THE SHIELDS-HARARY NUMBER OF GRAPHS

Except where reference is made to the work of others, the work described in this dissertation is my own or was done in collaboration with my advisory committee.

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Vita

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Dissertation Abstract

The Shields-Harary Number of Graphs

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The Shields-Harary numbers are a class of graph paramaters that measure the robustness of a graph in terms of network vulnerability, with reference to a given cost function.

Style manual or journal used Discrete Mathematics (together with the style known as "auphd"). Bibliograpy follows van Leunen's *A Handbook for Scholars*.

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Table of Contents

List of Figures

Chapter 1

The Shields-Harary Numbers

1.1 Introduction

Suppose we have a simple, finite non-empty graph $G = (V, E)$ whose vertices have non-negative weights $g(v)$, $v \in V$ where $g : V \to [0, \infty)$. We can consider this graph to be some network where the weight at each vertex represents some amount of harmful material stored at that particular location (vertex).

This network has an enemy who wishes to dismantle it by removing vertices from the network and thus destroying the material stored at the removed vertices. The network is considered dismantled when the sum of the weights on each remaining connected component of the network after vertex removal is less than or equal to some threshold, say 1. The defense that the network has against the enemy is that it costs the enemy a certain price to remove these vertices. The enemy pays $f(g(v))$ to remove $v \in V$ and $\sum_{v \in S} f(g(v))$ to remove a set S of vertices where f is some decreasing (or at least non-increasing) cost function on the range of g.

We will assume that the enemy is intelligent and has complete knowledge of the weighted network and so for any allocation of weights throughout the network, the enemy will always choose to remove the set of vertices that dismantles the network while giving rise to the least possible total cost. This least possible total cost that the enemy pays for a particular weighting g shall be denoted by $m_f(g, G)$. We endeavor to make the enemy pay as much as possible. In section [1.2], we define things a little more precisely, as well as defining what the Shields-Harary numbers are.

1.2 Definitions

Let $G = (V, E)$, $S \subseteq V$, $g : V \to [0, \infty)$ and let f be some non-increasing cost function on the range of g.

Why is the cost function f decreasing (or at least non-increasing)? The idea is that the more harmful material you have stored in one location, the harder it will be to defend and thus it will be easier (cheaper) for the enemy to remove from the network.

We call S a *g-dismantling* set if and only if $\sum_{v \in V(H)} g(v) \leq 1$ for each component H of $G - S$. The Shields-Harary number of a graph G with respect to the cost function f is given by

$$SH(G, f) = \sup_{g:V \to [0,\infty)} m_f(g, G).$$

Recall that $m_f(g, G)$ is the least possible cost to dismantle the graph G with the specific weighting g.

Now suppose we change the definition of dismantling so that the network is dismantled when the sums of the weights on each remaining connected component is strictly less than 1. We call S a *strict g-dismantling* set if and only if $\sum_{v \in V(H)} g(v) < 1$ for each component H of $G - S$. The least possible cost to strictly dismantle the graph G with weighting g is denoted by $\overline{m}_f(g, G)$. We denote the Shields-Harary number of a graph with this definition of dismantling by

$$\overline{SH}(G, f) = \sup_{g:V \to [0,\infty)} \overline{m}_f(g, G).$$

We define $SH_0(G, f)$ and $\overline{SH_0}(G, f)$ in the same manner as above except that our weighting functions are restricted to be constant. It is elementary that in the definitions of $\overline{SH}$ and $\overline{SH_0}$, the weighting functions may as well be into $[0, 1]$.

In [2] it was shown that if f is continuous from the right at each point of $[0,1]$ then the "sup" in the definition of $\overline{SH}(G,f)$ is always a maximum (i.e. we can actually make the enemy pay this amount). A weighting g which yields this "max" will be called *optimal* (for G and f).

1.3 An example of the Shields-Harary Numbers

Having now defined all of the terms that we need, (for any others, refer to [9]), let us now illustrate the concept of the Shields Harary Numbers (SH numbers for short) by looking at an example of how the SH number relates to a graph with a particular weighting and cost function. In later chapters, we will further explore the intricacies of these calculations by actually calculating the SH numbers of specific graphs with respect to various cost functions.

Consider the following graph G with the cost function $f(x) = \dfrac{1}{x}$ and weighting g_1 as illustrated by Figure 1.1.

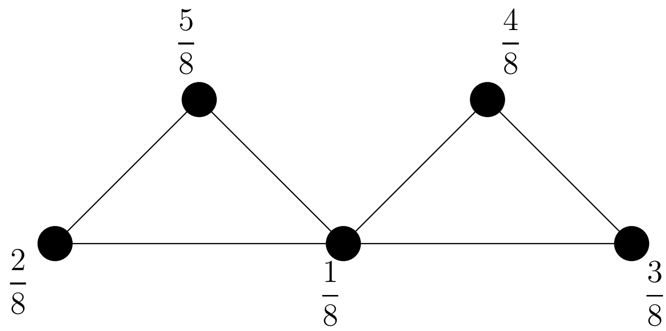


Figure 1.1: Our graph G with weighting g_1

The enemy whose intelligence is poor might yield to the temptation of dismantling this graph by removing the central vertex of weight $\frac{1}{8}$; thus disconnecting the graph with the resulting components shown in Figure 1.2. This dismantling would yield a dismantling cost of $f(\frac{1}{8}) = 8$. At a quick glance, this might seem a logical choice since the enemy would only have to remove a single vertex. Better intelligence, however, would make it clear that the enemy could more cheaply dismantle this graph by removing the two vertices of weights $\frac{5}{8}$ and $\frac{4}{8}$ with the resulting component shown in Figure 1.3. This dismantling yields a dismantling cost of $f(\frac{5}{8}) + f(\frac{4}{8}) = \frac{8}{5} + \frac{8}{4} = \frac{18}{5} < 8$. So dismantling does not always imply disconnecting the graph. Notice that in both cases, the sum of the weights on each remaining component of the graph is strictly less than one and so the enemy in this example would pay $\frac{18}{5}$ as this is the cheapest dismantling cost for this particular weighting. Without considering any other weightings, we would say that $\overline{SH}(G, \frac{1}{x}) \geq \frac{18}{5}$.

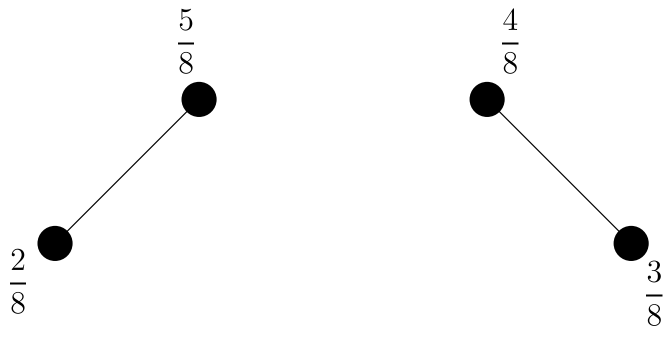


Figure 1.2: Dismantling (G,g_1) by disconnecting

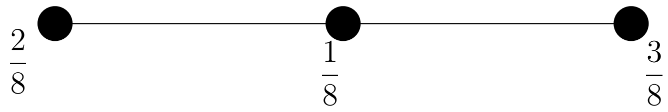


Figure 1.3: Dismantling but not disconnecting

Let us now consider some other weightings of G. Suppose we consider the following weighting of G:

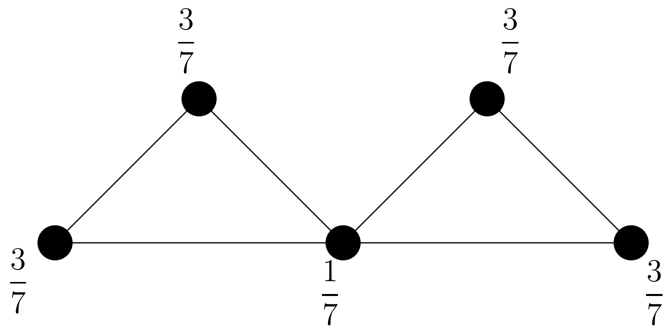


Figure 1.4: Graph G with weighting g_2

There are two ways in which we can dismantle G with this particular weighting. One way would be to remove the central vertex of weight $\frac{1}{7}$ and pay a cost of $f(\frac{1}{7}) = 7$. This dismantling leaves the components as shown in Figure 1.5. The other way in which we could dismantle this graph would be to use a greedy algorithm and keep removing the vertex of the largest weight until the graph is dismantled. With strict dismantling in mind, we are required to remove three of the vertices of weight $\frac{3}{7}$ which gives rise to a removal cost of $3f(\frac{3}{7}) = 3(\frac{7}{3}) = 7$ and is illustrated in Figure 1.6. (As one can see, unlike the first weighting, with this weighting we get the same cost whether we disconnect the graph or not.) So $\overline{SH}(G, \frac{1}{x}) \geq 7 > \frac{18}{5}$.

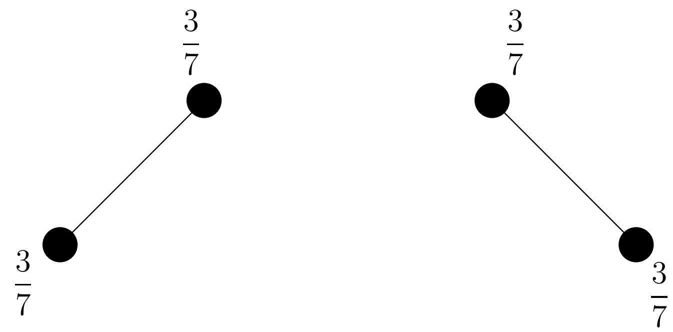


Figure 1.5: Dismantling G with weighting g_2 by disconnecting.

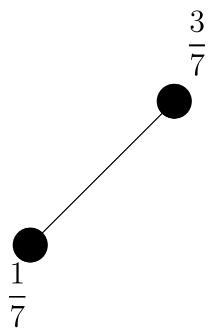


Figure 1.6: Dismantling G with weighting g_2 but not disconnecting.

Let us now consider a constant weighting of our graph G as shown in Figure 1.7. With this particular weighting and with strict dismantling in mind, we cannot leave two adjacent vertices. In this situation, we want to remove the fewest number of vertices that we can and so the structure of the graph comes into play. If we leave the center vertex in, then we must remove the four outer vertices. However, we can get away with only removing three as shown in Figure 1.8. This dismantling forces us to pay $3f(\frac{1}{2}) = 3(2) = 6$. Thus it remains that, $\overline{SH}(G, \frac{1}{x}) \geq 7 > 6$. However, we find also that $\overline{SH}_0(G) \geq 6$. (By results to be introduced later, it will be easy to see that $\overline{SH}_0(G) = 6$.)

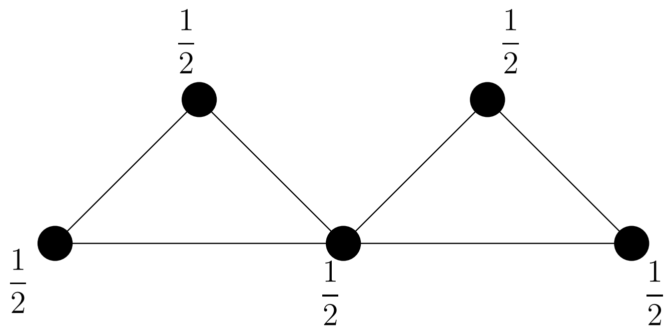


Figure 1.7: G with constant weighting g_3.

Figure 1.8: Cheapest dismantling of G with weighting g_3.

From these different weightings, one can see that the computation of $\overline{SH}\left(G, \frac{1}{x}\right)$ is not necessarily a trivial calculation. In fact, the problem of computing $\overline{SH}\left(G, \frac{1}{x}\right)$ is indeed a five variable optimization problem.

1.4 The History of the Shields-Harary Numbers and a Review of Literature.

The Shields-Harary numbers arose from a conjecture of the late Allen Shields. In 1972, at a University of Michigan Math Club meeting, Shields posed the following conjecture:

> For any sequence of positive numbers $d_1, d_2, \ldots, d_n$ there exists a set $S \subseteq \{1, 2, \ldots, n\}$ such that $\displaystyle\sum_{i \in S} \frac{1}{d_i} \leq n$ and for each set of consecutive integers (block) B of $\{1, \ldots, n\} \setminus S$, $\displaystyle\sum_{i \in B} d_i \leq 1$.

In terms of graph theory, Shields' conjecture was actually that $SH(P_n, f) \leq n$ where P_n is the path on n vertices and f is the cost function $f(x) = \dfrac{1}{x}$.

