_1\} \\
&\geq (st+s)q_1 + r_1 s - \min\{r_1,\, s\} \\
&= s\left\lceil \frac{(t+1)n}{t+2} \right\rceil - \min\{r_1, s\}.
\end{aligned}$$

Case 2. $r_1 \leq |\Gamma| \leq s$.

In this case, we have

$$n - |\Gamma| = (t+2)(q_1 - 1) + t + 2 + r_1 - |\Gamma|.$$

Thus,

$$\begin{aligned}
\sigma(G) &\geq (st+s)(q_1-1) + (t+2+r_1-|\Gamma|)s + \\
&\qquad + \min\{0,\, t-s+2-(t+2+r_1-|\Gamma|)\} + |\Gamma|(s-1) \\
&= (st+s)q_1 + r_1 s + \min\{s-|\Gamma|,\, -r_1\} \\
&= (st+s)q_1 + r_1 s - r_1 \\
&= s\left\lceil \frac{(t+1)n}{t+2} \right\rceil - \min\{r_1, s\}.
\end{aligned}$$

□

3.4.2 Minimum degree at most $s-2$

In this section, we will assume that $\delta(G) \le s-2$ and $n = (t+1)q_2 + r_2 + \lceil s/2 \rceil$, where $1 \le r_2 \le t+1$. Since Γ is a clique, we have

$$\sigma(G) \ge \sigma'(G) = \sigma'(V^*) + \sigma'(\Gamma) \ge \sigma'(V^*) + |\Gamma|(|\Gamma| - 1).$$

Now assume that

$$|V^*| = (t+1)q + r, \text{ where } 1 \le r \le t+1.$$

Then we have

$$(t+1)q + r = (t+1)q_2 + r_2 + \lceil s/2 \rceil - |\Gamma|. \tag{3.2}$$

Also, it follows by Lemma 3.12(ii) that

$$\sigma(G) \ge (st+1)q + rs + \min\{0,\, t-s+2-r\} + |\Gamma|^2 - |\Gamma|, \tag{3.3}$$

which is a quadratic function in $|\Gamma|$.

Theorem 3.14. *Let G be a partially $S_{s,t}$-saturated graph of order n with $\delta(G) \le s-2$. Assume $n - \lceil s/2 \rceil = (t+1)q_2 + r_2$, where $1 \le r_2 \le t+1$. Then*

$$\sigma(G) \ge (st+1)q_2 + r_2 s + \min\{0,\, t-s+2-r_2\} + \lceil s/2 \rceil \big(\lceil s/2 \rceil - 1\big).$$

Proof. We divide the proof into three cases.

Case 1. $1 \le r_2 + \lceil s/2 \rceil - |\Gamma| \le t+1$.

By Equation 3.2, we have

$$q = q_2 \text{ and } r = r_2 + \lceil s/2 \rceil - |\Gamma|.$$

It then follows by Inequality 3.3 that

$$\begin{aligned}\sigma(G) \geq (st+1)q_2 + (r_2 + \lceil s/2 \rceil - |\Gamma|)s + \\ + \min\{0,\, t - s + 2 - (r_2 + \lceil s/2 \rceil - |\Gamma|)\} + |\Gamma|^2 - |\Gamma|,\end{aligned} \tag{3.4}$$

which is minimized when $|\Gamma| = \lceil s/2 \rceil$. Hence,

$$\sigma(G) \geq (st+1)q_2 + r_2 s + \min\{0, t - s + 2 - r_2\} + \lceil s/2 \rceil \big(\lceil s/2 \rceil - 1\big).$$

Case 2. $r_2 + \lceil s/2 \rceil - |\Gamma| \leq 0$.

By Equation 3.2, we have

$$q = q_2 - 1 \text{ and } r = r_2 + \lceil s/2 \rceil - |\Gamma| + t + 1.$$

It then follows by Inequality 3.3 that

$$\begin{aligned}\sigma(G) \geq (st+1)(q_2 - 1) + (r_2 + \lceil s/2 \rceil - |\Gamma| + t + 1)s + \\ + \min\{0,\, t - s + 2 - (r_2 + \lceil s/2 \rceil - |\Gamma| + t + 1)\} + |\Gamma|^2 - |\Gamma|,\end{aligned}$$

which is minimized when $|\Gamma| = r_2 + \lceil s/2 \rceil$, since $|\Gamma| \geq r_2 + \lceil s/2 \rceil \geq \lceil s/2 \rceil + 1$. Thus,

$$\begin{aligned}\sigma(G) &\geq (st+1)(q_2 - 1) + (t+1)s + \min\{0,\, 1 - s\} + \\ &\quad + (r_2 + \lceil s/2 \rceil)^2 - (r_2 + \lceil s/2 \rceil) \\ &= (st+1)q_2 + (r_2 + \lceil s/2 \rceil)^2 - (r_2 + \lceil s/2 \rceil) \\ &\geq (st+1)q_2 + r_2 s + \lceil s/2 \rceil^2 - \lceil s/2 \rceil.\end{aligned}$$

Case 3. $r_2 + \lceil s/2 \rceil - |\Gamma| > t + 1$.

By Equation 3.2, we have

$$q = q_2 + 1 \text{ and } r = r_2 + \lceil s/2 \rceil - |\Gamma| - t - 1.$$

It then follows by Inequality 3.3 that

$$\sigma(G) \geq (st+1)(q_2+1) + (r_2 + \lceil s/2 \rceil - |\Gamma| - t - 1)s + \\ + \min\{0,\, t-s+2-(r_2+\lceil s/2\rceil - |\Gamma| - t - 1)\} + |\Gamma|^2 - |\Gamma|,$$

which is minimized when $|\Gamma| = r_2 + \lceil s/2 \rceil - t - 1$, since $|\Gamma| \leq r_2 + \lceil s/2 \rceil - t - 1 \leq \lceil s/2 \rceil$.

Thus,

$$\sigma(G) \geq (st+1)(q_2+1) + (r_2 + \lceil s/2 \rceil - t - 1)^2 - (r_2 + \lceil s/2 \rceil - t - 1),$$

which is exactly the same as Inequality 3.4 when $|\Gamma| = r_2 + \lceil s/2 \rceil - t - 1$. Therefore,

$$\sigma(G) \geq (st+1)q_2 + r_2 s + \min\{0, t-s+2-r_2\} + \lceil s/2 \rceil \big(\lceil s/2 \rceil - 1\big).$$

□

3.5 Main Results

We define

$$f_1(n) = s\left\lceil \frac{(t+1)n}{t+2} \right\rceil - \min\{r_1, s\}, \tag{3.5}$$

where $n \equiv r_1 \pmod{t+2}$, with $0 \leq r_1 \leq t+1$.

We also define

$$f_2(n) = (st+1)q_2 + r_2 s + \min\{0,\, t-s+2-r_2\} + \lceil s/2 \rceil \big(\lceil s/2 \rceil - 1\big), \tag{3.6}$$

where $n = (t+1)q_2 + r_2 + \lceil s/2 \rceil$, with $1 \leq r_2 \leq t+1$.

It can be seen that

$$f_2(n) = \frac{(st+1)n + \lceil s/2 \rceil (s-1) + \min\{r_2(s-1),\ (t+1-r_2)(t-s+2)\}}{t+1} - \left\lfloor \frac{(s+1)^2}{4} \right\rfloor.$$

We state our next remark without proof.

Remark 3.15. *We have*

(i) $f_2(n) \geq \dfrac{(st+1)n}{t+1} - \dfrac{(s+1)^2}{4}$.

(ii) $f_1(n) \leq f_2(n)$ *if* $n \geq \dfrac{(t+1)(t+2)(s+1)^2}{4(t-s+2)}$.

Our next two results follow directly from Theorems 3.6, 3.7, 3.13, and 3.14.

Theorem 3.16. *Assume* $3 \leq s < t$ *and* $n \geq (t+2)\lceil s/2 \rceil$. *If* $f_1(n) \leq f_2(n)$, *then*

$$\mathrm{psat}(n, S_{s,t}) = \mathrm{sat}(n, S_{s,t}) = \left\lceil \frac{f_1(n)}{2} \right\rceil.$$

Theorem 3.17. *Assume* $3 \leq s < t$ *and* $n \geq (t+2)\lceil s/2 \rceil$. *If* $n - \lceil s/2 \rceil \equiv r_2 \pmod{t+1}$, *with* $1 \leq r_2 \leq t-s+2$, *then*

$$\mathrm{psat}(n, S_{s,t}) = \min\left\{ \left\lceil \frac{f_1(n)}{2} \right\rceil, \left\lceil \frac{f_2(n)}{2} \right\rceil \right\}.$$

Chapter 4

Triangle-free graphs

4.1 Paths

We first include the following remark concerning the relationship between ceiling and floor functions.

Remark 4.1. *Let $a, b \in \mathbb{R}$ and $m, n \in \mathbb{Z}$.*

(i) If $a + b = n$, then $\lceil a \rceil + \lfloor b \rfloor = n$.

(ii) $\left\lfloor \frac{m}{n} \right\rfloor = \left\lceil \frac{m-n+1}{n} \right\rceil$ and $\left\lceil \frac{m}{n} \right\rceil = \left\lfloor \frac{m+n-1}{n} \right\rfloor$.

For every integer $k \geq 4$, we use T_k^* to denote the perfect binary tree with $\lfloor \frac{k}{2} \rfloor$ levels. The cases for $k = 6$ and $k = 7$ are illustrated in Figure 4.1. Note that T_k^* has a single root when k is even and double roots when k is odd. It can be easily checked that T_k^* has order a_k, where

$$a_k = \begin{cases} 3 \cdot 2^{t-1} - 2 & \text{if } k = 2t \\ 4 \cdot 2^{t-1} - 2 & \text{if } k = 2t + 1. \end{cases}$$

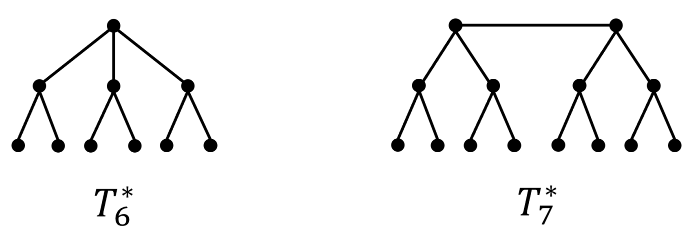

Figure 4.1: T_k^* when $k = 6$ and $k = 7$

In 1986, Kászonyi and Tuza [13] completely determined the saturation number of paths. Their result is given in the theorem below.

Theorem 4.2 (Kászonyi and Tuza [13]). *(i) For $n \geq 3$,* $\mathrm{sat}(n, P_3) = \lfloor n/2 \rfloor$.

(ii) For $n \geq 4$, $$\mathrm{sat}(n, P_4) = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (n+3)/2 & \text{if } n \text{ is odd.} \end{cases}$$

(iii) For $n \geq 6$, $\mathrm{sat}(n, P_5) = \lfloor \frac{5n+1}{6} \rfloor$.