Realizing that this conjecture was connected to graphs, Shields consulted Frank Harary. They soon proved that if $SH(P_n, \frac{1}{x}) = n$, then $SH(C_n, \frac{1}{x}) = 2n - 2$ where C_n is the cycle of length n. They also computed the values of $SH(K_n, \frac{1}{x})$ and $SH(K_{1,n-1}, \frac{1}{x})$. Both of these values are around $\frac{n^2}{4}$, which is interesting; by this measure of network robustness, stars are close to cliques and are much more robust than cycles. However, Shields and Harary never did prove Shields' original conjecture. Instead, it was proven, by a clever "cost analysis" argument, by Stephen Schanuel [7].

Initially, much of what was done with the Shields-Harary parameters dealt with the specific cost function $f(x) = \dfrac{1}{x}$, which was the cost function involved in Shields' original conjecture. Johnson [6] presented everything known at the time about the SH parameters with that particular cost function. Another paper by Wunsch [8], also on the original Shields-Harary number, obtains an infinite number of graphs G for which

$SH(G, \frac{1}{x})$ is not an integer, through asymptotic estimates of $SH(G)$ for G in a certain family of graphs. (Johnson had found one graph for which $SH(G)$ was not an integer.)

Some more recent work has dealt with the constant weight Shields-Harary numbers. The basis for the work on the SH_0 numbers is a result in [2] which reduces the computation of SH_0 or $\overline{SH}_0$ of a graph to computing the minimum cardinality of a set of vertices whose removal from the graph leaves components of a certain size. This result is as follows:

$$\overline{SH}_0(G, f) = \max_{1 \leq k \leq n} p(k, G) f\left(\frac{1}{k}\right)$$

where $p(k, G)$ is the minimum cardinality of a set $S \subset V(G)$ such that each component of $G - S$ has $k - 1$ or fewer vertices. In [5] it is shown that for all non-increasing cost functions f and all trees T on n vertices, $SH_0(K_{1,n-1}, f) \leq SH_0(T, f) \leq SH_0(P_n, f)$ with the same inequality holding for $\overline{SH_0}$. In [5] the authors also show that for all trees T on n vertices, $1 \leq p(k, T) \leq \left\lfloor \frac{n}{k} \right\rfloor$, $2 \leq k \leq n$. Another paper, [1], shows that $SH(G, f) \leq SH(G + e, f)$ for any edge e between non-adjacent vertices of G. This inequality holds for $\overline{SH}$, SH_0 and $\overline{SH}_0$. This result proved useful in the calculations of the values of $\overline{SH}_0(G, f)$ in terms of $p(k, G)$ when G is a special kind of Cayley Graph. (See [1]).

The inequality $SH(G, f) \leq SH(G + e, f)$ mentioned above is a special case of the monotonicity of the Shields-Harary numbers with respect to taking subgraphs. Surprisingly, this monotonicity is not completely trivial; we prove it here.

Proposition 1.1 *Suppose that $P \in \{SH, \overline{SH}, SH_0, \overline{SH_0}\}$, with respect to some cost function f, and that G_1 is a subgraph of G. Then $P(G_1) \leq P(G)$.*

Proof. Suppose that $\epsilon > 0$ and g is a weighting of $V(G_1)$ such that the minimum cost m of (strictly) dismantling (G_1, g) satisfies $P(G_1) - \epsilon < m$; and, of course, $m \leq P(G_1)$. (If $P \in \{SH_0, \overline{SH_0}\}$, g is a constant function.) In case $V(G) \setminus V(G_1)$ is non-empty, extend g to a weighting $\tilde{g}$ on $V(G)$ any which way; but $\tilde{g} = g$ on $V(G) \setminus V(G_1)$ if g is constant. We shall prove that the minimum cost $\tilde{m}$ of (strictly) dismantling $(G, \tilde{g})$ satisfies $\tilde{m} \geq m$. It will then follow that $P(G) \geq P(G_1) - \epsilon$ for all $\epsilon > 0$, and thence the desired conclusion.

Let $\tilde{S}$ be a $\tilde{g}$-dismantling set of vertices of minimum cost, say $\tilde{m}$ and let S_1 be the set of vertices defined as follows, $S_1 = \tilde{S} \cap V(G_1)$. Now suppose that $u, v \in V(G_1 - S_1)$ such that u, v are in the same component of $G_1 - S_1$. We can then find a path in $G_1 - S_1$ from u to v. Such a path will also be in $G - \tilde{S}$ since all of the edges and vertices of $G_1 - S_1$ are in $G - \tilde{S}$ as well, in view of the definition of S_1 and the fact that G_1 is a subgraph of G. Thus, any vertices of G_1 that lie in the same component of $G_1 - S_1$ must lie in the same component of $G - \tilde{S}$ and so every component of $G_1 - S_1$ is contained in a component of $G - \tilde{S}$.

Since the components of $G_1 - S_1$ are subgraphs of the components of $G - \tilde{S}$, it follows that S_1 dismantles G_1. Since S_1 is a subset of $\tilde{S}$, it follows that the cost of removing S_1 is no more than $\tilde{m}$ (the cost of removing $\tilde{S}$). Thus, the minimum dismantling cost m of (G_1, g) is such that $m \leq \tilde{m}$. $\square$

A paper from Harary and Johnson [2] presents more results on the Shields-Harary parameter, with arbitrary cost functions. Here are some of those results.

1. If f is a continuous function, then $SH(G, f) = \overline{SH}(G, f)$ and $SH_0(G, f) = \overline{SH}_0(G, f)$. (Incidentally, f is allowed to take the value ∞ and still be continuous throughout.)

2. If f is continuous, then the sup in the definition of $\overline{SH}(G,f)$ and $\overline{SH}_0(G,f)$ is always a max.

3. $SH(K_n,f) = SH_0(K_n,f) = \max_{1\le k\le n}(n-k+1)\, f\left(\frac{1}{k}\right)^+$, where $f\left(\frac{1}{k}\right)^+$ is the right hand limit of f at $\frac{1}{k}$.

4. $\max_{1\le k\le n}\left\lfloor\frac{n}{k}\right\rfloor f\left(\frac{1}{k}\right) \le \overline{SH}(P_n,f) \le n \sup_{0<x\le 1} xf(x)$

5. $\max_{1\le k\le n}\left\lceil\frac{n}{k}\right\rceil f\left(\frac{1}{k}\right) \le \overline{SH}(C_n,f) \le \max\left[f\left(\frac{1}{n}\right), f\left(\frac{1}{n-1}\right) + \overline{SH}(P_{n-1},f)\right]$

Result (2) has some practical significance: with "strict" dismantling and a continuous cost function, we can actually find a weighting of the nodes of the network that will force the enemy to pay the maximum amount. Finding that critical weighting is a problem that can usually be solved as a by-product of the search for the value of $\overline{SH}(G,f)$. For instance, by (1) and (3) we have, for continuous f, $\overline{SH}(K_n,f) = \max_{1\le k\le n}(n-k+1)f\left(\frac{1}{k}\right)$. Now, from this result alone, a certain amount of thought shows that one of the constant weightings $\frac{1}{k}$ is a critical weighting; but even if that is not clear, the proof of this result in [2] shows this to be the case, and more: if f is strictly decreasing, the only critical weightings are among those constant weightings.

In [2], Harary and Johnson also pose the following conjecture:

Conjecture 1.1 (Harary,Johnson, [2]) *If f is continuous and G is vertex-transitive then there is a constant optimal weighting of $V(G)$. (So $SH(G,f) = \overline{SH}(G,f) = \overline{SH}_0(G,f) = SH_0(G,f)$).*

This conjecture gives rise to the following related problem:

> For which continuous f is it the case that for every G, there is an optimal weighting of $V(G)$ which is constant on each orbit of $V(G)$ under $Aut(G)$, the group of graph automorphisms of G?

The conjecture that the answer to the question above is "all continuous f" we will call the *Constant Weights on Orbits Conjecture* (CWOC) which will be discussed further in Chapter 2. For now, it suffices to say that CWOC was of interest when work began on [4] which initially began to compute the exact $\overline{SH}$ values of two intersecting cliques for general cost functions f. The exact values, however, soon proved to be elusive and the results obtained in [4] tended to pertain more to continuous concave cost functions and optimal weightings of graphs with "clique-like" structures. As a result, doubts about the CWOC arose and counterexamples were found with $G = K_n - e$, a clique minus an edge. Counterexamples will be given and discussed in chapter 2 as will the exact values of $\overline{SH}(K_n - e, f)$ for continuous cost functions f (see [3]).

Chapter 2

The Shields-Harary Number of $K_n - e$

2.1 Full Solution of $\mathbf{K_n - e}$ with optimal weighting possibilities

Let $K_n - e$ stand for K_n minus an edge.

Theorem 2.1 *For any $f : [0, \infty) \to [0, \infty]$, non-increasing and continuous, and any $n \geq 3$, $\overline{SH}(K_n - e, f) = \max[M_1, M_2, M_3]$ where*

$$M_1 = \max_{k=1,3,\ldots,n} (n - k + 1) f(\frac{1}{k})$$

$$M_2 = f(1) + (n - 2) f(\frac{1}{2})$$

$$M_3 = \begin{cases} (n-1) f(1-x) & \text{if } f(0) > 2f(1), \text{ for some} \\ & x \in (0, \frac{1}{2}] \text{ s.t. } f(x) = 2f(1-x) \\ M_2 & \text{if } f(0) \leq 2f(1). \end{cases}$$

Proof. In the first part of this proof, we will show that there is a weighting of $K_n - e$ for each M_i, $1 \leq i \leq 3$, so that the (strict) dismantling of $K_n - e$ costs M_i. This will prove that $\overline{SH}(K_n - e, f) \geq \max(M_1, M_2, M_3)$. Let us denote the vertices of K_n by $v_1, \ldots, v_n$ and let $e = v_{n-1}\, v_n$.

Let us first consider the constant weighting $\frac{1}{k}, k \in \{1, 3, 4, \ldots, n\}$ on the vertices of $K_n - e$. When $k = 1$, we put a weight of 1 on all vertices. To dismantle, we must knock out all of the vertices, at a cost of $nf(1)$. When $k > 2$, we can dismantle the $K_n - e$

by either removing the K_{n-2} induced by $v_1, \ldots, v_{n-2}$ or by leaving some vertices in the K_{n-2} and then knocking out vertices until the sum of the weights on the remaining component is strictly less than one. If we remove the K_{n-2}, we do so at a cost of $(n-2)\ f(\frac{1}{k}) \geq (n-k+1)\ f(\frac{1}{k})$. If we knock out vertices until the sum of weights is less than 1, then we must knock out $n-(k-1) = (n-k+1)$ vertices at a cost of $(n-k+1)\ f(\frac{1}{k})$. When $k \geq 3$, this is the cheaper of the two dismantlings. Thus, for each $k \in \{1, 3, \ldots, n\}$, the dismantling cost is $(n-k+1) f(\frac{1}{k})$; therefore the cost M_1 is achieved by one of these weightings.

Now, consider the dismantling cost required by placing a weight of 1 on v_n and a weight of $\frac{1}{2}$ on $v_1, \ldots, v_{n-1}$. To dismantle, we must take out v_n at a cost of $f(1)$. We must also remove all but one of $v_1, \ldots, v_{n-1}$ at a cost of $(n-2)\ f(\frac{1}{2})$. Thus, the cost of dismantling with this weighting is $f(1) + (n-2)\ f(\frac{1}{2}) = M_2$.

Let $x \in (0, \frac{1}{2}]$. Consider the weighting where we put x on v_1 and $1-x$ on $v_2, \ldots, v_n$. If v_1 is not knocked out, then all the other vertices must be, for strict dismantling, at a cost of $(n-1)\ f(1-x)$. If v_1 is knocked out, then the dismantling is most economically achieved by further knocking out each of $v_2, \ldots, v_{n-2}$ (since $1-x \geq \frac{1}{2}$, two adjacent vertices with weight $1-x$ cannot be left, but since $1-x < 1$, v_n and v_{n-1} can be left); the cost of this particular dismantling is $f(x) + (n-3)\ f(1-x)$. Thus the cost of dismantling with these weights is $\min[(n-1)f(1-x), f(x) + (n-3)f(1-x)]$.

Claim 1 If $f(0) \leq 2f(1)$, then for each $x \in [0, \frac{1}{2}]$, $f(x) + (n-3)f(1-x) \leq f(1) + (n-2)f(\frac{1}{2})$.