(iv) Every P_k-saturated tree contains T_k^* as a subtree.

(v) If $n \geq a_k$ and $k \geq 6$, then $\mathrm{sat}(n, P_k) = n - \lfloor \frac{n}{a_k} \rfloor$.

When $k \geq 6$, there exist graphs H of order k such that $\mathrm{psat}(n, H) \ll \mathrm{sat}(n, H)$. For example, if $H = P_6$ and $n = 7q + r$, where $q \geq 1$ and $0 \leq r \leq 6$, Paul Horn demonstrated that $(q-1)P_7 + P_{7+r}$ is a partially P_6-saturated graph of order n. On the other hand, we have $\mathrm{sat}(n, P_6) = \lceil \frac{9n}{10} \rceil$ when $n \geq 10$ [13].

For the remainder of this section, we let $b_k = \lfloor \frac{3k-3}{2} \rfloor$.

Remark 4.3. *Let $k \geq 5$. Then*

(i) $a_5 = b_5 = 6$, and $a_k > b_k$ when $k \geq 6$.

(ii) $b_k \leq 2k - 4$.

(iii) $b_k \geq 2k - 6$ if and only if $k \leq 9$.

Lemma 4.4. *Assume $k \geq 5$. Let G be a partially P_k-saturated graph and T be a tree component of G of order at least 3. Then $|V(T)| \geq b_k$. In addition, we have $|V(T)| \geq 2k-4$ if K_1 is a component of G, and $|V(T)| \geq 2k - 6$ if K_2 is a component of G.*

Proof. If T is P_k-free, then $|V(T)| \geq a_k \geq b_k$ and we are done, by Theorem 4.2(iv). So assume T contains a copy of P_k. Let P be a longest path in T with vertex set $V(P) = \{v_1, v_2, ..., v_m\}$, where $m \geq k$. Define $i = \lfloor \frac{k-1}{2} \rfloor$ and $j = m - \lceil \frac{k-1}{2} \rceil + 1$. Note that the path $P' = v_1 v_2 ... v_{i-1} v_i v_j v_{j+1} ... v_m$ in $T + v_i v_j$ has $i + (m - j + 1) = k - 1$ vertices, and that $T + v_i v_j$ has a new copy of P_k, say P_k^*, containing edge $v_i v_j$.

Since P is a longest path in T, it follows that either $V(P_k^*) \cap \{v_1, ..., v_{i-1}\} = \emptyset$ or $V(P_k^*) \cap \{v_{j+1}, ..., v_m\} = \emptyset$ (otherwise, we would have $|V(P_k^*)| \leq |V(P')| = k-1$). So $|V(T)| \geq |V(P_k^*)| + \lfloor \frac{k-3}{2} \rfloor = k + \lfloor \frac{k-3}{2} \rfloor = \lfloor \frac{3k-3}{2} \rfloor = b_k$. The second statement in the lemma can be proved by considering $T + e$, where e is an edge joining a central vertex of P and a vertex in K_1 or K_2. □

A forest F is called *linear* if every component in F is a path.

Lemma 4.5. *Let F be a linear forest. Let $k \geq 5$ and ϵ be the order of a smallest component in F.*

(i) If $\epsilon \geq 3$, then F is partially P_k-saturated if and only if each component of F has order at least b_k.

(ii) If $\epsilon = 2$, then F is partially P_k-saturated if and only if one component of F has order 2 and every other component has order $\geq \max\{b_k, 2k-6\}$.

(iii) If $\epsilon = 1$, then F is partially P_k-saturated if and only if one component of F has order 1 and every other component has order $\geq 2k-4$.

Proof. We prove (i) only. The necessary condition follows directly from Lemma 4.4. For the other direction, it is sufficient to prove that every path P_m of order $m \geq b_k$ is partially P_k-saturated. Let u and v be nonadjacent vertices in P_m. Then $P_m - \{u, v\}$ consists of three disjoint subpaths. Among these three paths, let H_0 be the one with the smallest order. Then $V(P_m + uv) \setminus V(H_0)$ induces a subpath of order m_1 in $P_m + uv$, where

$$m_1 = m - |V(H_0)| \geq m - \frac{m-2}{3} = \frac{2m+2}{3} \geq \frac{2b_k+2}{3} \geq \frac{3k-2}{3},$$

which implies that $P_m + uv$ contains a new copy of P_k. □

Theorem 4.6. *Let* $b_k = \lfloor \frac{3k-3}{2} \rfloor$*. If* $k \geq 5$ *and* $n \geq b_k$*, then* $\operatorname{psat}(n, P_k) = n - f(n,k)$*, where*

$$f(n,k) = \begin{cases} \lceil \frac{n}{6} \rceil & \text{if } k = 5 \\ \lceil \frac{n-1}{b_k} \rceil & \text{if } 6 \leq k \leq 9 \\ \lfloor \frac{n}{b_k} \rfloor & \text{if } k \geq 10 \text{ and } \lfloor \frac{n}{b_k} \rfloor > \lfloor \frac{n-2}{2k-6} \rfloor \\ \lfloor \frac{n}{b_k} \rfloor + 1 & \text{if } k \geq 10 \text{ and } \lfloor \frac{n}{b_k} \rfloor = \lfloor \frac{n-2}{2k-6} \rfloor. \end{cases}$$

Proof. Define

$$c(n,k) = \max\left\{1 + \left\lfloor \frac{n-1}{2k-4} \right\rfloor,\ 1 + \left\lfloor \frac{n-2}{\max\{b_k, 2k-6\}} \right\rfloor,\ \left\lfloor \frac{n}{b_k} \right\rfloor\right\}.$$

For convenience, we write $c_k = 2k-4$ and $d_k = \max\{b_k,\ 2k-6\}$.