Proof of Claim 1 If $f(0) \leq 2f(1)$ and $0 \leq x \leq \frac{1}{2}$, then $\frac{1}{2} \leq 1-x \leq 1$ and so $2f(1) \geq f(0) \geq f(x) \geq f(\frac{1}{2}) \geq f(1-x) \geq f(1)$. From this long inequality, we get that $f(x)+(n-3)f(1-x) = f(x)-f(1-x)+(n-2)f(1-x) \leq 2f(1)-f(1-x)+(n-2)f(\frac{1}{2}) \leq 2f(1)-f(1)+(n-2)f(\frac{1}{2}) = f(1)+(n-2)f(\frac{1}{2})$ as in the statement of the claim. □

Suppose that $2f(1) \geq f(0)$. By definition, $M_3 = M_2 \leq \overline{SH}(K_n-e, f)$. The following observations will be helpful later. Because f is non-increasing, $2f(1-x) \geq f(x)$ for each $x \in (0, \frac{1}{2}]$, so $(n-1)f(1-x) \geq f(x)+(n-3)f(1-x)$, for each $x \in (0, \frac{1}{2}]$. Thus the cost of dismantling with this weighting is always $f(x)+(n-3)f(1-x)$ if $2f(1) \geq f(0)$, and the greatest of these costs is incurred when $f(x)+(n-3)f(1-x)$ achieves its max on $(0, \frac{1}{2}]$, if it does achieve a max there. If it doesn't, then $f(x)+(n-3)f(1-x)$, a continuous function, achieves its max on $[0, \frac{1}{2}]$ at 0, where its value is $f(0)+(n-3)f(1) \leq (n-1)f(1)$. In either case, the maximum value of $f(x)+(n-3)f(1-x)$ occurs on $[0, \frac{1}{2}]$ and by Claim 1 above, $f(x)+(n-3)f(1-x) \leq f(1)+(n-2)f(\frac{1}{2}) = M_2$.

If $f(0) > 2f(1)$ then, since $f(\frac{1}{2}) \leq 2f(\frac{1}{2})$, and $f(x) - 2f(1-x)$ is continuous, there is some $x \in (0, \frac{1}{2}]$ satisfying $f(x) = 2f(1-x)$, and since f is non-increasing, the values of $f(x)$ and $f(1-x)$ are the same for different values of x satisfying the equation $f(x) = 2f(1-x)$. For any such x, $f(x)+(n-3)f(1-x) = (n-1)f(1-x)$ is the cost of dismantling $K_n - e$ equipped with this particular weighting. Thus $M_3 \leq \overline{SH}(K_n-e, f)$ in the case $f(0) > 2f(1)$. Thus $M_3 \leq \overline{SH}(K_n-e, f)$ in any case. This completes the proof that $\max(M_1, M_2, M_3) \leq \overline{SH}(K_n-e, f)$.

Now we shall show that $\overline{SH}(K_n-e, f) \leq \max(M_1, M_2, M_3)$ by showing that for every $g : V(G) \to [0, 1]$, where $G = K_n - e$, there is a strict g-dismantling set S of vertices of $K_n - e$ such that $\sum_{v \in S} f(g(v)) \leq \max(M_1, M_2, M_3)$. (Note that we may as well suppose that all weights are ≤ 1.) So, suppose g is such a weighting function, and let $g(v_i) = x_i$, $1 \leq i \leq n$.

Claim 2 It suffices to consider only weighting functions g such that $g(v_{n-1}), g(v_n) \geq g(v_i), 1 \leq i \leq n-2$.

Proof of Claim 2 Suppose $g(v_n) = x_n < x_1 = g(v_1)$. Let us define $\hat{g}$ by $\hat{g}(v_i) = x_i$ for $2 \leq i \leq n-1$, $\hat{g}(v_n) = x_1$, and $\hat{g}(v_1) = x_n$. We will show that $\overline{m}(g, G) \leq \overline{m}(\hat{g}, G)$. Suppose that S is a cheapest strict $\hat{g}$-dismantling set of vertices; then $\sum_{v \in S} f(\hat{g}(v)) = \overline{m}(\hat{g}, G)$.

If $v_1, v_n \notin S$, or, if $v_1, v_n \in S$, then S is a strict g-dismantling set (in the case $v_1, v_n \notin S$, note that v_1 and v_n are in the same component of $G - S$; indeed, $G - S$ is connected in this case) and $\overline{m}(g, G) \leq \sum_{v \in S} f(g(v)) = \sum_{v \in S} f(\hat{g}(v)) = \overline{m}(\hat{g}, G)$. If $v_n \in S$, $v_1 \notin S$, then let $\tilde{S} = (S \setminus \{v_n\}) \cup \{v_1\}$. Note that $G - S$ is connected, since $v_1 \notin S$, so we have $\sum_{v \in V(G - \tilde{S})} f(g(v)) = \sum_{v \in V(G-S)} f(\hat{g}(v)) < 1$. Thus, $\tilde{S}$ is a strict g-dismantling set, and we have $\overline{m}(g, G) \leq \sum_{v \in \tilde{S}} f(g(v)) = \sum_{v \in S} f(\hat{g}(v)) = \overline{m}(\hat{g}, G)$. For the last case, if $v_1 \in S$, $v_n \notin S$, then S is clearly a strict g-dismantling set, and we have $\overline{m}(g, G) \leq \sum_{v \in S} f(g(v)) \leq \sum_{v \in S} f(\hat{g}(v)) = \overline{m}(\hat{g}, G)$. This establishes the claim. $\square$

So, we may as well assume that $x_1 \leq x_2 \leq \cdots \leq x_{n-1} \leq x_n \leq 1$. If $x_n = 1$, then v_n must be removed, for strict dismantling, and then the $(n-1)$-clique $G - v_n$ must be dismantled, which can be accomplished at a cost no greater than $\overline{SH}(K_{n-1}, f)$.

Thus, if $x_n = 1$, $\overline{m}(g, G) \leq f(1) + \max_{1 \leq k \leq n-1} (n-k) f(\frac{1}{k}) \leq \max[M_1, M_2]$. So assume that $x_1 \leq \cdots \leq x_n < 1$.

If $\sum_{i=1}^{n} x_i < 1$ take $S = \emptyset$. Otherwise, there is some $k \in \{2, \ldots, n\}$ such that $\sum_{i=1}^{k-1} x_i < 1 \leq \sum_{i=1}^{k} x_i \leq k x_k$ (so, $x_k \geq \frac{1}{k}$). Take $S = \{v_k, \ldots, v_n\}$ which is a strict g-dismantling set with cost $\sum_{i=k}^{n} f(x_i) \leq (n-k+1) f(\frac{1}{k}) \leq M_1$, unless $k = 2$. Let us now assume that $k = 2$. If $x_1 \geq \frac{1}{2}$, then $\frac{1}{2} \leq x_i < 1$, $1 \leq i \leq n$. Dismantle by knocking out $v_1, \ldots, v_{n-2}$ at a cost of $\sum_{i=1}^{n-2} f(x_i) \leq (n-2) f(\frac{1}{2}) \leq M_2$. Therefore, we may assume that $x_1 < \frac{1}{2}$. Since $k = 2$ we have $1 \leq x_1 + x_2$, so, $1 - x_1 \leq x_2 \leq \cdots \leq x_n$. Take $S_1 = \{v_2, \ldots, v_n\}$, which is a strict g-dismantling set with cost $\sum_{i=2}^{n} f(x_i) \leq (n-1) f(1 - x_1)$ and take $S_2 = \{v_1, \ldots, v_{n-2}\}$, which costs $\sum_{i=1}^{n-2} f(x_i) \leq f(x_1) + (n-3) f(1-x_1)$. So,

$$\overline{m}(g, G) \leq \min[(n-1) f(1-x_1), f(x_1) + (n-3) f(1-x_1)] \leq M_3,$$

by Claim 1 above, and associated arguments. [To see the inequality in the case when $f(0) > 2f(1)$ observe that $f(1-x)$ is a non-decreasing function of x, and $(f(x) + (n-3) f(1-x)) - (n-1) f(1-x) = f(x) - 2f(1-x)$ is therefore a non-increasing function of x. The value of the difference, $h(x) = f(x) - 2f(1-x)$, is positive when $x = 0$, and non-positive when $x = \frac{1}{2}$, and is therefore equal to zero at some $\tilde{x} \in (0, \frac{1}{2}]$, since f is continuous. Because $h(x)$ is non-increasing on $[0, \frac{1}{2}]$, for $x_1 < \frac{1}{2}$, $\min[(n-1) f(1-x_1)$, $f(x_1) + (n-3) f(1-x_1)] = \min[(n-1) f(1-x_1), (n-1) f(1-x_1) + h(x_1)] \leq (n-1) f(1-\tilde{x}) = M_3$.] This completes the proof that $\overline{SH}(K_n - e, f) \leq \max(M_1, M_2, M_3)$. $\square$

Corollary (of the proof). For any non-increasing, non-negative, continuous cost function f, and any $n \geq 3$, one of the following is a critical (most costly) weighting of $K_n - e$, with respect to f:

1. a constant weighting $1/k$, for some $k \in \{1, 3, \ldots, n\}$;

2. a weight of 1 on one of the vertices of degree $n-2$, with $\frac{1}{2}$ on all the remaining vertices;

3. a weight of x on one vertex of degree $n-1$ and a weight of $1-x$ on the other vertices, if $f(0) > 2f(1)$, where $x \in (0, \frac{1}{2}]$ satisfies $f(x) = 2f(1-x)$.

2.2 Examples of different cost functions for each $\mathbf{M_i}$

In this section, we present different cost functions f_i such that $\overline{SH}(K_n - e, f_i) = M_i$ for $i = 1, 2, 3$.

Let $f_1(x) = \frac{1}{x}$. This cost function gives us the following values for M_1, M_2 and M_3; $n \geq 4$.

$$M_1 = \left\lfloor \frac{(n+1)^2}{4} \right\rfloor, \qquad M_2 = 2n-3, \qquad M_3 = \frac{3}{2}(n-1).$$

It can easily be seen that for all $n \geq 4$, $M_1 > \max(M_2, M_3)$ and so by Theorem 2.1 we have that $\overline{SH}(K_n - e, f_1) = M_1$.

Now let $f_2(x) = \begin{cases} 3 & \text{if } 0 \leq x \leq \frac{1}{2} \\ -4x+5 & \text{if } \frac{1}{2} < x \leq 1 \end{cases}$. From this cost function, for $n \geq 3$, we get the following values:

$$M_1 = 3n-6, \qquad M_2 = 3n-5, \qquad M_3 = \tfrac{3}{2}(n-1).$$

Once again it can easily be checked that for all $n \geq 3$, $M_2 > \max(M_1, M_3)$ and thus by Theorem 2.1 we have that $\overline{SH}(K_n - e, f_2) = M_2$.

$$\text{Finally, let } f_3(x) = \begin{cases} -x(n+2)(n+1)+n+3 & \text{if } 0 \leq x < \frac{1}{n+1} \\ 1 & \text{if } \frac{1}{n+1} \leq x \leq \frac{n+1}{n+2} \\ -(n+2)(x-1) & \text{if } \frac{n+1}{n+2} < x \leq 1 \end{cases}$$

This cost function yields the following values:

$$M_1 = n-2, \qquad M_2 = n-2, \qquad M_3 = n-1 \qquad .$$

In this case, obtaining the values of the M's is a bit difficult but it is quite clear that $M_3 > \max(M_1, M_2)$. By Theorem 2.1, $\overline{SH}(K_n - e, f_3) = M_3$.

2.3 A counterexample to the constant weights on orbits conjecture

The constant weights on orbits conjecture (CWOC) has been posed as follows:

> For all continuous non-increasing cost functions f and all graphs G, there is an optimal weighting of $V(G)$ which is constant on each orbit of $V(G)$ under $Aut(G)$, the group of graph automorphisms of G.

It takes little imagination to see that this conjecture, if true, would be quite useful in computing the SH numbers of intersecting cliques. However, as stated, the CWOC is not true and a counterexample lies among the cost functions f on $K_n - e$. In fact, perhaps any cost function f that yields $\overline{SH}(K_n - e, f) = M_2$ is such a counterexample. The one which we present and prove is such a counterexample is the cost function f_2 which was first presented in section 2.2 (and which we will present again later in this section).

In section 2.2, we presented the values of M_1, M_2 and M_3 obtained from using f_2 as our cost function. From these values, it can easily be verified that $\max[M_1, M_2, M_3] =$

$M_2 = 3n-5$, for $n \geq 3$. We will now show that with f_2 as our cost function, any weighting that is constant on each orbit of $G = K_n - e$ is not an optimal weighting (i.e. $\overline{SH}(G, f_2) > m_{f_2}(g, G)$.

Proposition 2.1 *Let g be any weighting that is constant on each orbit of $G = K_n - e$. Then $\overline{SH}(G, f_2) > m_{f_2}(g, G)$ where f_2 is defined as follows:*

$$f_2(x) = \begin{cases} 3 & \textit{if } 0 \leq x \leq \frac{1}{2} \\ -4x+5 & \textit{if } \frac{1}{2} < x \leq 1 \end{cases}$$

Proof. Let, as before, the vertices of $K_n - e$ be $v_1, \ldots, v_n$ with $e = v_n, v_{n-1}$. Let $a, b \in [0,1]$ and let g be the weighting such that for each $1 \leq i \leq n$

$$g(v_i) = \begin{cases} b & \text{if} \quad 1 \leq i \leq n-2 \\ a & \text{if} \quad n-2 < i \leq n \end{cases}$$

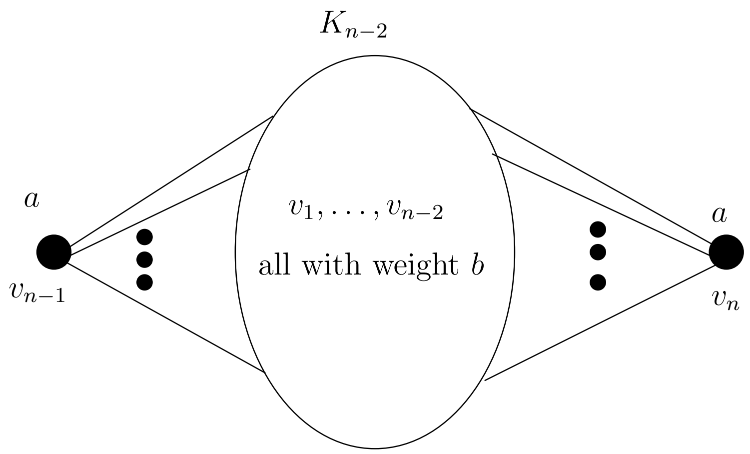


Figure 2.1: $K_n - e$ with constant on orbits weighting g

If $a = b$, then we find that $\overline{SH_0}(K_n - e, f_2) = \max_{1 \leq k \leq n} p(k, K_n - e) f_2(\frac{1}{k}) = \max[n, 3(n-2)] = 3n-6$. Clearly, $3n-6 < 3n-5 = M_2$ and so any weighting that is constant on the vertices of $K_n - e$ is not an optimal weighting. Otherwise, if $a \neq b$, then by Claim 2 in the proof of Theorem 2.1 we may as well assume that $a > b$. This leaves us a few cases to consider. In each of the following cases, we look at how we can choose values of a and b so as to maximize the cheapest dismantling cost of G with respect to the weighting g and cost function f_2.

Case 1: $b < a \leq \frac{1}{2}$ In this case, $f_2(g(v)) = 3$ for all $v \in V(K_n - e)$. So we want to force the removal of as many vertices as we can. Choosing values of a and b so that $2a + b \geq 1$ will force the removal of $n-2$ vertices which will be the best we can do (since $a + b < 1$). This results in $m_{f_2}(g, K_n - e) = 3(n-2) = 3n - 6 < 3n - 5 = M_2 = \overline{SH}(G, f_2)$.

Case 2: $b \leq \frac{1}{2} < a < 1$ Since for all $b \leq \frac{1}{2}$, $f(b) = f(\frac{1}{2})$ we may as well choose $b = \frac{1}{2}$. For any $\frac{1}{2} < a < 1$, we can force the removal of no more than $n-2$ vertices and so $m_{f_2}(g, K_n - e) = 3(n-2) = 3n - 6 < 3n - 5 = M_2 = \overline{SH}(G, f_2)$.

Case 3: $b \leq \frac{1}{2} < a = 14$ In this case, $f_2(b) = 3$ and $f_2(a) = 1$. The two vertices of weight a must be removed for strict dismantling and so we wish to force the removal of as many of the $n-2$ vertices of weight b. Choosing $b = \frac{1}{2}$ forces the removal of $(n-2)-1$ of the vertices of weight b which is the best we can do. $m_{f_2}(g, K_n - e) = 3(n-3) + 2 = 3n - 7 < 3n - 5 = M_2 = \overline{SH}(G, f_2)$.

Case 4: $\frac{1}{2} < b < a < 1$ For strict dismantling, all remaining components after vertex removal must contain no more than one vertex. Of all cheapest dismantling costs, the maximum is achieved when the K_{n-2} is removed and so $m_{f_2}(g, K_n - e) = f_2(b)(n-2) < 3(n-2) = 3n - 6 < 3n - 5 = M_2 = \overline{SH}(G, f_2)$.

Case 5: $\frac{1}{2} < b < a = 1$ For strict dismantling, v_{n-1} and v_n must be removed at a cost of $2f_2(a) = 2$. Since $b > \frac{1}{2}$, $(n-2) - 1 = n - 3$ vertices of the K_{n-2} must be removed at a cost of $f_2(b)(n-3)$ and so $m_{f_2}(g, K_n - e) = f_2(b)(n-3) + 2 < 3(n-3) + 2 = 3n - 7 < 3n - 5 = M_2 = \overline{SH}(G, f_2)$. $\square$

Chapter 3

Some General Results with Applications to Intersecting Cliques

In what follows, G will be an arbitrary finite simple graph with vertex set $V(G)$, of order $n(G) = |V(G)|$. For $u \in V(G)$, $\deg(u)$ will denote the degree of u in G and $N_G(u)$ will denote the set of vertices adjacent to u in G.

3.1 A useful proposition

The following is a generalization of Claim 2 in the proof of Theorem 2.1.

Proposition 3.1 *Suppose that $T \subseteq V(G)$ and for each $u \in T$ $deg(u) = n(G) - 1$. For any continuous f, there is an optimal weighting g of G satisfying $g(u) \leq g(v)$ for each $u \in T$ and each $v \notin T$.*

Proof. Let g be an optimal weighting of G, with respect to f (i.e., $\overline{m}(g, G) = \overline{SH}(G, f)$). Now suppose that for some $v \notin T, u \in T, g(v) < g(u)$. Define $\hat{g}$ by $\hat{g}(u) = g(v)$, $\hat{g}(v) = g(u)$, and $\hat{g} = g$ on $V(G)\backslash\{u, v\}$. We shall show that $\hat{g}$ is an optimal weighting of G with respect to f. This will prove the proposition, since, even if $\hat{g}$ does not satisfy the requirement of the conclusion, we can go on switching values until we arrive at a weighting that does satisfy that requirement, and this final weighting will be optimal.

Let $S \subseteq V(G)$ be a strict $\hat{g}$-dismantling set such that $\overline{m}(\hat{g}, G) =$ $\sum_{w \in S} f(\hat{g}(w))$. If neither u nor v, or if both u and v, belong to S, then S is a strict g-dismantling set, whence $\overline{SH}(G, f) \geq \overline{m}(\hat{g}, G) = \sum_{w \in S} f(\hat{g}(w)) = \sum_{w \in S} f(g(w)) \geq$

$\overline{m}(g, G) = \overline{SH}(G, f)$, and it follows that $\hat{g}$ is an optimal weighting of G, with respect to f. This leaves two cases to consider.

$v \in S,\ u \notin S$: in this case, because $u \in T$ is not in S, $G - S$ is connected, and $\sum_{w \in V(G)\backslash S} \hat{g}(w) < 1$. Let $\tilde{S} = (S\backslash\{v\}) \cup \{u\}$. Then $\sum_{w \in V(G)\backslash \tilde{S}} g(w) = \sum_{w \in V(G)\backslash S} \hat{g}(w) < 1$ so $\tilde{S}$ is a strict g-dismantling set, and so $\overline{m}(g, G) \leq$ $\sum_{w \in \tilde{S}} f(g(w)) = \sum_{w \in S} f(\hat{g}(w)) = \overline{m}(\hat{g}, G)$. The conclusion that $\hat{g}$ is an optimal weighting follows as before.

$v \notin S,\ u \in S$: In this case, S is a strict g-dismantling set and $\overline{m}(g, G) \leq \sum_{w \in S} f(g(w)) \leq \sum_{w \in S} f(\hat{g}(w)) = \overline{m}(\hat{g}, G)$ because $\hat{g}(u) < g(u)$ and f is non-increasing. The conclusion that $\hat{g}$ is optimal follows as before. □

For $0 < s < l, r$, denote by $G(l, r, s)$ the graph consisting of a K_l and a K_r intersecting in a K_s, as indicated in Figure 3.1 below.

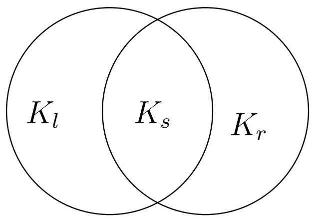


Figure 3.1: $G(l, r, s)$

Corollary 3.1 *For any continuous f, there is an optimal weighting g of $G(l, r, s)$ with $g(u) \leq g(v)$ for every u in the K_s and every v not in the K_s.*

Proof. The proof of this corollary follows immediately from Proposition 1 by taking $T = V(K_s)$. □

3.2 Limits of Optimal Weightings

Proposition 3.2 *If f is continuous and g_n is an optimal weighting of G for each $n = 1, 2, 3, \ldots$ and $g_n(v) \to g(v)$ as $n \to \infty$ for each $v \in V(G)$, then g is an optimal weighting of G.*

Proof. Since each weighting g_n is an optimal weighting of G, $\overline{m}_f(g_n, G) = \overline{SH}(G, f)$ for each n. Now, let S be a strict g-dismantling set of vertices of least cost with $\sum_{v \in S} f(g(v)) = \overline{m}_f(g, G) \leq \overline{SH}(G, f)$. We now show that S is a strict g_n-dismantling set for all n sufficiently large by showing that for such n and for each component H of $G - S$, $\sum_{v \in V(H)} g_n(v) < 1$.

Since S is a strict g-dismantling set, then for each component H of $G - S$ we have $\lim_{n \to \infty} \sum_{v \in V(H)} g_n(v) = \sum_{v \in V(H)} g(v) < 1$. Thus, there exists some integer N_H such that $n \geq N_H$ implies that $\sum_{v \in V(H)} g_n(v) < 1$. There are only finitely many such H; take $N = \max\limits_{H \text{ a component of } G\text{-}S} N_H$; then $n \geq N$ implies that for each such H , $\sum_{v \in V(H)} g_n(v) < 1$.

Now for all n sufficiently large we have the following:

$\overline{SH}(G, f) = \overline{m}_f(g_n, G) \leq \sum_{v \in S} f(g_n(v)) \to \sum_{v \in S} f(g(v)) = \overline{m}_f(g, G)$.

This gives us that $\overline{m}_f(g, G) \geq \overline{SH}(G, f)$. Since $\overline{m}_f(g, G) \leq \overline{SH}(G, f)$, g is an optimal weighting of G. □

<u>Definition</u> $f : I \to R$ is concave on an interval I if and only if for all $x, y \in I$ and $t \in [0, 1]$, $f(tx + (1 - t)y) \geq tf(x) + (1 - t)f(y)$.

Proposition 3.3 *Suppose f is continuous and concave on $[0,1]$ and $S \subseteq V(G)$ satisfies: for each $u, v \in S, N_G(u)\backslash\{v\} = N_G(v)\backslash\{u\}$. Suppose either that $f(1) = 0$ or that S induces a clique in G. Then for any optimal weighting g of G, there is another optimal weighting $\tilde{g}$ of G which is constant on S, agrees with g on $V(G)\backslash S$, and $\min_{v\in S} g(v) \leq \tilde{g}|_S \leq \max_{v\in S} g(v)$. Further, if $S_1, S_2, \ldots, S_k$ are disjoint sets of vertices of G each satisfying the suppositions above, then there is an optimal weighting $\hat{g}$ of G which is constant on each $S_i, i = 1, 2, \ldots, k$, agrees with g at vertices not in any S_i, and, for each i, $\min_{v\in S_i} g(v) \leq \hat{g}|_{S_i} \leq \max_{v\in S_i} g(v)$.*

Proof. Suppose we have an optimal weighting g of G with respect to f, so $\overline{m}_f(g, G) = \overline{SH}(G, f)$. Let $S \subseteq V(G)$ be such that for each $u, v \in S$ with $u \neq v$, $N_G(u) \setminus \{v\} = N_G(v) \setminus \{u\}$. We further suppose that either $f(1) = 0$ or S induces a clique in G. If g is constant on S we can take $\tilde{g} = g$, so assume that g is not constant on S. Let u_0, $u_1 \in S$ such that $g(u_0) = \min\limits_{v\in S} g(v) < g(u_1) = \max\limits_{v\in S} g(v)$.

Now let us define a weighting $\hat{g}$ by $\hat{g} = g$ except at u_0 and u_1 where $\hat{g}(u_0) = \hat{g}(u_1) = \frac{g(u_0)+g(u_1)}{2}$. We will show that $\overline{m}_f(\hat{g}, G) \geq \overline{m}_f(g, G)$, which implies $\overline{m}_f(\hat{g}, G) = \overline{SH}(G, f)$.