First we will show that $\operatorname{psat}(n, P_k) = n - c(n,k)$. For the upper bound, we assume $n - 1 = c_k q_1 + r_1$ with $0 \leq r_1 < c_k$, $n - 2 = d_k q_2 + r_2$ with $0 \leq r_2 < d_k$, and $n = b_k q_3 + r_3$ with $0 \leq r_3 < b_k$. Define $F_1 = (q_1 - 1)P_{c_k} + P_{c_k + r_1} + K_1$, $F_2 = (q_2 - 1)P_{d_k} + P_{d_k + r_2} + K_2$, and $F_3 = (q_3 - 1)P_{b_k} + P_{b_k + r_3}$. Then each F_i is a partially P_k-saturated linear forest of order n, by Lemma 4.5. In addition, we have $|E(F_1)| = n - c(F_1) = n - (1 + q_1) = n - (1 + \lfloor \frac{n-1}{c_k} \rfloor)$, $|E(F_2)| = n - c(F_2) = n - (1 + q_2) = n - (1 + \lfloor \frac{n-2}{d_k} \rfloor)$, and $|E(F_3)| = n - c(F_3) = n - q_3 = n - \lfloor \frac{n}{b_k} \rfloor$. This proves that $\operatorname{psat}(n, P_k) \leq \min\left\{|E(F_1)|,\ |E(F_2)|,\ |E(F_3)|\right\} = n - c(n,k)$.

For the lower bound, let G be a minimum partially P_k-saturated graph of order n with t tree components. By Lemma 4.4, we have

$$n \geq \min\{b_k t,\ 1 + (2k-4)(t-1),\ 2 + (\max\{b_k, 2k-6\})(t-1)\},$$

which implies that $t \leq c(n,k)$. Thus, by Remark 2.10,

$$\operatorname{psat}(n, P_k) = |E(G)| \geq n - t \geq n - c(n,k).$$

Thus we have shown that $\text{psat}(n, P_k) = n - c(n,k)$. It remains to prove that $c(n,k) = f(n,k)$.

If $k = 5$, then

$$c(n,5) = 1 + \left\lfloor \frac{n-1}{2k-4} \right\rfloor = \left\lfloor \frac{n+5}{6} \right\rfloor = \left\lceil \frac{n}{6} \right\rceil = f(n,5).$$

If $6 \le k \le 9$, then $b_k \ge 2k-6$, and it can be verified that

$$c(n,k) = 1 + \left\lfloor \frac{n-2}{b_k} \right\rfloor = \left\lfloor \frac{n+b_k-2}{b_k} \right\rfloor = \left\lceil \frac{n-1}{b_k} \right\rceil = f(n,k).$$

If $k \ge 10$, then $b_k \le 2k-6$ and $\dfrac{n}{b_k} \ge \dfrac{n-2}{2k-6} \ge \dfrac{n-1}{2k-4}$. This implies that

$$c(n,k) = \max\left\{1 + \left\lfloor \frac{n-2}{2k-6} \right\rfloor,\ \left\lfloor \frac{n}{b_k} \right\rfloor\right\} = \begin{cases} \left\lfloor \dfrac{n}{b_k} \right\rfloor & \text{if } \left\lfloor \dfrac{n}{b_k} \right\rfloor > \left\lfloor \dfrac{n-2}{2k-6} \right\rfloor \\ \left\lfloor \dfrac{n}{b_k} \right\rfloor + 1 & \text{if } \left\lfloor \dfrac{n}{b_k} \right\rfloor = \left\lfloor \dfrac{n-2}{2k-6} \right\rfloor. \end{cases}$$

This proves that $c(n,k) = f(n,k)$. □

Theorem 4.6 states that $\text{psat}(n, P_5) = n - \lceil \frac{n}{6} \rceil = \lfloor \frac{5n}{6} \rfloor$ when $n \ge 6$. For $n = 5$, we can easily check that $\text{psat}(5, P_5) \ge 4$. Since $K_2 + K_3$ is P_5-saturated, we have that $\text{psat}(5, P_5) = \text{sat}(5, P_5) = 4$. Since $\text{sat}(5, P_5) = 4$ by Theorem 4.2, we have $\text{psat}(5, P_5) = 4$ as well. Also note that $\lfloor \frac{5n+1}{6} \rfloor = \lfloor \frac{5n}{6} \rfloor$ if and only if $n \not\equiv 1 \pmod 6$. Thus, the following corollary holds.

Corollary 4.7. *Let $n \ge 5$. Then $\text{psat}(n, P_5) = \lfloor \frac{5n}{6} \rfloor$. Furthermore, $\text{psat}(n, P_5) = \text{sat}(n, P_5)$ if and only if $n \not\equiv 1 \pmod 6$.*

4.2 Lower bound

Our next result improves upon Theorem 1.7 when H is a triangle-free graph with weight t, containing neither S_{t+1} nor $S_{t,t}$ as a component.

Theorem 4.8. *Let H be a triangle-free graph with weight t which contains neither S_{t+1} nor $S_{t,t}$ as a component. Then* $\mathrm{psat}(n,H) \geq \frac{1}{2}(t-1+\frac{1}{t^2-t+1})n - \frac{t^2+4}{8}$.