Let $T \subseteq V(G)$ be a strict $\hat{g}$-dismantling set of least cost, so $\overline{m}_f(\hat{g}, G) = \sum_{v\in T} f(\hat{g}(v))$. We have four cases to consider:

<u>Case 1: $u_0 \notin T$, $u_1 \in T$</u> In this case, for any connected component H of $G - T$, $\sum_{u\in V(H)} g(u) \leq \sum_{u\in V(H)} \hat{g}(u) < 1$ since $g(u_0) < \hat{g}(u_0)$ and $u_0 \notin T$. Thus T is a strict g-dismantling set, so $\overline{m}_f(g, G) \leq \sum_{u\in T} f(g(u)) \leq \sum_{u\in T} f(\hat{g}(u)) = \overline{m}_f(\hat{g}, G)$ because f is non-increasing.

<u>Case 2: $u_0 \in T$, $u_1 \notin T$</u> If this occurs, we can find another set $T_1 = (T \setminus \{u_0\}) \cup \{u_1\}$ with a dismantling cost equal to $\sum_{v \in T} f(\hat{g}(v))$, because $\hat{g}(u_0) = \hat{g}(u_1)$. T_1 is a strict $\hat{g}$-dismantling set of vertices because T is, and u_0 and u_1 have the same neighbors other than themselves. Since, by assumption, T is a cheapest strict $\hat{g}$-dismantling set, T_1 must be one as well. Further, T_1 satisfies the requirement defining Case 1, so we are done in this case.

<u>Case 3: $u_0 \notin T$, $u_1 \notin T$</u> If u_0 and u_1 are adjacent, then u_0 and u_1 will be in the same component of $G - T$. Then for every connected component H of $G - T$, $\sum_{u \in V(H)} g(u) = \sum_{u \in V(H)} \hat{g}(u) < 1$. So T is a g-dismantling set, whence $\overline{m}_f(g, G) \leq \sum_{v \in T} f(g(v)) = \sum_{v \in T} f(\hat{g}(v)) = \overline{m}_f(\hat{g}, G)$. Now if u_0 and u_1 are not adjacent, then S does not induce a clique in G, so $f(1) = 0$. Now u_0 and u_1 may possibly not be in the same component of $G - T$. If they are in the same component, then T is a strict g-dismantling set and we are done. If they are not, then u_0 and u_1 are isolated vertices in $G - T$ because they have the same neighbor sets in G. We know that $\sum_{u \in V(H)} g(u) \leq \sum_{u \in V(H)} \hat{g}(u) < 1$ for every connected component H of $G - T$ except the H consisting of the vertex u_1. Now if $g(u_1) < 1$ then T is a g-dismantling set and we are done. If $g(u_1) = 1$ then $f(g(u_1)) = 0$ and so $T \cup \{u_1\}$ is a strict g-dismantling set with the same cost as T. We have that $\overline{m}_f(g, G) \leq \sum_{v \in T \cup \{u_1\}} f(g(v)) = \sum_{v \in T} f(g(v)) = \sum_{v \in T} f(\hat{g}(v)) = \overline{m}_f(\hat{g}, G)$.

<u>Case 4: $u_0 \in T$, $u_1 \in T$</u> In this case, it is clear that for every connected component H of $G - T$, $\sum_{u \in V(H)} g(u) = \sum_{u \in V(H)} \hat{g}(u) < 1$ and so T is a strict g-dismantling set. Now, $\overline{m}_f(\hat{g}, G) = \sum_{v \in T} f(\hat{g}(v)) = \sum_{v \in T} f(g(v)) - [f(g(u_0)) + f(g(u_1))] + 2[f(\frac{g(u_0)+g(u_1)}{2})]$ $\geq \overline{m}_f(g, G) - f(g(u_0)) - f(g(u_1)) + 2[f(\frac{1}{2}g(u_0) + \frac{1}{2}g(u_1))] \geq \overline{m}_f(g, G)$ since f is concave. This completes the proof that $\hat{g}$ is optimal.

Now for any weighting h of G let $d(h) = \max_{u\in S} h(u) - \min_{u\in S} h(u)$. We will show that for every optimal weighting h of G with $d(h) > 0$ there is another optimal weighting $\tilde{h}$ satisfying the following: $d(\tilde{h}) < d(h)$, $\tilde{h}(v) = h(v)$ for all $v \in V(G)\backslash S$, and $\min_{v\in S} h(v) \le \tilde{h}|_S \le \max_{v\in S} h(v)$.

Let h be any optimal weighting of G with $d(h) > 0$ and let $h_1 = \hat{h}$, obtained as above. By the definition of $\hat{h}$, $h(v) = h_1(v)$ for all $v \in V(G-S)$, and clearly $\min_{v\in S} h(v) \le \min_{v\in S} h_1(v) \le \max_{v\in S} h_1(v) \le \max_{v\in S} h(v)$. Therefore $d(h_1) \le d(h)$. If $d(h_1) < d(h)$, take $\tilde{h} = h_1$. Otherwise, we have $d(h_1) = d(h) > 0$, which implies that $\max_{v\in S} h_1(v) = \max_{v\in S} h(v)$ and $\min_{v\in S} h_1(v) = \min_{v\in S} h(v)$. Note that the set of vertices in S where h_1 achieves its maximum is the set of vertices in S where h achieves its maximum, minus one vertex, and the same holds for the sets of points where h and h_1 achieve their minimum on S.

Let $h_2 = \hat{h_1}$. If $d(h_2) < d(h)$, take $\tilde{h} = h_2$. Otherwise, continue, letting $h_3 = \hat{h_2}$, and so on. In going from h_{i-1} to h_i, one vertex of S at which h_{i-1} is maximal has its weight decreased, and one vertex at which h_{i-1} is minimal has its weight increased, and these are vertices at which h is maximal, respectively minimal. Since there are only a finite number of such vertices, we must have $d(h_i) < d(h)$ eventually. It is straightforward to see that $\tilde{h} = h_i$ has the desired properties.

Suppose that g is an optimal weighting of G and suppose that $W = \{h : V(G) \to [0,1]; h$ is an optimal weighting of G, $h \equiv g$ on $V(G)\backslash S$, and $\min_{v\in S} g(v) \le h|_S \le \max_{v\in S} g(v)\}$ contains no weightings which are constant on S. Let $d = \inf[d(h); h \in W]$. By the meaning of inf, for each positive integer k, there is a weighting $h_k \in W$ with $d \le d(h_k) < d + \frac{1}{k}$. Then (h_k) is a sequence of optimal weightings. Since the h_k are bounded functions on a finite set, $V(G)$, the sequence (h_k) has a convergent subsequence; to avoid proliferation of subscripts, let us suppose that (h_k) itself is convergent, i.e., for each $v \in V(G)$, $(h_k(v))$ converges to some value $h(v)$.

By Proposition 3.2, the weighting h is optimal, and it clearly satisfies the other requirements for membership in W. We claim that $d(h) = d$. It is certainly clear by the definition of d that $d \leq d(h)$. Let $u_0, u_1 \in S$ be such that $h(u_1) = \max_{u \in S} h(u)$ and $h(u_0) = \min_{u \in S} h(u)$. Then $d(h) = h(u_1) - h(u_0) = \lim_{k\to\infty}(h_k(u_1) - h_k(u_0)) \leq \lim_{k\to\infty} d(h_k)$ since for each h_k, $d(h_k)$ is the maximum distance between the values of h_k at two vertices in S. Thus, $d(h) \leq d$. Since $d \leq d(h)$ and $d(h) \leq d$ then $d(h) = d$ as claimed. If $d = 0$ then h is an optimal weighting with $d(h) = 0$, so such a weighting satisfying all conditions of the proposition must exist after all, contrary to supposition. If $d(h) > 0$ then by previous remarks there is another optimal weighting $\tilde{h} \in W$ with $d(\tilde{h}) < d(h) = d$. But this contradicts the definition of d, by which d is a lower bound of a collection of numbers of which $d(\tilde{h})$ is one. So there must be an optimal weighting of G which is constant on S satisfying all the conditions of the proposition after all.

Now suppose that $S_1, \ldots, S_k$ are pairwise disjoint sets of vertices, each satisfying the conditions of the proposition. We proceed by induction on k. We may as well suppose that $k > 1$. By the induction hypothesis, there is an optimal weighting $\hat{g}_{k-1}$ of G, with respect to f, which is constant on each of $S_1, \ldots, S_{k-1}$, agrees with g off $\bigcup_{i=1}^{k-1} S_i$, and whose constant value on S_i is between the max and min values of g on S_i, for each $i = 1, \ldots, k-1$. If we let S_k play the role of S and $\hat{g}_{k-1}$ replace g in the argument above we get an optimal weighting $\hat{g}$ that satisfies the conclusion of the proposition. $\square$

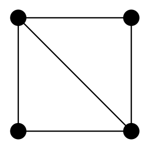
Corollary 3.2 *If f is continuous and concave on $[0, 1]$, there is an optimal weighting of $G(l, r, s)$ satisfying the conclusion of Corollary 3.1 which is constant on each of $K_l \backslash K_s$ and $K_r \backslash K_s$, and K_s.*

Proof. Let g be an optimal weighting of $G(l, r, s)$ satisfying the conclusion of Proposition 3.1 and which is possibly not constant on $K_l - K_s, K_r - K_s$ and/or K_s. Now let $S_1 = V(K_l - K_s), S_2 = V(K_r - K_s)$ and $S_3 = V(K_s)$. S_1, S_2 and S_3 are disjoint sets of vertices which satisfy the conditions of Proposition 3.3. Applying Proposition 3.3 to $G(l, r, s)$ with the weighting g of Corollary 1 will then yield a new optimal weighting $\hat{g}$ which will satisfy the conclusion of the corollary. $\square$

3.3 An application of Proposition 3.3 and other useful results

The goal of this section is to present an application of some of the results presented here in Chapter 3. We do this with special emphasis on an application of Proposition 3.3 in order to illustrate how it can be used to simplify and reduce the number of calculations that are often required in computing the Shields-Harary numbers. It should be noted that Proposition 3.3 arose out of an attempt to compute the SH numbers of two intersecting cliques for general continuous cost functions with non-constant weightings. The main usefulness of Proposition 3.3 lies in the fact that for any graph G containing a set or disjoint sets of vertices satisfying the requirements of the proposition, if the cost function is concave we can restrict the search for optimal weightings to weightings constant on that set or those sets. This reduces the number of unknowns in the calculation of $SH(G, f)$. We illustrate this in proving the following corollary.

Corollary 3.3 *Let* $G =$



.

Then $\overline{SH}(G, 1-x) = \frac{4}{3}$

Proof. Let S_1 consist of the two vertices of G of degree 2 and let S_2 consist of the two vertices of G of degree 3. Since $1 - x$ is concave on $[0, 1]$ and S_1 and S_2 are disjoint sets of vertices satisfying the requirements of Proposition 3.3, then as a result of the proposition, the computation of $\overline{SH}(G, 1-x)$ requires only considering those weightings which are constant on each of S_1 and S_2.

Now suppose that our weighting g puts weights a and b on the vertices of S_1 and S_2 respectively. Since $G = K_4 - e$, then by Corollary 3.1, we know that there is an optimal weighting with $f(x) = 1 - x$ such that $b \leq a$. So we may as well assume that $a \geq b$.

Suppose that S is a cheapest g-dismantling set of vertices of G. We have several cases to consider:

<u>Case 1: $a, b \geq \frac{1}{2}$</u> We may as well let $a = b = \frac{1}{2}$ in this case. This gives us that $S = S_2$ with a g-dismantling cost of $2f(\frac{1}{2}) = 2(1 - \frac{1}{2}) = 1$.

<u>Case 2: $a \geq \frac{1}{2}, b < \frac{1}{2}$</u> We can drive the cheapest dismantling cost as high as possible by choosing $a = \frac{1}{2}$. This yields $S = S_1$ with a g-dismantling cost of $2f(\frac{1}{2}) = 1$.

<u>Case 3: $a, b < \frac{1}{2}$</u> If $2b + a < 1$ then $S \subsetneq S_1$ with a g-dismantling cost of $f(a) = 1 - a < 1$. In this event, Cases 1 and 2 yield a higher cheapest dismantling cost. Otherwise, $2b + a \geq 1 \implies a \geq 1 - 2b$. Again, we can drive the cheapest dismantling cost as high

as possible by choosing $a = 1 - 2b$. This yields $S = S_1$ with a g-dismantling cost of $2f(a) = 2(1-a) = 2(1-1+2b) = 4b$. However, $a \geq b \implies 1-2b \geq b \implies b \leq \frac{1}{3}$. So to make $4b$ as high as possible, we choose $b = \frac{1}{3}$. Thus, if $2b + a \geq 1$, S has a g-dismantling cost of $\frac{4}{3}$.