Proof. Note that $t \geq 2$ by assumption. Let G be a minimum partially H-saturated graph of order n. We define the following sets:

$$\mathrm{S} = \{v \in V(G) \mid d(v) \leq t-2\}, \quad M = \{v \in V(G) \mid d(v) = t-1\}, \quad L = \{v \in V(G) \mid d(v) \geq t\},$$

$$\mathrm{M}_1 = \{v \in M \mid d(v,L) = 1\}, \quad M_2 = \{v \in M \mid d(v,L) = 2\}, \quad \mathrm{M}_3 = \{v \in M \mid d(v,L) \geq 3\}.$$

Claim 1. $|M_1 \cup M_2| \leq (t-1)\sigma(L)$.

Proof. For $i = 1,2$, define $A_i = \{v \in V(G) \setminus L \mid d(v,L) = i\}$. Clearly, we have $M_1 \cup M_2 \subseteq A_1 \cup A_2$, $|A_1| \leq \sigma(L)$, and $|A_2| \leq (t-2)|A_1|$ by definition. Thus,

$$|M_1 \cup M_2| \leq |A_1| + |A_2| \leq \sigma(L) + (t-2)\sigma(L) = (t-1)\sigma(L).$$

Claim 2. M_3 forms a clique of size at most t.

Proof. Suppose on the contrary that M_3 is not a clique. Then there exist two nonadjacent vertices u and v in M_3. So $G+uv$ contains a new copy of H, say H^*. Let H_0 be the component of H^* containing uv. Then every edge in H_0 is incident with either u or v; otherwise, H_0 would contain an edge with weight at most $t-1$ by the definition of M_3. Since $\mathrm{wt}(H_0) \geq t$, $d_{H_0}(u) \leq t$, and $d_{H_0}(v) \leq t$, it follows that $H_0 = S_{t+1}$ or $S_{t,t}$, which is impossible.

Claim 3. $$\sum_{v \in L \cup M_1 \cup M_2} d(v) \geq \left(t-1+\frac{1}{t^2-t+1}\right)\big|L \cup M_1 \cup M_2\big|.$$

Proof. By Claims 1 and 2, we have

$$
\begin{aligned}
&\sum_{v\in L\cup M_1\cup M_2} d(v) - \left(t-1+\frac{1}{t^2-t+1}\right)\big|L\cup M_1\cup M_2\big| \\
&= \sum_{v\in L} d(v) - \left(t-1+\frac{1}{t^2-t+1}\right)\big|L\big| - \frac{1}{t^2-t+1}\big|M_1\cup M_2\big| \\
&\geq \sigma(L) - \left(t-1+\frac{1}{t^2-t+1}\right)\big|L\big| - \frac{t-1}{t^2-t+1}\sigma(L) \\
&\geq \left(1-\frac{t-1}{t^2-t+1}\right)t\big|L\big| - \left(t-1+\frac{1}{t^2-t+1}\right)\big|L\big| \\
&= 0.
\end{aligned}
$$

It is easy to see that S forms a clique in G. So

$$
\begin{aligned}
2\,\mathrm{psat}(n,H) &= \sum_{v\in V(G)} d(v) \\
&\geq \left(t-1+\frac{1}{t^2-t+1}\right)\big(n-|M_3|-|S|\big) + (t-1)|M_3| + |S|\big(|S|-1\big) \\
&= \left(t-1+\frac{1}{t^2-t+1}\right)n - \frac{|M_3|+|S|}{t^2-t+1} + |S|^2 - t|S| \\
&\geq \left(t-1+\frac{1}{t^2-t+1}\right)n - \frac{2t-1}{t^2-t+1} + \big(|S|-\frac{t}{2}\big)^2 - \frac{t^2}{4} \\
&\geq \left(t-1+\frac{1}{t^2-t+1}\right)n - 1 - \frac{t^2}{4}.
\end{aligned}
$$

□

Let $t\geq 3$ and let $S^*_{t,t}$ be the graph obtained from $S_{t,t}$ by subdividing its central edge. We shall refer to the unique path of order 3 in $S^*_{t,t}$ joining two vertices of degree t as the *central path* of $S^*_{t,t}$. The following proposition shows that the lower bound given in Theorem 4.8 is nearly sharp.

Proposition 4.9. *If* $n\geq 2(t^2-t+1)$, *then* $\mathrm{sat}(n,S^*_{t,t})\leq \frac{1}{2}\left(t-1+\frac{2}{t^2-t+1}\right)n+\frac{3t}{2}$.

Proof. Assume $n=2(t^2-t+1)q+r$, where $q\geq 1$ and $1\leq r\leq 2(t^2-t+1)$. Also assume $r=(t-1)q_1+r_1$, where $q_1\leq 2t$ and $1\leq r_1\leq t-1$. Let A,B,C, and D be disjoint sets such that $|A| = 2q$, $|B| = 2tq + q_1$, $|C| = (t-2)(2tq+q_1)$, and $|D| = r_1$. We can easily check that $|A \cup B \cup C \cup D| = n$. We now define G with vertex set $A \cup B \cup C \cup D$ so that

(i) $G[A] = qK_2$, one vertex v_0 in A is adjacent to $t + q_1$ vertices in B and every vertex in D, and every other vertex in A is adjacent to exactly t vertices in B,

(ii) every vertex in B is adjacent to one vertex in A and $t - 2$ vertices in C,

(iii) every vertex in C is adjacent to one vertex in B, and $G[C]$ is almost $(t-2)$-regular, such that the possible vertex u_0 of degree $t-3$ is also adjacent to v_0 in A, with $N(u_0) \cap N(v_0) = \emptyset$,

(iv) D forms a clique.