As a result, $\overline{SH}(G, 1-x) = \max[1, 1, \frac{4}{3}] = \frac{4}{3}$. □

Note: Since $G = K_4 - e$, we could just as easily have used Theorem 2.1 to compute $\overline{SH}(G, 1-x)$. If so, the results would be that $\overline{SH}(G, 1-x) = \max[M_1, M_2, M_3] = \max[\max[0, \frac{4}{3}, \frac{3}{4}]], 1, 1] = \frac{4}{3}$ as computed above.

Clearly, Theorem 2.1 provides a shorter route in this particular case. Since $K_n - e$ is just a special case of two intersecting cliques (namely two K_3's intersecting in a K_2), this serves as no big surprise. However, in the more general setting, where Theorem 2.1 will not always apply, we can see how useful Proposition 3.3 can be in its use in the calculation above. In this particular case we only had to consider two unknown quantities as opposed to four.

We now consider some other applications of these propositions. If $G = G(3, 3, 1)$ and $f(x) = 1-x$, then $\overline{SH}(G, f) = \frac{3}{2}$. Corollary 3.2 tells us that there will be an optimal weighting of G as illustrated in Figure 3.2 where either (1)$a \geq b \geq x$ or (2)$b \geq a \geq x$. Clearly, we may look for a weighting satisfying (1) without loss of generality.

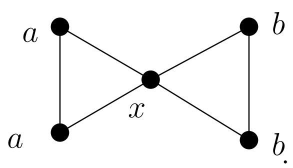


Figure 3.2: $G(3, 3, 1)$

The simplification provided by Corollary 3.2 in the problem of determining $\overline{SH}(G(l,r,s),f)$ for any concave cost function f is mainly to reduce the number of variables involved from $l+r-s(=5$ in this case) to 3. The additional restriction that $f(1)=0$ allows us the assumption that $a,b,x<1$. Even with these simplifications, and even with a particular cost function f, the analysis necessary to determine $\overline{SH}(G(l,r,s),f)$ and (what may be more important) an optimal weighting of the vertices of G, will be rather tedious and involved and we leave an illustration of the proof of $\overline{SH}(G(3,3,1),1-x)=\frac{3}{2}$ to the work done in [4].

We can explore this situation even further by considering the graph $G(3,3,1)$ with the cost function $f(x)=2-x$. In this case, it turns out that $\overline{SH}(G(3,3,1),2-x)=5$. Now, the analysis is complicated by the possibility of using 1 as a weight. Any vertex with weight 1 must be removed in strict dismantling and with $f(x)=2-x$ it turns out to be optimal to use 1 as a weight. In fact, two optimal weightings with $f(x)=2-x$ are given by $a=b=x=1$ and $a=1,b=x=\frac{1}{2}$.

In the case of a complete r-partite graph $K_{n_1,\ldots,n_r}, r\geq 2$, and a concave function satisfying $f(1)=0$, the application of Proposition 3.3 allows us to look for optimal weightings which are constant on each part of size $n_i\geq 2$, and constant on the clique formed by the parts with only one vertex. Thus, the number of variables is reduced from $n=\sum_{i=1}^{r} n_i$ either to $r-s+1$ or to r, if $s=\{i;n_i=1\}=0$. Thus, for the complete bipartite graphs $K_{m,n}$, except for $K_{1,1}=K_2$ and such a cost function, there are only two variables to worry about, the constant weights on each part.

Chapter 4

The Shields-Harary Numbers of $K_{m,n}$ for Some Cost Functions

4.1 Some generalities

Throughout, let $K_{m,n}$ have parts M and N with $|M| = m \leq n = |N|$, and let f be some continuous non-increasing cost function.

Lemma 4.1 *Suppose $a, b \in (0,1)$ and g is the weighting of the vertices of $G = K_{m,n}$ constantly equal to a on M and to b on N. Then there is a strict g-dismantling set S of least cost (with respect to f) satisfying one of the following:*

(i) $S = M$;

(ii) $S = N$;

(iii) $S \subsetneq M$;

(iv) $S \subsetneq N$.

Further, if $a > b$ then possibility (iv) can be excluded, and if $a \leq b$ then possibility (iii) can be excluded.

Proof. Since $a, b < 1$, choosing either $S = M$ or $S = N$ strictly g-dismantles G. Also, any set of vertices of G whose removal disconnects G will contain either M or N. Therefore, one of M or N is a strict g-dismantling set of least cost among those (if any) whose removal disconnects G.

Now we consider the cases in which there is a strict g-dismantling set $\hat{S}$ of minimum cost such that $G - \hat{S}$ is connected, and $\hat{S}$ is neither M nor N ($G - N$ is connected if $m = 1$, and $G - M$ is connected if $m = n = 1$).

<u>Case 1: $a > b$</u> If $\hat{S} \subsetneq M$, we are done, so suppose that $\hat{S} = \hat{M} \cup \hat{N}$, $\hat{M} \subsetneq M$, $\emptyset \neq \hat{N} \subsetneq N$. If $|\hat{N}| \geq |M - \hat{M}|$, then we can find a no more expensive g-dismantling set S of vertices by defining $S = M$. Otherwise, we can still find a cheaper g-dismantling set S of vertices by defining $S = (\hat{S} - \hat{N}) \cup \hat{M}_2$ where $\hat{M}_2 \subsetneq M - \hat{M}$ and $|\hat{M}_2| = |\hat{N}|$. Since, in either case, S consists only of vertices with weight a and $|S|$ is never more than $|\hat{S}|$, we can certainly do no better.

Thus, if $a > b$, then there is a strict g-dismantling set S of least cost such that either $S = M$ or $S \subsetneq M$.

<u>Case 2: $a \leq b$</u> If $\hat{S} \subsetneq N$ we are done, so, as in Case 1, suppose that $\hat{S} = \hat{M} \cup \hat{N}$, $\hat{N} \subsetneq N$, $\emptyset \neq \hat{M} \subsetneq M$. If $|\hat{M}| \geq |N - \hat{N}|$, then we can find a no more expensive g-dismantling set S of vertices by defining $S = N$. Otherwise, we can still find a no more expensive g-dismantling set S by defining $S = (\hat{S} - \hat{M}) \cup \hat{N}_2$ where $\hat{N}_2 \subsetneq N - \hat{N}$ and $|\hat{N}_2| = |\hat{M}|$. Since, in either case, S consists only of vertices of weight b and $|S|$ is never more than $|\hat{S}|$, we can certainly do no better.

So, if $a \leq b$, then there is a strict g-dismantling set S of least cost such that either $S = N$ or $S \subsetneq N$. $\square$

Lemma 4.2 *Among weightings constant on M and on N, there is one of greatest strict dismantling cost whose value on M is less than or equal to its value on N.*

Proof. Throughout, suppose $a < b \leq 1$ and let f be some continuous, non-increasing cost function and $G = K_{m,n}$. Let $\tilde{g}$ be a weighting of the vertices of G constantly equal to a on M and b on N. Now let g be a weighting of the vertices of G constantly equal to b on M and a on N. We will show $\overline{m}_f(\tilde{g}, G) \geq \overline{m}_f(g, G)$, which implies the result.

Let S be a cheapest strict g-dismantling set of vertices of G and let $\tilde{S}$ be a cheapest strict $\tilde{g}$ dismantling set of vertices of G . If $b = 1$, then because of the requirement of strict dismantling, $S = M$ and $\tilde{S} = N$; since $n \geq m$, our result follows in this case. Otherwise, $b < 1$ and so by Lemma 4.1, we only need consider the cases where $S = M$, $S = N$ or $S \subsetneq M$ and for $\tilde{S}$, $\tilde{S} = M$, $\tilde{S} = N$ or $\tilde{S} \subsetneq N$.

<u>Case 1: $S = M$</u> If $S = M$, then S has g-dismantling cost of $mf(b)$, so $\overline{m}_f(g, G) = mf(b)$. If $G - \tilde{S}$ is disconnected, then either $\tilde{S} = M$ and so $\overline{m}_f(\tilde{g}, G) = mf(a) \geq mf(b) = \overline{m}_f(g, G)$ or $\tilde{S} = N$ in which case $\overline{m}_f(\tilde{g}, G) = nf(b) \geq mf(b) = \overline{m}_f(g, G)$. Otherwise, $G - \tilde{S}$ is connected and so we may assume that $\tilde{S} \subsetneq N$. So let $|\tilde{S}| = k$. If $k \geq m$, then $\overline{m}_f(\tilde{g}, G) = kf(b) \geq mf(b) = \overline{m}_f(g, G)$. Otherwise, $k < m$.

Since S is a strict g-dismantling set, then $na + b \geq 1$. Now, since $\tilde{S}$ is a cheapest $\tilde{g}$ dismantling set, then it must be the case that $(n-k)b+ma < 1$. However, $(n-k)b+ma \geq na + b \geq 1$ whenever $b(n - (k + 1)) \geq a(n - m)$ which is true when $k < m$ since $n - (k+1) \geq n - m$ whenever $k < m$. This implies that $k \geq m$ and so we are done with this case.

<u>Case 2: $S = N$</u> If $S = N$, then S has g-dismantling cost of $nf(a)$. Clearly $mf(b) \leq nf(a)$, and M is a dismantling set for g, so this case can be dismissed.

<u>Case 3: $S \subsetneq M$</u> If $S \subsetneq M$, then $\overline{m}_f(g, G) = lf(b)$ where $l = |S|$. Now suppose that $\tilde{S} = N$, in which case $\overline{m}_f(\tilde{g}, G) = nf(b) \geq lf(b) = \overline{m}_f(g, G)$ or if $\tilde{S} = M$, then $\overline{m}_f(\tilde{g}, G) = mf(a) \geq lf(b) = m_f(g, G)$. So if $G - \tilde{S}$ is disconnected, then $\overline{m}_f(\tilde{g}, G) \geq \overline{m}_f(g, G)$. Now, if $G - \tilde{S}$ is connected, then we may assume that $\tilde{S} \subsetneq N$ and so let $|\tilde{S}| = k$. If $k \geq l$, then $\overline{m}_f(\tilde{g}, G) = kf(b) \geq lf(b) = \overline{m}_f(g, G)$. Since $\tilde{S}$ is a strict $\tilde{g}$-dismantling set, $(n - k)b + ma < 1$. Now, since S is a cheapest strict g-dismantling

set, we know that $(m-l+1)(b)+na \geq 1$. Therefore, $(m-l+1)b+na > (n-k)b+ma$, which implies that $0 \geq (n-m)(a-b) > (l-(k+1))b$. Thus, $l \geq k$. $\square$

Lemma 4.3 *Among all weightings of $G = K_{m,m}$ constantly equal to a on one partition of G and constantly equal to b on the other partition of G, there is one of greatest strict dismantling cost such that $a = b$.*

Proof. Let g be a weighting of $G = K_{m,m}$ such that weight a is assigned to the vertices of one partition, which we shall call M_a and weight b is assigned to the vertices of the other partition of G which we shall call M_b. Without loss of generality, we may as well assume that $a > b$. Now let $\tilde{g}$ be the weighting of G such that weight a is assigned to all of the vertices of G. We shall show that $\overline{m}_f(\tilde{g}, G) \geq \overline{m}_f(g, G)$.

Let $\tilde{S}$ be a cheapest strict $\tilde{g}$-dismantling set of vertices. By Lemma 4.1 and since $a > b$ either $\tilde{S} = M_a$ or $\tilde{S} \subsetneq M_a$ (likewise for the cheapest g-dismantling set of vertices). These are the two cases that we must consider.

If $\tilde{S} = M_a$, then it must be the case that $m(a)+a \geq 1$ and $\underline{m}_f(\tilde{g}, G) = mf(a)$. Since $\tilde{S}$ disconnects G, then $\tilde{S}$ is also a g-dismantling set of vertices, though not necessarily the cheapest g-dismantling set of vertices. If $mb+a \geq 1$ then since $ma+a \geq mb+a$, $\tilde{S}$ is also the cheapest g-dismantling set of vertices and so $m_f(g, G) = \underline{m}_f(\tilde{g}, G)$. Otherwise, the cheapest g-dismantling set of vertices is a proper subset of $\tilde{S}$ and so $m_f(g, G) < \underline{m}_f(\tilde{g}, G)$.