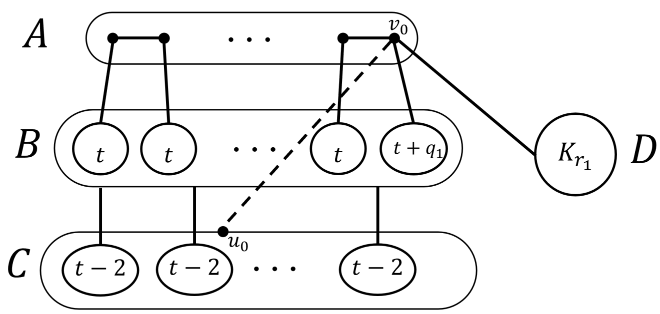

Figure 4.2: $S^*_{t,t}$-saturated graph G

Thus, we have that (i) every vertex in A has degree at least $t+1$, (ii) every vertex in $B \cup C$ has degree $t-1$, and (iii) every vertex in D has degree r_1. The graph G is depicted in Figure 4.2. So

$$\begin{aligned} 2|E(G)| &\leq (t-1)(n-r_1) + 4q + q_1 + 1 + r_1 + r_1^2 \\ &= (t-1)n + 4q + q_1 + r_1^2 - (t-2)r_1 + 1 \\ &\leq (t-1)n + 4q + 2t + t \\ &\leq \left[(t-1) + \frac{2}{t^2 - t + 1}\right]n + 3t. \end{aligned}$$

Thus, $|E(G)| \leq \frac{1}{2}(t - 1 + \frac{2}{t^2-t+1})n + \frac{3t}{2}$. It now remains to prove that G is $S^*_{t,t}$-saturated.

Note that no two vertices of degree at least t in G have a common neighbor, which implies that G does not contain a copy of $S^*_{t,t}$. Next we consider $G + e$, where $e = uv$ is an edge in $\overline{G}$. Then, since D is a clique, e must contain a vertex in $A \cup B \cup C$, say u.

Case 1. $u \in A$.

If v is adjacent to a vertex $u^* \in A \setminus \{u\}$, then G contains a copy of $S^*_{t,t}$ with central path uvu^*. Now assume $v \not\sim A$. Let u' be the unique neighbor of u in A. Then G contains a copy of $S^*_{t,t}$ with central path $u'uv$.

Case 2. $u \in B$ and $v \notin A$.

Let a be the unique neighbor of u in A and a' be the unique neighbor of a in A. Then G contains a copy of $S^*_{t,t}$ with central path uaa'.

Case 3. $u \in C$ and $v \notin A \cup B$.

Let b be the unique neighbor of u in B and a be the unique neighbor of b in A. Then G contains a copy of $S^*_{t,t}$ with central path uba.

This proves that G is $S^*_{t,t}$-saturated. Thus, $\text{sat}(S^*_{t,t}, n) \leq |E(G)| \leq \frac{1}{2}(t-1+\frac{2}{t^2-t+1})n+\frac{3t}{2}$, when $n \geq 2(t^2 - t + 1)$. □

4.3 Psat-sharp graphs

From [3], a graph H is called *sat-sharp* if $\lim_{n\to\infty} \frac{\text{sat}(n,H)}{n} = \frac{wt(H)-1}{2}$. We remark that it is not known, in general, whether the limit $\lim_{n\to\infty} \frac{\text{sat}(n,H)}{n}$ even exists, although the existence of this limit was stated as a conjecture by Tuza [16]. Similarly, we say H is *psat-sharp* if $\lim_{n\to\infty} \frac{\text{psat}(n,H)}{n} = \frac{wt(H)-1}{2}$. BY Theorem 1.7', every sat-sharp graph is psat-sharp. Also note that a graph H is psat-sharp if for every large n, there exists a partially H-saturated graph G of order n such that $|E(G)| \leq \frac{\text{wt}(H)-1}{2}n + o(n)$.

A natural class of sat-sharp graphs is the class of *threshold graphs*. A simple graph G with vertex set $\{v_1, ..., v_n\}$ is a threshold graph if there exist weights $x_1, ..., x_n \in \mathbb{R}$ such that, for all $i \neq j$, we have $v_iv_j \in E(G)$ if and only if $x_i + x_j \geq 0$. Threshold graphs were first introduced by Chvátal and Hammer [7], who proved that a simple graph G is a threshold graph if and only if G can be obtained from K_1 by iteratively adding a new vertex which is either an isolated vertex, or is one that dominates all previous vertices. Cameron and Puleo [3] showed that every threshold graph is sat-sharp. Therefore, every threshold graph is psat-sharp as well.

A connected graph H with weight t is called *special* if H contains a cut-edge uv such that both components of $H - uv$ have at most t vertices.

Remark 4.10. *Let H be a special graph with weight t, cut-edge uv, and $d(u) \leq d(v)$. Then*

(i) $d(v) = t$.

(ii) the component of $H - uv$ containing v has exactly t vertices.