If $\tilde{S} \subsetneq M_a$ then $ma+a < 1$ and there is some $0 < k < m$ such that $(m-k)a+ma < 1$ and $(m-k+1)a+ma \geq 1$. This gives us that $m_f(\tilde{g}, G) = kf(a)$. Since $a \geq b$, $(m-k)a+mb \leq (m-k)a+ma < 1$ which implies that the cardinality of the cheapest g-dismantling set is no greater than the cardinality of $\tilde{S}$. As the vertices of both of these sets are coming from M_a (because $a \geq b$), $\underline{m}_f(g, G) \leq \underline{m}_f(\tilde{g}, G)$. $\square$

Corollary 4.1 *If f is non-increasing, continuous and concave on [0,1] and $f(1) = 0$, then $\overline{SH}(K_{m,m}, f) = \overline{SH}_0(K_{m,m}, f)$.*

Proof. Since M and N satisfy the requirements on the S_i in Proposition 3.3, it follows from that proposition that there is an optimal weighting of $K_{m,n}$ which is constant on each of M and N. By lemma 4.3, therefore, there is an optimal weighting of $K_{m,n}$ which is constant, which proves the claim. $\square$

Corollary 4.2 $\overline{SH}(G = K_{m,m}, 1 - x) = \dfrac{m^2}{m+1}$ *and the constant weighting $\frac{1}{m+1}$ is optimal.*

Proof. Since $f(x) = 1 - x$ is non-increasing, continuous and concave on $[0, 1]$, and $f(1) = 0$, then we can apply corollary 4.1 and so only need worry with weightings which are constant on G.

By the above paragraph, $\overline{SH}(G, 1 - x) = \overline{SH}_0(G, 1 - x) = \max\limits_{1 \le k \le n} p(k, G) f\left(\dfrac{1}{k}\right)$ (a result stated in Chapter 1). Now $p(k, G)$ is the minimum cardinality of a set of vertices of $V(G)$ such that each component of $G - S$ has $k-1$ or fewer vertices). In fact, $p(k, G)f(\frac{1}{k})$ is the dismantling cost when the vertices bear the constant weight $\frac{1}{k}$. For $G = K_{m,m}$, we get the following, with $n = 2m$:

$$p(1,G) = 2m$$
$$p(2,G) = m$$
$$p(3,G) = m$$
$$\vdots$$
$$p(m-1,G) = m$$
$$p(m,G) = m$$
$$p(m+1,G) = m$$
$$p(m+2,G) = m-1$$
$$p(m+3,G) = m-2$$
$$\vdots$$
$$p(m+k,G) = m-k+1$$
$$\vdots$$
$$p(2m-1,G) = 2$$
$$p(2m,G) = 1$$

So $\max\limits_{1\leq k\leq n} p(k,G)f(\frac{1}{k}) = \max\limits_{1\leq k\leq m}\left[p(k,G)f\left(\frac{1}{k}\right), p(m+k,G)f\left(\frac{1}{m+k}\right)\right]$

$$= \max_{1\leq k\leq m}\left[m\left(1-\frac{1}{k}\right), (m-k+1)\left(1-\frac{1}{m+k}\right)\right].$$

Now the value $m(1-\frac{1}{k})$ attains its maximum value when $k=2$ to get a value of $\frac{m}{2}$. The value $(m-k+1)(1-\frac{1}{m+k})$ attains its maximum value at $k=1$ (as it is a decreasing function of positive k) and that value is $(m)(1-\frac{1}{m+1}) = \frac{m^2}{m+1} > \frac{m}{2}$. As a result, $\overline{SH}(K_{m,m}, 1-x) = \max\limits_{1\leq k\leq n} p(k,G)f(\frac{1}{k}) = \dfrac{m^2}{m+1}$, and the constant weighting $\frac{1}{m+1}$ is optimal. $\square$

Lemma 4.4 *Suppose that $1 \leq m \leq n$. There is an integer $k \in \{1, \ldots, n-1\}$ satisfying $\frac{k(n-k)}{n-k+1} \geq m$ if and only if $n \geq m + 2\sqrt{m}$.*

Proof. Solving $\frac{x(n-x)}{n-x+1} \geq m$ for x, assuming $x \in [1, n-1]$, so that $n-x+1 \geq 2 > 0$, we get that $x^2 - (n+m)x + m(n+1) \leq 0$ which has a real solution x, $\frac{n+m-\sqrt{(n+m)^2-4m(n+1)}}{2} \leq x \leq \frac{n+m+\sqrt{(n+m)^2-4m(n+1)}}{2}$, if and only if $(n+m)^2 - 4m(n+1) = (n-m)^2 - 4m \geq 0$, i.e. if and only if $n - m \geq 2\sqrt{m}$ (recollect that $n - m \geq 0$), or $n \geq m + 2\sqrt{m}$.

Now suppose that $n \geq m + 2\sqrt{m}$. We want to know if $\left[\frac{n+m-\sqrt{(n-m)^2-4m}}{2}, \frac{n+m+\sqrt{(n-m)^2-4m}}{2}\right] \bigcap \{1, \ldots, n-1\} \neq \emptyset$.

First of all, it can easily be verified that $1 \leq \frac{n+m-\sqrt{(n-m)^2-4m}}{2} \leq n-1$, so $[c, d] = \left[\frac{n+m-\sqrt{(n-m)^2-4m}}{2}, \frac{n+m+\sqrt{(n-m)^2-4m}}{2}\right]$ will contain an integer among $1, \ldots, n-1$ unless $d < n-1$ and $d - c < 1$. If $d = c$, i.e. $(n-m)^2 - 4m = 0$, meaning $n = m + 2\sqrt{m}$ (note m is a perfect square), then $n \equiv m \bmod 2$ so $c = d = \frac{n+m}{2}$ is an integer, and is clearly in the right range (i.e., is among $1, \ldots, n-1$).

If $d > c$ then $(n-m)^2 - 4m > 0 \implies (n-m)^2 - 4m \geq 1 \implies \sqrt{(n-m)^2 - 4m} = d - c \geq 1$, and we are done. □

Lemma 4.5 $\overline{SH}(K_n, 1-x) = \max\limits_{k=1,\ldots,n-1} \dfrac{k(n-k)}{n-k+1}$.

Proof. From [2], we know that $\overline{SH}(K_n, 1-x) = \max\limits_{k=1,\ldots,n} (n-k+1)\, f\left(\frac{1}{k}\right)$ (where $f(x) = 1 - x$) $= \max\limits_{k=2,\ldots,n} (n-k+1) f\left(\dfrac{1}{k}\right) = \max\limits_{k=1,\ldots,n-1} kf\left(\dfrac{1}{n-k+1}\right) = \max\limits_{k=1,\ldots,n-1} k\left(1 - \dfrac{1}{n-k+1}\right)$
$= \max\limits_{k=1,\ldots,n-1} \dfrac{k(n-k)}{n-k+1}$. □

<u>Remark:</u> Applying a little calculus to $g(x) = \frac{x(n-x)}{n-x+1}$ shows that it achieves its max on $[1, n-1]$, $n \geq 3$, at $n + 1 - \sqrt{n+1}$ (and that max is $(\sqrt{n+1} - 1)^2$). It follows from this and the analysis of g that g achieves its max on the integers $1, \ldots, n-1$ at

either $n+1-\sqrt{n+1}$, if $n+1$ is a perfect square, or at one of the 2 integers closest to $n+1-\sqrt{n+1}$, namely, $n+1-\lceil\sqrt{n+1}\rceil$ or $n+1-\lfloor\sqrt{n+1}\rfloor$.

Proposition 4.1

$$\overline{SH}(K_{m,n},1-x)=\begin{cases} m & \textit{if} \quad n \geq m+2\sqrt{m} \\ \max[\frac{1}{n+1}\lfloor\frac{(n+m)^2}{4}\rfloor, \overline{SH}(K_n,1-x)] < m & \textit{if} \quad m \leq n < m+2\sqrt{m} \end{cases}$$

Proof. For any weighting of $K_{m,n}$ with weights from $[0,1]$, take M together with all vertices of N which are weighted 1, for a dismantling cost $\leq m$. Thus $\overline{SH}(K_{m,n},1-x) \leq m$.

Suppose that $m \leq n < m+2\sqrt{m}$. Let $k=\lfloor\frac{n+m}{2}\rfloor$. Weight each vertex of N with $b=\frac{k+1-m}{n+1}$, and each vertex of M with $a=1-\frac{k(n+m-k)}{m(n+1)}$. Note that a is positive because $k(n+m-k) \leq \frac{(n+m)^2}{4}$, so $\frac{k(n+m-k)}{m(n+1)} \leq \frac{(n+m)^2}{4m(n+1)} < 1$ because $m \leq n < m+2\sqrt{m}$ implies $(n-m)^2 < 4m$, so $(n+m)^2=(n-m)^2+4mn < 4m(n+1)$.

It is straightforward to verify that $m(1-a)=k(1-b)$ and that $ma+(n-k+1)b=1$. Therefore $K_{m,n}$ with this weighting can be dismantled by removing M, or k vertices of N, at the identical (minimum) cost of $k(1-b)=\frac{k(n+m-k)}{n+1}=\frac{1}{n+1}\lfloor\frac{(m+n)^2}{4}\rfloor$. Thus $\overline{SH}(K_{m,n},1-x) \geq \frac{1}{n+1}\lfloor\frac{(n+m)^2}{4}\rfloor$ if $m \leq n < m+2\sqrt{m}$.

Now suppose that $n \geq m+2\sqrt{m}$. By Lemma 4.4, for some $k \in \{1,\ldots,n-1\}$, $\frac{k(n-k)}{n-k+1} \geq m$. Weight M with 0 and each vertex of N with $\frac{1}{n-k+1}$. Then the only dismantling sets competing with M are the k-subsets of N; but each of these costs $k(1-\frac{1}{n-k+1})=\frac{k(n-k)}{n-k+1} \geq m$. Thus M is the cheapest dismantling set, at a cost of m, so we have $\overline{SH}(K_{m,n},1-x) \geq m$; thus $\overline{SH}(K_{m,n},1-x)=m$ if $n \geq m+2\sqrt{m}$.

Now suppose that $m \leq n < m+2\sqrt{m}$. We want to show that $\overline{SH}(K_{m,n},1-x) = \max[\frac{1}{n+1}\lfloor\frac{(n+m)^2}{4}\rfloor, \overline{SH}(K_n,1-x)]$ and that this value is $< m$.

By Proposition 3.3 and Lemma 4.2, there is an optimal weighting of the vertices of $K_{m,n}$ constant on each of M and N, with the weight on M no greater than that on N. Let a denote the weight on M and b the weight on N; keep in mind that $a \le b$.

Suppose that $a = 0$. We must have that $b < 1$ (if $b = 1$, we can dismantle by removing N, at no cost) and $nb \ge 1$ (otherwise, if $nb < 1$, the empty set is dismantling, again at no cost). Therefore, for some $k \in \{1, \ldots, n-1\}$, $(n-k)b < 1 \le (n-k+1)b$.

By Lemma 4.1, the cheapest dismantling set is either M (at a cost of m) or $S \subseteq N$, $|S| = k$. Since reducing b down to $\frac{1}{n-k+1}$ will not change that fact, but will increase the cost of S, by the fact that our weighting is optimal we may as well suppose that $b = \frac{1}{n-k+1}$. Then the cost of S is $k(1 - \frac{1}{n-k+1}) = \frac{k(n-k)}{n-k+1} < m$, by Lemma 4.4 (since $m \le n < m+2\sqrt{m}$), so S is the cheapest (or, cheaper) dismantling set, and $\overline{SH}(K_{m,n}, 1-x) = \frac{k(n-k)}{n-k+1}$. Since we may as well take $k \in \{1, \ldots, n-1\}$ so that $\frac{k(n-k)}{n-k+1}$ is maximized, we conclude that $\overline{SH}(K_{m,n}, 1-x) = \overline{SH}(K_n, 1-x)$, by Lemma 4.5, if $a = 0$ in the optimal weighting.

Also, weighting with $a = 0$ and $b = \frac{1}{n-k+1}$, where $k \in \{1, \ldots, n-1\}$ maximizes $\frac{k(n-k)}{n-k+1}$, shows that $\overline{SH}(K_{m,n}, 1-x) \ge \overline{SH}(K_n, 1-x)$, if $m \le n < m + 2\sqrt{m}$. Therefore, $\overline{SH}(K_{m,n}, 1-x) \ge \max[\frac{1}{n+1}\lfloor\frac{(n+m)^2}{4}\rfloor, \overline{SH}(K_n, 1-x)]$ if $m \le n < m+2\sqrt{m}$, and it remains to demonstrate the reverse inequality. We have disposed of the case when there is an optimal weighting with $a = 0$ on M.

Now suppose that $a > 0$ (and $m \le n < m + 2\sqrt{m}$). Again, from Lemma 4.1, we have that our cheapest dismantling set S will be one of three, either $S = M$, $S = N$ or $S \subsetneq N$. We now consider each of these three cases.