Proposition 4.11. *If H is a triangle-free special graph with weight t, then H is either S_{t+1} or $S_{t,t}$.*

Proof. Let uv be the cut-edge of H such that $s = d(u) \leq d(v) = t$. Note that every vertex in $V(H) \setminus \{u, v\}$ has degree at most $t - 1$, as both components of $H - uv$ have at most t vertices. Since H is triangle-free, for any edge xy in H, we have $t = \mathrm{wt}(H) \leq \mathrm{wt}(xy) = \max\{d(x), d(y)\} \leq t$. So every vertex in H either has degree t or is adjacent to a vertex of degree t. This condition is satisfied only when either $s = t$ and $H = S_{t,t}$, or $s = 1$ and $H = S_{t+1}$. □

Theorem 4.12. *Let H be a graph of order k containing a special graph H_0 as a component such that $\mathrm{wt}(H) = \mathrm{wt}(H_0) = t \geq 1$. Then H is sat-sharp. More specifically, for $n \geq 2k - 2$, we have*

$$\mathrm{sat}(n, H) \leq \frac{t-1}{2}n + \frac{(k-1)(k-t-1)}{2}.$$

Proof. Assume $|V(H_0)| = s + t$, where $1 \leq s \leq t$. Let H_1 be the union of all nontrivial components in H of order at most t, and H_2 be the union of all components in H of order at least $t + 1$. Define $k_i = |V(H_i)|$ for $i = 1, 2$. Then we have $k - k_1 \geq k_2 \geq s + t$. By Remark

1.6, every component in H_1 contains at least $\frac{t+3}{2}$ vertices. Hence, $k_1 \geq \frac{t+3}{2}q_1$, where q_1 is the number of components in H_1.

Now let $n = qt + r$ so that $k_2 - t \leq r \leq k_2 - 1$. Then $q \geq 1$ and $r \geq s$. Define $G = qK_t + K_r$. Then G is H-free, as G contains no copy of H_2. We claim that G is H-saturated.

First assume $q = 1$. Then we have

$$k_2 + t - 1 \leq 2k_2 - 2 \leq 2k - 2 \leq n = t + r \leq k_2 - 1 + t.$$

Thus, $H = H_0$ has order $k = k_2 = t + 1$. It is then easily seen that $G = K_t + K_r$ is H-saturated.

Now assume $q \geq 2$. Since $n \geq 2k-2$, $2k_1 \geq (t+3)q_1$, $r \leq k_2-1$, and $k_2 \geq |V(H_0)| \geq t+1$, it follows that

$$(q-1)t = n - t - r \geq (2k_1 + 2k_2 - 2) - t - (k_2 - 1) = 2k_1 + k_2 - t - 1 \geq 2k_1 \geq (t+3)q_1.$$

This implies that $q_1 \leq q - 2$ since $q \geq 2$. So $(q-2)K_t$ contains a copy of H_1 and thus, G is H-saturated. A simple counting yields

$$\mathrm{sat}(n, H) \leq |E(G)| = \frac{(t-1)n + r(r-t)}{2} \leq \frac{t-1}{2}n + \frac{(k-1)(k-t-1)}{2}.$$

This completes the proof of Theorem 4.12. □

Corollary 4.13. *Let H be a graph of order k and weight t, containing either $S_{t,t}$ or S_{t+1} as a component. Then H is sat-sharp. More specifically, for $n \geq 2k - 2$,*

$$\mathrm{sat}(n, H) \leq \frac{t-1}{2}n + \frac{(k-1)(k-t-1)}{2}.$$

The result below follows immediately from Theorem 4.8 and Corollary 4.13.

Theorem 4.14. *Let H be triangle-free graph with weight $t \geq 1$. Then the following three statements are equivalent:*

(i) H is sat-sharp.

(ii) H is psat-sharp.

(iii) H contains either S_{t+1} or $S_{t,t}$ as a component.

Chapter 5

Summary and future work

From Chapter 1, we have that $\mathrm{wsat}(n, H) \leq \mathrm{psat}(n, H) \leq \mathrm{sat}(n, H)$ for every graph H. Both of the functions $\mathrm{psat}(n, H)$ and $\mathrm{sat}(n, H)$ are not, in general, monotone with respect to H. On the other hand, it is easy to see that $\mathrm{wsat}(n, H_1) \leq \mathrm{wsat}(n, H_2)$ whenever $H_1 \subseteq H_2$. So $\mathrm{wsat}(n, H)$ behaves quite differently from the other two functions. The following question was first raised by Tuza in [18].

Question 5.1 (Tuza [18])**.** *Are there necessary and/or sufficient conditions for* $\mathrm{wsat}(n, H)$ *to equal* $\mathrm{sat}(n, H)$*?*

Any result on $\mathrm{sat}(n, H)$ remains true for $\mathrm{psat}(n, H)$ provided that the original proof does not make use of the condition that an H-saturated graph is H-free. In particular, we pointed out that Theorem 1.1 on complete graphs, Theorem 1.2 on stars, and Theorem 1.7 on the general lower bound are all true for both $sat(n, H)$ and $\mathrm{psat}(n, H)$. Thus, it is natural to ask the following question.

Question 5.2. *Are there succinct necessary and/or sufficient conditions for* $\mathrm{psat}(n, H)$ *to equal* $\mathrm{sat}(n, H)$*?*

In Chapter 2, we characterize all minimum partially C_4-saturated graphs of order n for all $n \geq 4$ (Theorem 2.6). We also showed that $\mathrm{psat}(n, H) = \mathrm{sat}(n, H)$ for every $n \geq |V(H)|$ and every nontrivial graph H of order 4 or less, with the exception that $\mathrm{psat}(5, P_4) = 3$ and $\mathrm{sat}(5, P_4) = 4$ (Theorem 2.12).

For the saturation of all graphs of order 5, the cycle C_5 has been of particular interest. In 1995, Fisher, Fraughnaugh, and Langley [12] gave an upper bound of $\left\lceil \frac{10}{7}(n-1) \right\rceil$ for C_5.

Later, in [5] and [6], Chen proved that this upper bound serves as the lower bound as well for all $n \geq 21$, and also characterized all minimum C_5-saturated graphs of order n.

Our next step is to consider the following two problems.

Problem 5.3. *Characterize all minimum partially C_5-saturated graphs of order n for all $n \geq 5$.*

Problem 5.4. *Determine* $\mathrm{psat}(n, H)$ *for every graph H of order* 5.