<u>Case 1: $S = M$</u> We may as well assume that $\overline{SH}(K_{m,n}, 1-x) = m(1-a) > \frac{1}{n+1}\lfloor\frac{(n+m)^2}{4}\rfloor$. Noting that $\frac{1}{n+1}\lfloor\frac{(n+m)^2}{4}\rfloor \geq \frac{mn}{n+1}$, we have that $m(1-a) > \frac{mn}{n+1}$ and thus that $a < \frac{1}{n+1}$.

<u>Case 1A: $ma + b \geq 1$</u> If $ma+b \geq 1$, then $b \geq 1-ma > 1-\frac{m}{n+1} = \frac{n-m+1}{n+1}$ and (since S is a cheapest g-dismantling set) $m(1-a) \leq n(1-b) \implies m+nb \leq n+ma < n+\frac{m}{n+1} \implies b < \frac{n-m}{n} + \frac{m}{n(n+1)}$. So if $m(1-a) > \frac{mn}{n+1}$, b satisfies $\frac{n-m+1}{n+1} < b < \frac{n-m}{n} + \frac{m}{n(n+1)}$. However, $\frac{n-m+1}{n+1} < \frac{n-m}{n} + \frac{m}{n(n+1)} \implies 0 < 0$. This contradiction disposes of this case.

<u>Case 1B: $ma + (n-k)b < 1 \leq ma + (n-k+1)b$ for some $k \in \{1, \ldots, n-1\}$.</u> Since S is a cheapest g-dismantling set, we also have that $m(1-a) \leq k(1-b) \implies b \leq 1 - \frac{m(1-a)}{k}$ and from the original inequality here in the case 1B, we have that $\frac{1-ma}{n-k+1} \leq b < \frac{1-ma}{n-k}$.

Claim If $a > 0$ then it must be the case that $ma + (n-k+1)b = 1$ and $m(1-a) = k(1-b)$.

Proof of Claim If both $ma + (n-k+1)b > 1$ and $m(1-a) < k(1-b)$, reduce a slightly without changing b, so that the new value $\tilde{a}$, still positive, satisfies $m\tilde{a} + (n-k+1)b \geq 1$ and $m(1-\tilde{a}) \leq k(1-b)$. Then $S = M$ is still a cheapest dismantling set with respect to the new weighting, but then $\tilde{a} < a \implies \overline{SH}(K_{m,n}, 1-x) = m(1-a) < m(1-\tilde{a}) \leq \overline{SH}(K_{m,n}, 1-x)$, a contradiction. Therefore, it is not possible that both inequalities are strict.

If $ma + (n-k+1)b > 1$ and $m(1-a) = k(1-b)$ then $b = 1 - \frac{m}{k}(1-a)$. Now if we decrease the value of a just a little to a value $\tilde{a}$ then b will decrease by a small amount to a new value $\tilde{b} = 1 - \frac{m}{k}(1-\tilde{a})$, and we can still have the inequality $m\tilde{a} + (n-k+1)\tilde{b} > 1$. However, this implies that $S = M$ is still a cheapest g-dismantling set with respect to the new weighting with a dismantling cost of $m(1-\tilde{a})$. Since $a > \tilde{a}$ we get that $\overline{SH}(K_{m,n}, 1-x) = m(1-a) < m(1-\tilde{a}) \leq \overline{SH}(K_{m,n}, 1-x)$, a contradiction. Therefore, if $m(1-a) = k(1-b)$ then $ma + (n-k+1)b = 1$.

Now if $m(1-a) < k(1-b)$ and $ma + (n-k+1)b = 1$ then $b = \frac{1-ma}{n-k+1}$. If we lower a just a little to a value say $\tilde{a}$ then b will increase slightly to a value $\tilde{b}$ and the inequality $m(1-\tilde{a}) < k(1-\tilde{b})$ can be maintained. Then we can actually increase the cost of removing M which will still be a cheapest dismantling set. We get a contradiction once again, as above. $\square$

Since $a > 0$, then from claim 1 we have that $ma + (n-k+1)b = 1$ and $m(1-a) = k(1-b)$ and so we want the a and/or the b that satisfy this system of equations. Finding both of these values gives us our optimal weighting. Since $ma+(n-k+1)b = 1$ and $m(1-a) = k(1-b)$ we can solve this system by rewriting $m(1-a) = k(1-b)$ as $m-k = ma-kb$ and eliminating the ma from both equations. This leaves us with $1+k-m = (n+1)b \implies b = \frac{k-m+1}{n+1} \implies k \geq m$ in this case. Now $m(1-a) = k(1-b) = k(1-\frac{1+k-m}{n+1}) = k(\frac{n-k+m}{n+1})$. However, $\frac{k(n+m-k)}{n+1}$ achieves its maximum value $\frac{1}{n+1}\lfloor\frac{(n+m)^2}{4}\rfloor$, for $k \in \{1, \ldots, n-1\}$, at $k = \lfloor\frac{n+m}{2}\rfloor$, so $\overline{SH}(K_{m,n}, 1-x) = \frac{k(n+m-k)}{n+1} \leq \frac{1}{n+1}\lfloor\frac{(n+m)^2}{4}\rfloor$, contrary to assumption, which finishes this case.

<u>Case 2: $S \subsetneq N$</u> Let $|S| = k \in \{1, \ldots, n-1\}$ and since S is a cheapest g-dismantling set then we have that $ma+(n-k)b < 1 \leq ma+(n-k+1)b$ and $k(1-b) \leq m(1-a)$. We also may as well assume that $\overline{SH}(K_{m,n}, 1-x) = k(1-b) > \frac{1}{n+1}\lfloor\frac{(n+m)^2}{4}\rfloor$ (keeping in mind that $m \leq n < m + 2\sqrt{m}$). If we consider the possibility of lowering b to try to exact a

slightly higher cheapest dismantling cost, we see that at least one of the two inequalities, $ma + (n - k + 1)b \geq 1$ and $k(1 - b) \leq m(1 - a)$ must be equality. Now suppose that we have $ma + (n - k + 1)b > 1$ and $k(1 - b) = m(1 - a)$. Then M is also a cheapest dismantling set, i.e., we are in Case 1, and so we are done in this case. Now suppose that $ma + (n - k + 1)b = 1$ and $k(1 - b) < m(1 - a)$. From the equation, we get that $a = \frac{1-(n-k+1)b}{m}$ and so if we lower b, a increases. Pushing b down very slightly, we get new weights $\tilde{b} < b$ and $\tilde{a} > a$ so that $k(1 - \tilde{b}) < m(1 - \tilde{a})$ and $m\tilde{a} + (n - k + 1)\tilde{b} = 1$. Clearly, S is still a cheapest g-dismantling set but with a dismantling cost of $k(1 - \tilde{b}) > k(1 - b)$. This yields a contradiction to the supposition that $\overline{SH}(K_{m,n}, 1 - x) = k(1 - b)$, and finishes the disposal of Case 2.

<u>Case 3: $S = N$</u> We may as well assume that $\overline{SH}(K_{m,n}, 1 - x) = n(1 - b) > \frac{1}{n+1}\lfloor\frac{(n+m)^2}{4}\rfloor$ (keeping in mind that $a > 0$ and $m \leq n < m+2\sqrt{m}$). Since S is a cheapest g-dismantling set, we have that $ma+b \geq 1$ and $m(1-a) \geq n(1-b)$. We want to consider the possibility of lowering b a little in order to exact a higher cheaper dismantling cost. As in previous cases, it turns out that it must be the case that at least one of the inequalities $ma+b \geq 1$ and $m(1-a) \geq n(1-b)$ must be equality. Suppose that $ma+b = 1$ and $m(1-a) > n(1-b)$. This implies that $a = \frac{1-b}{m}$ and so if we decrease b slightly, we increase a slightly and the inequality $m(1-a) > n(1-b)$ can still hold. So N is still the cheapest g-dismantling set, but its new dismantling cost is actually higher than the assumed value of $\overline{SH}(K_{m,n}, 1-x)$ and so we have a contradiction leading us to the conclusion that b cannot be lowered at all; and in fact, it must be the case that if $ma + b = 1$, then $m(1 - a) = n(1 - b)$.

Therefore, it must be that $m(1 - a) = n(1 - b)$, which implies that M is also a cheapest dismantling set, which returns us to case 1 and finishes the proof. $\square$

4.2 The constant-on-each-part Shields-Harary numbers of $K_{m,n}$

Let $P_0(m,n,f)$ be defined as $\overline{SH}(K_{m,n},f)$ is, except that only weightings constant on each of M,N are allowed. The arguments in [2] show that $P_0(m,n,f)$ is achieved by some such weighting (assuming f is continuous), and Lemma 4.2 guarantees that $P_0(m,n,f)$ will be achieved with a weight on M less than or equal to that on N. Proposition 3.3 implies that if f is concave on $[0,1]$ and $f(1) = 0$ then $P_0(m,n,f) = \overline{SH}(K_{m,n},f)$. We will make some general remarks about $P_0(m,n,f)$ and then attempt to calculate if for particular cost functions f, especially $f(x) = (1-x)^p$, $0 < p \leq 1$, and $f(x) = 1 - x^p$, $p \geq 1$, since these are concave on $[0,1]$ and $f(1) = 0$.

Corollary 4.1 shows that as long as f is concave on $[0,1]$ and $f(1) = 0$ then $P_0(m,m,f) = \overline{SH}_0(K_{m,m},f)$ and as a direct result, Corollary 4.2 gives the value of $P_0(m,m,1-x) = \frac{m^2}{m+1}$. Proposition 4.1 gives the value of $P_0(m,n,1-x)$ for all $m \leq n$.

Now in terms of different cost functions, it is not difficult to see that the following inequalities should hold:

$$P_0(m,n,1-x) \leq P_0(m,n,1-x^p) \text{ where } p \geq 1$$

$$P_0(m,n,1-x) \leq P_0(m,n,(1-x)^q) \text{ where } 0 < q \leq 1$$

In fact, we conclude with the following generalization of these two inequalities.

Proposition 4.2 *Let $f_1(x)$ and $f_2(x)$ be non-increasing, continuous functions on $[0,\infty)$ satisfying $f_1(x) \leq f_2(x)$ for all $x \in [0,\infty)$. Then for any of the Shields-Harary parameters $P = P(G,f)$, and any G, $P(G,f_1) \leq P(G,f_2)$.*

Proof. Suppose g is an admissible (depending on how P is defined) weighting of the vertices of G and suppose S is a g-dismantling set (of whichever type is called for in the definition of P) of least cost, with respect to f_2. Then S is a g-dismantling set and $\sum_{u\in S} f_1(g(u)) \leq \sum_{u\in S} f_2(g(u))$. Consequently, the minimum cost of dismantling (g, G) with respect to f_1 is no greater than that of dismantling with respect to f_2. Since this holds for all admissible g, and since $P(G, f_i)$ is the supremum of these minimum costs, $i = 1, 2$, it follows that $P(G, f_1) \leq P(G, f_2)$. $\square$

Corollary 4.3 *If f_1, f_2 are as in the Proposition above, and if $f_1(x) \leq f_2(x)$ for all $x \in [0, 1]$, then $P_0(m, n, f_1) \leq P_0(m, n, f_2)$.*

Proof. In the definition of P_0, only the values of the cost function on $[0, 1]$ matter. To apply Proposition 4.1, extend f_i to $[0, \infty)$ by setting $f_i(x) = f_i(1)$, for $x > 1$, $i = 1, 2$. $\square$

Corollary 4.4 *For $0 < p \leq 1 \leq q$, $m \leq n < m+2\sqrt{m}$, $\frac{1}{n+1}\lfloor\frac{(n+m)^2}{4}\rfloor \leq \min[\overline{SH}(K_{m,n}, 1-x^q), \overline{SH}(K_{m,n}, (1-x)^p)]$, and for $n \geq m+2\sqrt{m}$, $m = \min[\overline{SH}(K_{m,n}, 1-x^q), \overline{SH}(K_{m,n}, (1-x)^p)]$.*

Proof. In the definition of $\overline{SH}$, only the values of the cost function on $[0, 1]$ matter. $1 - x \leq 1 - x^q$ for $q \geq 1$ and $x \in [0, 1]$ and $1 - x \leq (1 - x)^p$ for $0 < p \leq 1$ and $x \in [0, 1]$, thus proving the corollary, for $m \leq n < m + 2\sqrt{m}$, and proving $m \leq \min[\overline{SH}(K_{m,n}, 1 - x^q), \overline{SH}(K_{m,n}, (1 - x)^p)]$ if $n > m + 2\sqrt{m}$.

By the same argument that was applied in the case $f(x) = 1 - x$, it is easy to see that for any cost function f satisfying $f(0) = 1$, $f(1) = 0$, $\overline{SH}(K_{m,n}, f) \leq m$. This completes the proof of the Corollary. $\square$

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