In 1986, Kászonyi and Tuza [13] showed that the star S_k has the largest saturation number among all trees of order k (Theorem 1.2). Faudree et al., [9] showed that $S_{2,k-2}$ has the smallest saturation number among all trees of order k (Theorem 1.4), and also raised the following question.

Question 5.5. *Among all trees of order k, which is the tree(s) of second highest and the tree(s) of second lowest saturation number?*

We pointed out that S_k has the largest partial saturation number among all trees of order k (Theorem 1.2'). We also showed that $S_{2,k-2}$ has the smallest partial saturation number among all trees of order k (Theorem 3.4). So it is natural to ask the following question.

Question 5.6. *Among all trees of order k, which is the tree(s) of second highest and the tree(s) of second lowest partial saturation number?*

Assume $3 \leq s < t$. Recall from Chapter 3 that

$$f_1(n) = s\left\lceil \frac{(t+1)n}{t+2} \right\rceil - \min\{r_1, s\},$$

where $n \equiv r_1 \pmod{t+2}$, with $0 \leq r_1 \leq t+1$.

Also, we have

$$f_2(n) = (st+1)q_2 + r_2 s + \min\{0,\, t-s+2-r_2\} + \lceil s/2 \rceil \big(\lceil s/2 \rceil - 1\big),$$

where $n=(t+1)q_2+r_2+\lceil s/2\rceil$, with $1\le r_2\le t+1$.

In Theorem 3.16, we showed that $\operatorname{psat}(n,S_{s,t})=\operatorname{sat}(n,S_{s,t})=\left\lceil\frac{f_1(n)}{2}\right\rceil$ when $n\ge(t+2)\lceil s/2\rceil$ and $f_1(n)\le f_2(n)$. In particular, our result holds when $n\ge\frac{(t+1)(t+2)(s+1)^2}{4(t-s+2)}$ by Remark 3.15. In Theorem 3.17, we showed that $\operatorname{psat}(n,S_{s,t})=\min\left\{\left\lceil\frac{f_1(n)}{2}\right\rceil,\ \left\lceil\frac{f_2(n)}{2}\right\rceil\right\}$ when $n\ge(t+2)\lceil s/2\rceil$ and $n-\lceil s/2\rceil\equiv r_2\pmod{t+1}$, with $1\le r_2\le t-s+2$.

We believe that the following conjecture is plausible.

Conjecture 5.7. *Assume $3\le s<t$ and $n\ge(t+2)\lceil s/2\rceil$. If $n-\lceil s/2\rceil\equiv r_2\pmod{t+1}$, with $1\le r_2\le t+1$, then*

$$\operatorname{psat}(n,S_{s,t})=\min\left\{\left\lceil\frac{f_1(n)}{2}\right\rceil,\ \left\lceil\frac{f_2(n)}{2}\right\rceil\right\}.$$

In Chapter 4, we completely determined $\operatorname{psat}(n,P_k)$ for all $n\ge\left\lfloor\frac{3k-3}{2}\right\rfloor$ (Theorem 4.6). For any graph H, we define $\epsilon(H)=\limsup_{n\to\infty}\frac{\operatorname{sat}(n,H)-\operatorname{psat}(n,H)}{n}$. We then define $\epsilon(k)$ to be the supremum of $\epsilon(H)$ among all graphs H of order k. Clearly, by Theorem 2.12, we have that $\epsilon(k)=0$ when $k\le 4$. For $k\ge 5$, it follows by Theorems 4.2 and 4.6 that $\epsilon(k)\ge\epsilon(P_k)=\frac{1}{b_k}-\frac{1}{a_k}$.

Problem 5.8. *Determine $\epsilon(k)$ for $k\ge 5$.*

Let H be a triangle-free graph with weight t which contains neither S_{t+1} nor $S_{t,t}$ as a component. Then we have shown that $\operatorname{psat}(n,H)\ge\frac{1}{2}(t-1+\frac{1}{t^2-t+1})n+O(1)$ (Theorem 4.9). We also found a triangle-free graph H with weight t that contains neither S_{t+1} nor $S_{t,t}$ as a component such that $\operatorname{psat}(n,H)\le\frac{1}{2}(t-1+\frac{2}{t^2-t+1})n+O(1)$. This implies that the upper bound in Theorem 4.9 is nearly sharp. It is then natural to ask the following question.

Question 5.9. *Does there exist a triangle-free graph H with weight t which contains neither S_{t+1} nor $S_{t,t}$ as a component such that $\operatorname{psat}(n,H)\le\frac{1}{2}(t-1+\frac{1}{t^2-t+1})n+O(1)$?*

In 2022, Cameron and Puleo [3] introduced the concept of sat-sharp graphs. We define psat-sharp graphs in a similar fashion. In Theorem 4.14, we proved that for any triangle-free graph H with weight $t \geq 1$, the following three statements are equivalent: (i) H is sat-sharp, (ii) H is psat-sharp, and (iii) H contains either S_{t+1} or $S_{t,t}$ as a component.

Cameron and Puleo also conjectured that if H_1 and H_2 are two disjoint sat-sharp graphs, then $H_1 + H_2$ is sat-sharp as well. We end this dissertation with the following stronger conjecture.

Conjecture 5.10. *Let H_1 and H_2 be disjoint graphs such that $1 \leq \mathrm{wt}(H_1) \leq \mathrm{wt}(H_2)$ and H_1 is sat-sharp (psat-sharp). Then $H_1 + H_2$ is sat-sharp (psat-sharp).*

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